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ON SELF-ADJOINTNESS OF THE TWO-DIMENSIONAL MAGNETIC SCHRODINGER OPERATOR

Abstract

In the paper, under definite conditions on magnetic and electric potentials, the self-adjointness of the two-dimensional Schrodinger operator in electromagnetic field is proved.

It is known that the Hamiltonian of a number of physical problems (see for example [1]) in the two-dimensional space R_2 is given formally by the magnetic differential Schrodinger expression

$$H_{a,V}=\sum_{k=1}^{2}\left(\frac{1}{i}\frac{\partial}{\partial x_k}+a_k\left(x\right)\right)^2+V\left(x\right), \tag{1}$$

where $i=\sqrt{-1}$ is the imaginary unit, $x=(x_1,x_2)\in R_2$, $a\left(x\right)=\left(a_1\left(x\right),a_2\left(x\right)\right)$ and $V\left(x\right)$ are magnetic and electric potentials, respectively, and these potentials are real functions. Note that if the magnetic field is perpendicular to the plane x_1Ox_2 and retains the three-dimensional charged particle in this plane, then after insulating the free motion along the axis x_3 we get a Hamiltonian of the form $H_{a,V}$ in the state space $L_2\left(R_2\right)$ (see [2] or [3]).

In the present paper, in the space $L_2\left(R_2\right)$ we study the self-adjointness of the two-dimensional magnetic Schrodinger operator generated by the differential expression $H_{a,V}$, where the real magnetic and electric potentials $a\left(x\right)$ and $V\left(x\right)$ satisfy the following conditions:

1) $\int\limits_{R_2}\left|a\left(x\right)\right|^{\nu}dx<+\infty$, where $\nu>2$, $\left|a\left(x\right)\right|=\sqrt{a_1^2\left(x_1,x_2\right)+a_2^2\left(x_1,x_2\right)}$;

2) $\int\limits_{R_2}\left|\Phi\left(x\right)\right|^{\mu}dx<+\infty$, where $\mu>1$, $\Phi\left(x\right)\equiv\Phi\left(x_1,x_2\right)=a^2\left(x_1,x_2\right)+V\left(x_1,x_2\right)+idiva\left(x_1,x_2\right)$, $a^2\left(x\right)\equiv a^2\left(x_1,x_2\right)=a_1^2\left(x_1,x_2\right)+a_2^2\left(x_1,x_2\right)$, $diva\left(x_1,x_2\right)=\frac{\partial a_1(x_1,x_2)}{\partial x_1}+\frac{\partial a_2(x_1,x_2)}{\partial x_2}$.

Note that the similar issues were studied in one-dimensional case in [4], in three-dimensional case in [5], [6].

Subject to conditions 1) and 2) we can write differential equation (1) in the form

$$\Delta_{a,V}=-\Delta+W,$$

where Δ is a two-dimensional Laplace operator

$$W=-2idiva\left(x\right)+\Phi\left(x\right). \tag{2}$$

It is known that if $a\left(x\right)$ and $V\left(x\right)$ are sufficiently smooth bounded functions, then the minimal operators (in this case they are maximal) H_0 and $H=H_0+W$

that correspond to differential expressions $-\Delta$ and $-\Delta_{a,V}$ respectively, are self-adjoint operators in $L_2(R_2)$ with identical domains of definition $W_2^2(R_2)$ (second order Sobolev space). Generally speaking, under conditions 1) and 2) the differential expression $\Delta_{a,V}$ doesn't define the minimal operator on a linear manifold $C_0^\infty(R_2)$. Therefore, for constructing a self-adjoint operator with the help of this expression, we'll use the method of quadratic forms. To this end, recall some denotation and notation (detailed information in the books [7, p. 303], [8, p. 185], [9, p. 386]).

Let E be a Hilbert space and the linear manifold $Q(q)$ be dense in E. Denote by $q(\varphi,\psi)$ a complex-valued one-and-a half linear form with domain of definition $Q(q)$, and by $q(\varphi)=q(\varphi,\varphi)$ a quadratic form associated with $q(\varphi,\psi)$.

It the one-and-a half linear form $q(\varphi,\varphi)$ is generated by some linear operator A i.e

$$\forall\varphi\in Q(q), \quad \forall\psi\in D(A)\Longrightarrow q(\varphi,\psi)=(\varphi,A\psi),$$

then its domain of definition is denoted by $Q(q)=Q(A)$.

Definition. *Let the operator A be self-adjoint and lower bounded. The symmetric operator B is said to be A-bounded in the sense of forms if*

i) $Q(A)\subseteq Q(B)$,

ii*) $\exists a,b>0, \quad \forall\varphi\in Q(A)\Rightarrow|(\varphi,B\varphi)|\le a(\varphi,A\varphi)+b(\varphi,\varphi)$.*

The greatest lower bound of all such a is called A-bound of the operator B in the sense of forms.

Consider in $L_2(R_2)$ the quadratic forms

$$h_0(\varphi)=\int\limits_{-\infty}^{+\infty}|\nabla\varphi|^2\,dx,$$

$$h_{a,V}(\varphi)=h_0(\varphi)+(W\varphi,\varphi),$$

where $\nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right)$ is Hamilton's symbolic vector, W an operator acting by formula (2). Obviously, $h_0(\varphi)$ corresponds to the selfadjoint operator $H_0:=-\Delta$ with domain of definition $W_2^2(R_2)$. It is known that $Q(h_0)=W_2^1(R_2)=D\left(H_0^{1/2}\right)$ (here $W_2^1(R_2)$ is the Sobolev space of first order), and $\forall\varphi\in Q(h_0)=W_2^1(R_2)=D\left(H_0^{1/2}\varphi,H_0^{1/2}\varphi\right)$.

The following theorem is valid.

Theorem. *Let conditions 1) and 2) be fulfilled. Then there exists a lower bounded self-adjoint operator $H=H_0+W$ responsible for the form $h_{a,V}(\varphi)=h_0(\varphi)+(W\varphi,\varphi)$ with $Q(H_0)=Q(H)$ such that any essential domain of the operator H_0 is an essential domain for the operator H as well. In particular, the space of the basic functions $C_0^\infty(R_2)$ is the essential domain of the operator H.*

Proof. Obviously, the operator W acting according to formula (2), is symmetric. Show that $Q(H_0)\subseteq Q(W)$. Take an arbitrary element φ from $Q(H_0)\subseteq W_2^1(R_2)$. Apply to the integral

$$\int\limits_{R_2}\Phi(x)\,\varphi(x)\,\overline{\varphi(x)}dx=\int\limits_{R_2}\Phi(x)\,|\varphi(x)|^2\,dx$$

the Holder inequality

$$\left|\int\limits_{R_2} \Phi(x)\varphi(x)\overline{\varphi(x)}dx\right| \le \left\{\int\limits_{R_2} |\Phi(x)|^{\mu} dx\right\}^{\frac{1}{\mu}} \left\{\int\limits_{R_2} |\varphi(x)|^{2\mu'} dx\right\}^{\frac{1}{\mu'}}, \qquad (3)$$

where $\mu' = \frac{\mu}{\mu-1} > 1$. From the Sobolev-Il'in imbedding theorem with a limiting exponent (see [10] or [11, p. 273, point 6.1]) we have

$$\left\{\int\limits_{R_2} |\varphi(x)|^{2\mu'} dx\right\}^{\frac{1}{\mu'}} = \left(\left\{\int\limits_{R_2} |\varphi(x)|^{2\mu'} dx\right\}^{\frac{1}{2\mu'}}\right) \le c\,\|\varphi\|^2_{W_2^1(R_2)}, \qquad (4)$$

where c is independent of φ (in the course of the paper we'll denote by the letter c a constant, not necessary one and the same). From (3) and (4) we find

$$\left|\int\limits_{R_2} \Phi(x)\varphi(x)\overline{\varphi(x)}dx\right| < +\infty. \qquad (5)$$

Now, using the equality

$$div\,(a(x)\varphi(x))\overline{\varphi(x)} = (div\,(x))\,|\varphi(x)|^2 + a(x)\,div\,(\varphi(x))\,\overline{\varphi(x)},$$

we estimate the integral

$$\int\limits_{R_2} div\,(a(x)\varphi(x))\overline{\varphi(x)}dx.$$

If we take into account that $div a(x) \in L_{\mu}(R_2)$ follows from condition 2) then applying the reasoning similar in the estimation of the integral $\int\limits_{R_2} \Phi(x)\varphi(x)\overline{\varphi(x)}dx$, we get

$$\left|\int\limits_{R_2} div(a(x)\,|\varphi(x)|^2\,dx\right| \left\{\int\limits_{R_2} |div(a(x)|^{\mu} dx\right\}^{\frac{1}{\mu}} \left(\left\{\int\limits_{R_2} |\varphi(x)|^{2\mu'} dx\right\}^{\frac{1}{2\mu'}}\right)^2 \le$$

$$\le c\left\{\int\limits_{R_2} |div\,(a(x))|^{\mu} dx\right\}^{\frac{1}{\mu}} \|\varphi\|^2_{W_2^1(R_2)} < +\infty. \qquad (6)$$

Now we estimate the integral

$$\int\limits_{R_2} a(x)\frac{\partial\varphi(x)}{\partial x_j}\overline{\varphi(x)}dx, \quad j = 1,2.$$

Using the general Holder inequality for several functions (see [12], p. 13), we have:

$$\left|\int\limits_{R_2} a(x)\frac{\partial\varphi(x)}{\partial x_j}\overline{\varphi(x)}dx\right| \le \left\{\int\limits_{R_2}|a(x)|^{\nu}dx\right\}^{\frac{1}{\nu}}\times$$

$$\times\left\{\int\limits_{R_2}\left|\frac{\partial\varphi(x)}{\partial x_j}\right|^2dx\right\}^{\frac{1}{2}}\left\{\int\limits_{R_2}|\varphi(x)|^{\nu'}dx\right\}^{\frac{1}{\nu'}},\quad j=1,2, \tag{7}$$

where $\frac{1}{\nu}+\frac{1}{\nu'}+\frac{1}{2}=1$ i.e. $\nu'=\frac{2\nu}{\nu-2}$. The right integral in the right part of inequality (7), is finite by condition 1) and the second integral is finite by $\varphi(x)\in W_2^1(R_2)$. Note that $\nu'=\frac{2\nu}{\nu-2}>2$, therefore, from the Sobolev Il'in imbedding theorem with a limiting exponent it follows that

$$\left\{\int\limits_{R_2}|\varphi(x)|^{\nu'}dx\right\}^{\frac{1}{\nu'}}\le c\,\|\varphi\|_{W_2^1(R_2)}.$$

Obviously, from the obtained estimations it follows

$$\left|\int\limits_{R_2} a(x)\frac{\partial\varphi(x)}{\partial x_j}\overline{\varphi(x)}dx\right|<+\infty,\quad j=1,2. \tag{8}$$

Thus, from inequalities (5), (6) and (8) it follows that $\forall\varphi\in Q(H_0)$ expression

$$(W\varphi,\varphi)=\int\limits_{-\infty}^{+\infty}(W\varphi(x))\,\overline{\varphi(x)}dx$$

makes sense. This means that $\varphi\in Q(W)$, hence it follows that $Q(H_0)\subseteq Q(W)$.

Prove that the integrals

$$\int\limits_{|x-y|\le\delta}\frac{|a(y)|}{|x-y|}dy$$

and

$$\int\limits_{|x-y|\le\delta}\ln\frac{1}{|x-y|}\,|\Phi(y)|\,dy$$

uniformly on R_2 tend to zero as $0<\delta\to 0$. Apply to the integral

$$\int\limits_{|x-y|\le\delta}\frac{|a(y)|}{|x-y|}dy$$

the Holder inequality

$$\int\limits_{|x-y|\le\delta}\frac{|a(y)|}{|x-y|}dy\le\left\{\int\limits_{|x-y|\le\delta}|a(y)|^{\nu}dy\right\}^{\frac{1}{\nu}}\left\{\int\limits_{|x-y|\le\delta}\frac{1}{|x-y|^{p}}dy\right\}^{\frac{1}{p}}, \tag{9}$$

where $\frac{1}{\nu}+\frac{1}{p}=1$. From $\nu>2$ it follows that $p=\frac{\nu}{\nu-1}<2$. Since the integral

$$\int\limits_{|x-y|\leq\delta}\frac{1}{|x-y|^{p}}dy$$

for $p<2$ converges uniformly with respect to $x\in R_2$, then from inequality (9) and absolute continuity of the Lebesgue integral it follows that

$$\lim_{0<\delta\to 0}\left\{\sup_{x\in R_2}\frac{|a\,(y)|}{|x-y|}dy\right\}=0. \tag{10}$$

Similarly, using the Holder inequality, we get

$$\left|\int\limits_{|x-y|\leq\delta}\ln\frac{1}{|x-y|}\,|\Phi\,(y)|\,dy\right|\leq\left\{\int\limits_{|x-y|\leq\delta}|\Phi\,(y)|^{\mu}\,dy\right\}^{\frac{1}{\mu}}\left\{\int\limits_{|x-y|\leq\delta}|\ln|x-y||^{p}\,dy\right\}^{\frac{1}{p}},$$

where $\frac{1}{\mu}+\frac{1}{p}=1$. If we take into account that for any positive number ε

$$\lim_{r\to 0}r^{\varepsilon}\ln r=0,$$

then we get

$$\lim_{0<\delta\to 0}\left\{\sup_{x\in R_2}\int\limits_{|x-y|\leq\delta}\ln\frac{1}{|x-y|}\,|\Phi\,(y)|\,dy\right\}=0. \tag{11}$$

From conditions (10) and (11) it follows that the operator

$$W=-2idi\nu a\,(x)+\Phi\,(x)=-2ia\,(x)\cdot\nabla+\overline{\Phi\,(x)},$$

where $a\,(x)\cdot\nabla$ is a scalar product of the vectors $a\,(x)$ and ∇, belongs to the Kato class (see [13], p. 16). From the Schechter theorem [14, theorem 7.3] we get that the relative H_0 bound of the operator W equals zero. If we take into account that the space of basic functions $C_0^{\infty}\,(R_2)$ is the essential domain of the operator H_0, then we see that all the statements of the theorem follows from KLMN theorem (see e.i. 13, p. 11). The theorem is proved.

Remark. Note that the sum H_0+W is understood in the sense of forms, and may differ from the operator sum.

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Received September 25, 2013; Revised December 17, 2013.