Soltan A. ALIEV, Yaroslav I. YELEYKO,
Andriy Y. DREBOT

LIMIT THEOREMS AND TRANSITORY PHENOMENA IN THE RENEWAL EQUATION IN RANDOM ENVIRONMENTS

Abstract

This publication provides a research in which obtained the asymptotic of generalized solution of multidimensional renewal equation.

Problem formulation

It is the researching of the asymptotical behaviour of generalized solution of a multidimensional renewal equation. And was made comparison its behaviour with the asymptotical behaviour of ordinal solution of a multidimensional renewal equation.

Analysis of previous researching and publications

In articles [1] and [2] was researched the asymptotical behaviour the renewal matrix in ordinary, not generalized case. Also there were determined normalizing multiplier ρ^ε. It is responsible for time's scope. In our case it was found the asymptotical behaviour of matrix valued family of renewal equations under uncertainty.

Task formulation and solution

Consider the matrix renewal equation in series scheme:

$$X^\varepsilon(t) = A^\varepsilon(t) + \int_0^t F^\varepsilon(du)\, X^\varepsilon(t-u),\ t \geqslant 0,\, \varepsilon > 0$$

where $A^\varepsilon(t)$, $X^\varepsilon(t)-$ are the families of non negative matrix valued functions and unknown matrix valued functions respectively (matrix $d \times d-$ dimension). Its elements are finite non negative measures. They are depends on infinitesimal parameter $\varepsilon > 0$. $F^\varepsilon(dt) -$ is the family of preset non negative measures. Assume that we observe the renewal processes. They can described by matrix renewal equation in series scheme with variable free term $A^\varepsilon(t)$. Thus we obtain N different free terms $A_r^\varepsilon(t)$, $r = \overline{1 \ldots N}$. There is discrete distribution p_r, $r = \overline{1..N}$. It means that we we observe the renewal process with free term $A_r^\varepsilon(t)$, $r = \overline{1..N}$. Also $\sum\limits_{r=1}^{N} p_r = 1$.

Therefore the solution $\bar{X}(t) = \sum\limits_{i=1}^{N} p_i \int\limits_0^t H^\varepsilon(t-du)A_i^\varepsilon(u)$ corresponds to generalized solution of a renewal equation:

$$X^{\varepsilon *}(t) = A^{\varepsilon *}(t) + \int\limits_0^t F^\varepsilon(du)X^{\varepsilon *}(t-u).$$

Here is marked:

$$X^{\varepsilon *}(t) = \sum_{i=1}^{N} p_i X_i^\varepsilon(t);$$
$$A^{\varepsilon *}(t) = \sum_{i=1}^{N} p_i A_i^\varepsilon(t).$$

The underlying of family of non negative measures $F^\varepsilon(dt)-$ is that, it tends weakly to $F(dt),$ as $\varepsilon \to 0$. Such conditions are made:

1) $F^\varepsilon(dt)-$ is tends weakly to $F(dt),$ as $\varepsilon \to 0$:

$$\int\limits_0^\infty g(t)F^\varepsilon(dt) \underset{\varepsilon\to 0}{\longrightarrow} \int\limits_0^\infty g(t)F(dt) \tag{1}$$

for any continuous and limited function $g(t)$;

2) for any $\alpha > 0$, $\beta < 1$

$$\overline{\lim_{\varepsilon\to 0}} \max_i \sum_{j=1}^{d} F_{ij}^\varepsilon(\alpha) \leqslant \beta; \tag{2}$$

3) for any ε :

$$\max_i \sum_{j=1}^{d} F_{ij}^\varepsilon[0,\infty) \leqslant 1; \tag{3}$$

4)

$$\sum_{j=1}^{d} F_{ij}[0,\infty) = 1. \tag{4}$$

$F^\varepsilon(dt)-$ we can represent in block-diagonal form: $F = \mathrm{diag}\{F^1, F^2, \ldots, F^r\}$.

The set of indexes $E = \{1, 2, \ldots, d\}$ can be represented as $E_1, E_2, \ldots, E_r$ where $F_{ij} = \delta_{sk} F_{ij}^s$, when $i \in E_s$, $j \in E_k$, and $\delta_{sk} -$ Kroneker's symbol.

Each of the matrices F^i, $i = \overline{1,r}$ along the matrix's diagonal F is indivisible. Such condition is supposed:

$$\lim_{t\to\infty} \sup_\varepsilon \int\limits_t^\infty y F_{ij}^\varepsilon(dy) = 0, \tag{5}$$

and as result:

$$a_{ij} = \int_t^\infty y F_{ij}(dy) < \infty,$$

and let

$$\underset{i}{min} \sum_{j=1}^{d} a_{ij} > 0. \tag{6}$$

We should mark through $H^\varepsilon(t)$ renewal matrix, built with $F^\varepsilon(t)$:

$$H^\varepsilon(t) = \sum_{n=0}^{\infty} F^{\varepsilon(*n)}(t),$$

where

$$F^{\varepsilon(*0)}(t) = I,$$

$$F^{\varepsilon(*n)}(t) = F^\varepsilon(t) * F^{\varepsilon(*(n-1))} = \int_0^t F^\varepsilon(dy) F^{\varepsilon(*(n-1))}(t-y).$$

Here $I-$ identity matrix.

Matrix $F(dt)$ is called trellised matrix with step h, if $h > 0$ is exist and each of measures $F_{ij}(dt)$, $i, j \in E$ we can focus on the set $\{c_{ij} + nh,\ n = 0, \pm 1, \pm 2, \dots\}$ with the same step h.

In article [1] it has been proven uniqueness of normalizing factor ρ^ε for equation with each free term $A_r^\varepsilon(t)$, $r = \overline{1 \dots N}$. It was shown that asymptotic of this equation with each $A_r^\varepsilon(t)$, $r = \overline{1 \dots N}$ we can build on common time interval in the distant future. Also was used the theorem about asymptotic of generalized solution of a multidimensional renewal equation in series scheme. It was shown that this normalizing factor ρ^ε satisfies the conditions about finding the asymptotic of generalized solution $\bar{X}$.

The main material research

The principal results of this research are:

Assertion 1. Consider the conditions (1)-(6) are true for equation with each free term $A_r^\varepsilon(t)$, $r = \overline{1 \dots N}$ and every matrix $F^s(dt)$ is not trellised. Then the choice of normalizing factor ρ^ε for the renewal matrix $H^\varepsilon(t)$ of generalized equation depend only on $F^\varepsilon(t)$ and does not depend on $A^\varepsilon(t)$.

Substantiation 1. We know that solution of renewal equation

$$X^\varepsilon(t) = A^\varepsilon(t) + \int_0^t F^\varepsilon(du) X^\varepsilon(t-u), t \geqslant 0, \varepsilon > 0$$

has the form

$$X^{\varepsilon}(t)=\int\limits_{0}^{t}H^{\varepsilon}(t-du)A^{\varepsilon}(t)$$

$H^{\varepsilon}(t)$ was denoted earlier. So for every of N renewal equations we can obtain individual result. Then we get a system:

$$X_1^{\varepsilon}(t)=\int\limits_{0}^{t}H^{\varepsilon}(t-du)A_1^{\varepsilon}(u)$$

$$X_2^{\varepsilon}(t)=\int\limits_{0}^{t}H^{\varepsilon}(t-du)A_2^{\varepsilon}(u)$$

$$\cdots$$

$$X_N^{\varepsilon}(t)=\int\limits_{0}^{t}H^{\varepsilon}(t-du)A_N^{\varepsilon}(u)$$

where $t\geqslant 0,\varepsilon>0$. So we get:

$$\bar{X}(t)=p_1X_1^{\varepsilon}(t)+p_2X_2^{\varepsilon}(t)+\cdots+p_NX_N^{\varepsilon}(t);$$

$$\bar{X}(t)=\sum_{i=1}^{N}p_iX_i^{\varepsilon}(t);$$

$$\bar{X}(t)=\sum_{i=1}^{N}p_i\int\limits_{0}^{t}H^{\varepsilon}(t-du)A_i^{\varepsilon}(u).$$

Now we can find the normalizing factor ρ^{ε} for generalized solution $\bar{X}(t)$. We can summarize this equations:

$$p_1X_1^{\varepsilon}(t)+p_2X_2^{\varepsilon}(t)+\cdots+p_NX_N^{\varepsilon}(t)=$$
$$=p_1A_1^{\varepsilon}(t)+p_2A_2^{\varepsilon}(t)+\cdots+p_NA_N^{\varepsilon}(t)+$$
$$+p_1\int\limits_{0}^{t}F^{\varepsilon}(du)X_1^{\varepsilon}(t-u)+\cdots+p_N\int\limits_{0}^{t}F^{\varepsilon}(du)X_N^{\varepsilon}(t-u).$$

Or it can written in reduced form:

$$\sum_{i=1}^{N}p_iX_i^{\varepsilon}(t)=\sum_{i=1}^{N}p_iA_i^{\varepsilon}(t)+\sum_{i=1}^{N}p_i\int\limits_{0}^{t}F^{\varepsilon}(du)X_i^{\varepsilon}(t-u);$$

$$\sum_{i=1}^{N}p_iX_i^{\varepsilon}(t)=\sum_{i=1}^{N}p_iA_i^{\varepsilon}(t)+\int\limits_{0}^{t}F^{\varepsilon}(du)\sum_{i=1}^{N}p_iX_i^{\varepsilon}(t-u).$$

Or

$$X^{\varepsilon *}(t) = A^{\varepsilon *}(t) + \int_0^t F^{\varepsilon}(du) X^{\varepsilon *}(t-u)$$

where:

$$X^{\varepsilon *}(t) = \sum_{i=1}^{N} p_i X_i^{\varepsilon}(t);$$
$$A^{\varepsilon *}(t) = \sum_{i=1}^{N} p_i A_i^{\varepsilon}(t).$$

Therefor our solution $\bar{X}(t)$ corresponds to this renewal equation:

$$X^{\varepsilon *}(t) = A^{\varepsilon *}(t) + \int_0^t F^{\varepsilon}(du) X^{\varepsilon *}(t-u).$$

Now pay attention how multiplier ρ^{ε} was founded due to the proof of theorem 1 [] . It was determined by the next way:

$$\rho^{\varepsilon} = \sum_{s=1}^{r} \rho_s^{\varepsilon}$$

$$\rho_s^{\varepsilon} = \frac{1 - L^{\varepsilon}_{\omega_s \omega_s}(\infty)}{m_s}, s = \overline{1, r}.$$

r-is the number of blocks along the main diagonal of matrix $F(dt)$ where $F^{\varepsilon}(dt) \to F(dt)$. The sequence $L_{ij}^{\varepsilon(n)}(t)$ is monotonous relative to t and was denoted as:

$$L_{ij}^{\varepsilon(0)}(t) = F_{ij}^{\varepsilon}(t),$$
$$L_{ij}^{\varepsilon(n)}(t) = F_{ij}^{\varepsilon}(t) + \sum_{m \notin D} F_{im}^{\varepsilon} * L_{mj}^{\varepsilon(n-1)}(t).$$

Also consider that: $L_{ij}^{\varepsilon(n-1)}(t) = 0, n = 0$. As we can see, sequence $L_{ij}^{\varepsilon(n)}(t)$ depend only on $F^{\varepsilon}(t)$. Set D was denoted as:

$$D = \{w_1, w_2, \ldots, w_r\}.$$

Where $w_i \in E_i, i = \overline{1..r}$. $E_i, i = \overline{1..r}$. Whereas $L_{ij}^{\varepsilon(n)}(t) \leqslant H_{ij}^{\varepsilon}(t)$ for any n and sequence $L_{ij}^{\varepsilon(n)}(t)$ is nondecreasing relative to n, then there are these boundaries:

$$L_{ij}^{\varepsilon}(t) = \lim_{\varepsilon \to 0} L_{ij}^{\varepsilon(n)}(t).$$

Also we can remember how m_s was denoted. In the proof of Theorem 1 [2] m_s was defined as:

$$m_{ij}^{\varepsilon} = \int_0^{\infty} tL_{ij}^{\varepsilon}(dt);$$

$$m_{ij} = \lim_{\varepsilon \to 0} m_{ij}^{\varepsilon};$$

$$m_s = m_{w_s w_s} = \sum_{i,j \in E_s} \frac{p_i^{(s)}}{p_{w_s}^{(s)}} \cdot a_{ij}$$

Here: $p_i^{(s)}$, $p_{w_s}^{(s)}$-are coordinates of positive left eigenvector of matrix E_s. w_s-is fixed index.

$$a_{ij}^{\varepsilon} = \int_0^{\infty} tF_{ij}^{\varepsilon}(dt);$$

$$a_{ij} = \lim_{\varepsilon \to 0} a_{ij}^{\varepsilon}.$$

The choice of coefficient m_s also completely depend on $F^{\varepsilon}(t)$.

Therefor we shown that choice of normalizing factor ρ^{ε} depends only on $F^{\varepsilon}(t)$. It means that for generalized renewal equation:

$$X^{\varepsilon *}(t) = A^{\varepsilon *}(t) + \int_0^t F^{\varepsilon}(du) X^{\varepsilon *}(t-u)$$

with solution:

$$\bar{X}(t) = \sum_{i=1}^{N} p_i \int_0^t H^{\varepsilon}(t-du) A_i^{\varepsilon}(u);$$

$$\bar{X}(t) = \int_0^t H^{\varepsilon}(t-du) \sum_{i=1}^{N} p_i A_i^{\varepsilon}(u);$$

$$\bar{X}(t) = \int_0^t H^{\varepsilon}(t-du) A^{\varepsilon *}(u)$$

we can pick up normalizing factor ρ^{ε}, by the method described above. Also it will be satisfy each of equftion from this system:

$$X_1^\varepsilon(t) = A_1^\varepsilon(t) + \int_0^t F^\varepsilon(du) X_1^\varepsilon(t-u);$$

$$X_2^\varepsilon(t) = A_2^\varepsilon(t) + \int_0^t F^\varepsilon(du) X_2^\varepsilon(t-u);$$

$$\dots$$

$$X_N^\varepsilon(t) = A_N^\varepsilon(t) + \int_0^t F^\varepsilon(du) X_N^\varepsilon(t-u).$$

It is because every renewal equation from the system differs only in free term

$$A_i^\varepsilon(t), i = \overline{1..N}.$$

Further we need the definition of uniformly direct integrability by Riemann.

Definition. *Family of functions $A^\varepsilon(t)$ is uniformly direct integrable by Riemann on $[0,\infty)$, if:*

1) Sum $\sum\limits_{k=0}^{\infty} \sup\limits_{k\leqslant t\leqslant k+1} |A^\varepsilon(t)|$ *coincides uniformly with respect to* ε;

2) $\sup\limits_{\varepsilon} h \cdot \sum\limits_{k=0}^{\infty} \left[\sup\limits_{kh\leqslant t\leqslant (k+1)h} |A^\varepsilon(t)| - \inf\limits_{kh\leqslant t\leqslant (k+1)h} |A^\varepsilon(t)| \right] \xrightarrow[h\to 0]{} 0.$

Assertion 2. Let the families of functions $\left[[A_{ij}^e]_r(t),\ i,\ j\in E \right]$ are uniformly integrable by Riemann on $[0,\infty)$ and there are a boundary:

$$\lim_{\varepsilon\to 0} \int_0^\infty [A_{ij}^\varepsilon]_r(t)dt \equiv [D_{ij}]_r,\ r = \overline{1..N}.$$

If for generalized solution $\bar{X}(t)$, of renewal equation we have that each of matrix $F^s(dt)$ along the main diagonal of matrix $F(dt)$ is not trellised then nonzero matrix C $r\times r$-dimension and normalizing factor $\rho^\varepsilon \xrightarrow[\varepsilon\to 0]{} 0$ are exist. Where: $i\in E_s$, $j\in E_k$,

$$\bar{X}_{ij}\left(\frac{t}{\rho^\varepsilon}\right) \xrightarrow[\varepsilon\to 0]{} \frac{q_{sk}(t)}{\pi_k} \left[\vec{1}^{(s)} \otimes \vec{p}^{(k)} \bar{D}^k \right]_{ij},$$

$$q_{sk}(t) = \left[e^{tC}\right]_{sk}, \quad \pi_k = \sum_{i,j\in E_k} p_i^{(k)} \cdot \int_0^\infty t F_{ij}(dt),$$

$$\bar{D}^k = \sum_{r=1}^N p_r \cdot [D_{mj}]_r,$$

$$[D_{mj}]_r = \left[[D_{ij}]_r,\ i\in E_k,\ j\in E \right],$$

$$\lim_{\varepsilon\to 0} \int_0^\infty [A_{ij}^\varepsilon]_r(t)dt \equiv [D_{ij}]_r,\ r = \overline{1..N}.$$

Substantiation 2. As it was denoted earlier

$$\bar{X}(t)=\int_0^t H^\varepsilon(t-du)\sum_{i=1}^N p_i A_i^\varepsilon(u);$$

$$\bar{X}(t)=\int_0^t H^\varepsilon(t-du)A^{\varepsilon*}(u),$$

is the solution of this renewal equation:

$$X^{\varepsilon*}(t)=A^{\varepsilon*}(t)+\int_0^t F^\varepsilon(du)X^{\varepsilon*}(t-u),$$

where

$$X^{\varepsilon*}(t)=\sum_{i=1}^N p_i X_i^\varepsilon(t);$$

$$A^{\varepsilon*}(t)=\sum_{i=1}^N p_i A_i^\varepsilon(t).$$

As for each of the N equations from our system Theorem from [1] is true we can use its result for every equation and due to Assertion 1 we can found the asymptotic in the time scope $\frac{t}{\rho^\varepsilon}$. So matrix C and normalizing factor ρ^ε remain the same.

In this way:

$$\lim_{\varepsilon\to 0}\bar{X_{ij}}\left(\frac{t}{\rho^\varepsilon}\right)=\lim_{\varepsilon\to 0}\sum_{r=1}^N p_r\cdot[X_{ij}^\varepsilon]_r\left(\frac{t}{\rho^\varepsilon}\right)=$$

$$=\sum_{r=1}^N p_r\cdot\lim_{\varepsilon\to 0}[X_{ij}^\varepsilon]_r\left(\frac{t}{\rho^\varepsilon}\right)=\sum_{r=1}^N p_r\cdot\frac{q_{sk}(t)}{\pi_k}\left[\vec{1}^{(s)}\otimes\vec{p}^{(k)}D_r^k\right]_{ij}=$$

$$=\sum_{r=1}^N p_r\cdot q_{sk}(t)\cdot\sum_{m\in E_k}\frac{p_m^{(k)}}{\sum\limits_{i,j\in E_k}p_i^{(k)}a_{ij}}[D_{mj}]_r=$$

$$=\sum_{r=1}^N p_r\cdot\left[\frac{q_{sk}(t)}{\sum\limits_{i,j\in E_k}p_i^{(k)}a_{ij}}\cdot\sum_{m\in E_k}p_m^{(k)}\cdot[D_{mj}]_r\right]=$$

$$=\frac{q_{sk}(t)}{\pi_k}\cdot\sum_{r=1}^N p_r\cdot\left[\sum_{m\in E_k}p_m^{(k)}\cdot[D_{mj}]_r\right]=$$

$$=\frac{q_{sk}(t)}{\pi_k}\cdot\sum_{m\in E_k}p_m^{(k)}\cdot\left[\sum_{r=1}^N p_r\cdot[D_{mj}]_r\right]=\frac{q_{sk}(t)}{\pi_k}\left[\vec{1}^{(s)}\otimes\vec{p}^{(k)}\bar{D}^k\right]_{ij}.$$

We denote $\bar{D^k} = \sum_{r=1}^{N} p_r \cdot [D_{mj}]_r$.

Conclusions. Therefore we obtained that:

1) Normalizing multiplier ρ^ε for renewal equation with each free term $A_r^\varepsilon(t)$, $r = \overline{1 \ldots N}$ is unique.

2) We got the asymptotic of generalized solution of multidimensional renewal equation.

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Soltan A.Aliev

Institute of Mathematic and Mechanics of NAS of Azerbaijan.

9 B.Vahabzade st., Az1141, Baku, Azerbaijan

Tel.: (99412) 538-24-44 (off.).

Yaroslav I. Yeleyko, Andriy Y. Drebot
Ivan Franko National University of Lviv
1, Universitetska str., 79000, Lviv, Ukraine
Tel.: (38032) 239-45-31

Received September 24, 2013; Revised December 20, 2013.