Elnara B. SULTANOVA

TO THEORY OF FOURTH ORDER OPERATOR BUNDLES

Abstract

In the paper the conditions on the coefficients of a fourth order operator bundle, providing four-fold completeness of the system of eigen and associated vectors in separable Hilbert space are obtained.

Let H be a separable Hilbert space and assume that

1) C is a completely continuous self-adjoint operator in H with a spectrum in the angular sector $S_\varepsilon = \{\lambda : |arq\lambda| \le \varepsilon\}$, $0 \le \varepsilon < \pi/4$;

2) The operators B_j $(j = \overline{0,3})$ are bounded in H.

Consider in H the operator bundle

$$L(\lambda) = \lambda^4 C^4 + E + \sum_{j=0}^{3} \lambda^j B_j C^j, \tag{1}$$

where λ is a spectral parameter, E is a unit operator in H.

Definition 1. *If for some $\lambda \in \mathbb{C}$ there exists, $L^{-1}(\lambda)$ is determined in all of the space H and is bounded, then λ is said to be a regular point of the operator bundle $L(\lambda)$, and $L^{-1}(\lambda)$ is a resolvent of the operator bundle $L(\lambda)$.*

If for some $\lambda \in \mathbb{C}$ the equation $P(\lambda_0)\varphi_0 = 0$ has the solution $\varphi_0 \ne 0$, then φ_0 is said to be an eigen number, and φ_0 is called an eigen vector of the bundle $L(\lambda)$ responding to λ_0. If the vectors $\varphi_0, \varphi_1, ..., \varphi_{m_0}$ satisfy the equations

$$\sum_{m=0}^{s} \frac{L^{(m)}(\lambda_0)}{m!} \varphi_{s-m} = 0, \quad s = 0, ..., m_0,$$

then $\{\varphi_0, \varphi_1, ..., \varphi_{m_0}\}$ is said to be the system of eigen and associated vectors responding to λ_0.

Definition 2. *Let $\{\varphi_0, \varphi_1, ..., \varphi_{m_0}\}$ be the system of eigen and associated vectors of the bundle $L(\lambda)$ responding to the eigen number λ_0. Define the vectors*

$$\varphi_q^{(0)} = \varphi_q,$$

$$\varphi_q^{(j)} = \frac{d^j}{dt^j} e^{\lambda t} \left(\varphi_q^{(0)} + \varphi_{q-1}^{(0)} \frac{t}{1!} + ... + \frac{t^q}{q!} \varphi_0^{(0)} \right) \Bigg|_{t=0}, \quad (q = 0, ..., m_0)$$

in H and construct the system $\{\widetilde{\varphi}_q\} \subset H^4$, where $\widetilde{\varphi}_q = \left(\varphi_q^{(0)}, \varphi_q^{(1)}, \varphi_q^{(2)}, \varphi_q^{(3)}\right) \in H^4$, and H^4 is the product of four copies of the space H.

If the system $\{\widetilde{\varphi}_q\} \subset H^4$ formed with respect to all eigen numbers and eigen vectors is complete in H^4, it is said that the system of eigen and associated vectors of the bundle $L(\lambda)$ is four-fold complete in H.

The theorems on multiple completeness of eigen and associated vectors and the spectrum of such operator bundles were proved for instance in [1-6].

In the present paper we study analytic properties of the resolvent and prove theorems on four-fold completeness of the system of eigen and associated vectors of the bundle $L(\lambda)$ in H.

At first prove a theorem on behavior of the resolvent on some rays.

Theorem 1. *Let conditions 1), 2) be fulfilled, and it hold the inequality*

$$\sum_{j=0}^{3} b_j(\varepsilon)\|B_j\| < \varepsilon, \tag{2}$$

where

$$b_0(\varepsilon) = \begin{cases} 1, & 0 \le \varepsilon \le \pi/8 \\ \frac{1}{\sin 4\varepsilon}, & \pi/8 \le \varepsilon < \pi/4, \end{cases} \quad b_1^{(\varepsilon)} = b_3(\varepsilon) = \frac{3^{3/4}}{4}\frac{1}{\cos 2\varepsilon}, \quad b_2(\varepsilon) = \frac{1}{2\cos 2\varepsilon}.$$

Then on the rays $\Gamma_k = \left\{\lambda : \lambda = r^{\frac{\pi}{4}} e^{\frac{i\pi k}{2}} r > 0, k = \overline{0,3}\right\}$ the operator bundle $L(\lambda)$ is invertible and on these rays it holds the inequality

$$\left\|L^{-1}(\lambda)\right\| \le const, \quad \lambda \in \Gamma_k, \quad k = \overline{0,3}.$$

Proof. Let $\lambda = re^{\frac{\pi ik}{2}},\ k = \overline{0,3},\ r > 0$. Then from the equality

$$L(\lambda) = L_0(\lambda) + L_1(\lambda) = \left(E + L_1(\lambda) L_0^{-1}(\lambda)\right) L_0(\lambda)$$

we get that for the invertibility of $L(\lambda)$ on the rays Γ_k the invertibility of the operator $E + L_1(\lambda) L_0^{-1}(\lambda)$ is enough, since on the rays Γ_k the operator bundle

$$L_0(\lambda) = E + \lambda^4 C^4 = \prod_{j=1}^{4}(E - \lambda\omega_j C)$$

is invertible in H, and

$$L_0^{-1}(\lambda) = \left(E + \lambda^4 C^4\right)^{-1} = \prod_{j=1}^{4}(E - \lambda\omega_j C)^{-1}.$$

Here

$$\omega_1 = -\frac{1}{\sqrt{2}}(1+i)\,\omega_2 = -\frac{1}{\sqrt{2}}(1-i),\ \omega_3 = -\frac{1}{\sqrt{2}}(1+i),\ \omega_4 = -\frac{1}{\sqrt{2}}(1-i).$$

Show that subject to inequality (3) $E + L_1(\lambda) L_0^{-1}(\lambda)$ is invertible on the rays $\Gamma_k\ \left(k = \overline{0,3}\right)$. Since on the rays Γ_k

$$\left\|L_1(\lambda) L_0^{-1}(\lambda)\right\| = \left\|\sum_{j=0}^{3}\lambda^j B_j C^j L_0^{-1}(\lambda)\right\| \le \sum_{j=0}^{3}\|B_j\|\left\|\lambda^j C^j L_0^{-1}(\lambda)\right\|. \tag{3}$$

On the other hand,

$$\left\|\lambda^j C^j L_0^{-1}(\lambda)\right\| = \left\|\lambda^j C^j \left(E + \lambda^4 C^4\right)^{-1}\right\| = \left\|r^j C^j \left(E + r^4 e^{2\pi ki} C^4\right)^{-1}\right\| =$$

$$=\left\| r^j C^j \left(E+r^4C^4\right)^{-1}\right\|.$$

Note that for $\lambda_n \in S_\varepsilon$, $\lambda_n = |\lambda_n| e^{i\psi_n}$, $|\psi_n| < \varepsilon$ and if $\{e_n\}$ is an orthonornmalized basis of eigen vectors of the operator $\mathbf{C}$, then

$$Ce_n=\lambda_n e_n,\quad (e_n,e_m)=\delta_{nm}=\begin{cases}1, & n=m\\ 0, & n\neq m\end{cases},\ |\lambda_0|>|\lambda_1|>...|\lambda_n|...,$$

and from the spectral expansion of the operator C it follows that

$$\left\| r^j C^j \left(E+r^4C^4\right)^{-1}\right\| = \sup_n \left| r^j |\lambda_n|^j \left(1+r^4|\lambda_n|^4 e^{i4\psi_k}\right)^{-1}\right| =$$

$$=\sup_n \left| r^j |\lambda_n|^j \left(E+r^4|\lambda_n|^4 (\cos 4\psi_k + i\sin 4\psi_k)\right)^{-1}\right| =$$

$$=\sup_n \left| r^j |\lambda_n|^j \left(1+r^8|\lambda_n|^8 + 2|\lambda_n|^4 r^4 \cos 4\psi_k\right)\right|^{-1} =$$

$$=\sup_n \left(|r|\,|\lambda_n|^j\right)\left(1+r^8|\lambda_n|^8 + 2|\lambda_n|^4 r^4 \cos 4\varepsilon\right)^{-1/2} \leq$$

$$\leq \sup_{\tau\geq 0} \tau^j \left(1+\tau^8+2\tau^4\cos 4\varepsilon\right)^{-1/2}.$$

Let $j=0$. Then for $\lambda \in \Gamma_k \left(k=\overline{0,3}\right)$

$$\left\|\left(E+r^4C^4\right)^{-1}\right\| \leq \sup_{\tau\geq 0}\frac{1}{\left(1+\tau^8+2\tau^4\cos 4\varepsilon\right)^{1/2}}.$$

Since the function $f(\tau)\,\frac{1}{1+\tau^8+2\tau^4\cos 4\varepsilon}$ monotonically decreases for $0\leq\varepsilon\leq\pi/8$, then $\sup\limits_{\tau\geq 0}\frac{1}{1+\tau^8+2\tau^4\cos 4\varepsilon}=1$, and for $\pi/8\leq\varepsilon\leq\pi/4$ the function $f(\tau)$ accepts its maximum value for $r^4=-\cos 4\varepsilon$ $(\cos 4\varepsilon<0)$, therefore

$$\sup_{\tau\geq 0}\frac{1}{1+\tau^8+2\tau^4\cos 4\varepsilon}=\frac{1}{\sin^2 4\varepsilon}.$$

Consequently,

$$\left\|\left(E+r^4C^4\right)^{-1}\right\| \leq b_0(\varepsilon)=\begin{cases}1, & 0\leq\varepsilon\leq\pi/8\\ \frac{1}{\sin 4\varepsilon}, & \pi/8\leq\varepsilon\leq\pi/4\end{cases}. \tag{4}$$

Let $j=2$. In this case

$$\left\| r^2C^2\left(E+\lambda^4C^4\right)^{-1}\right\| \leq \sup_{\eta\geq 0}\left| r^2\left(1+r^8+2\tau^4\cos 4\varepsilon\right)^{-1/2}\right| \leq$$

$$\leq \sup_{\eta\geq 0}\left(\frac{\tau^4}{1+\tau^8+2\tau^4\cos 4\varepsilon}\right)^{1/2}=\frac{1}{2\cos 2\varepsilon}=b_2(\varepsilon). \tag{5}$$

Let $j=1$. In this case

$$\left\| rC\left(E+\lambda^4C^4\right)^{-1}\right\| \leq \sup_{\tau\geq 0}\left|\tau\left(1+r^8+2\tau^4\cos 4\varepsilon\right)^{-1/2}\right| \leq$$

$$\leq \sup_{\tau\geq 0}\left(\frac{\tau^2}{1+\tau^8+2\tau^4\cos 4\varepsilon}\right)^{1/2}\leq \sup_{\tau\geq 0}\left(\frac{\tau^2}{1+\tau^8+2\tau^4}\right)^{1/2}\times$$

$$\times\left(\frac{1+\tau^8+2\tau^4}{1+\tau^8+2\tau^4\cos 4\varepsilon}\right)^{\frac{1}{2}}=\frac{3^{3/4}}{4}\cdot\left(1+\frac{2\tau^4\left(1-\cos 4\varepsilon\right)}{1+\tau^8+2\tau^4\cos 4\varepsilon}\right)^{1/2}\leq$$

$$\leq\frac{3^{3/4}}{4}\cdot\left(1+\frac{4\tau^4\sin^2 2\varepsilon}{1+\tau^8+2\tau^4\cos 4\varepsilon}\right)^{1/2}\leq\frac{3^{3/4}}{4}\cdot\left(1+\frac{4\sin^2 2\varepsilon}{4\cos^2 2\varepsilon}\right)=$$

$$=\frac{3^{3/4}}{4}\frac{1}{\cos 2\varepsilon}=b_1\left(\varepsilon\right). \tag{6}$$

For $j=3$, similarly we have

$$\left\|r^2C^3\left(E+\lambda^4C^4\right)^{-1}\right\|\leq\sup_{\eta\geq 0}\left|\tau^3\left(1+r^8+2\tau^4\cos 4\varepsilon\right)^{-1/2}\right|\leq$$

$$\leq\sup_{\eta\geq 0}\left(\frac{\tau^6}{1+\tau^8+2\tau^4\cos 4\varepsilon}\right)^{1/2}\leq\sup_{\eta\geq 0}\left(\frac{\tau^6}{1+\tau^8+2\tau^4\varepsilon}\right)^{1/2}\times$$

$$\times\left(\frac{1+\tau^8+2\tau^4}{1+\tau^8+2\tau^4\cos 4\varepsilon}\right)^{1/2}\leq\frac{3^{3/4}}{4}\cdot\frac{1}{\cos 2\varepsilon}=b_3\left(\varepsilon\right). \tag{7}$$

Thus, taking into account inequalities (4-7) in inequality (3), we get that for $\lambda\in\Gamma_k$, $k=\overline{0,3}$

$$\left\|L_1L_0^{-1}\left(\lambda\right)\right\|\leq\sum_{j=0}^{3}b_j\left(\varepsilon\right)\left\|B_j\right\|\leq\alpha\left(\varepsilon\right)<1.$$

Then $L^{-1}\left(\lambda\right)=L_0^{-1}\left(\lambda\right)\left(E+L\left(\lambda\right)L_0^{-1}\right)^{-1}$, for $\lambda\in\Gamma_k$, $k=\overline{0,3}$

Thus, for $\lambda\in\Gamma_k$, $k=\overline{0,3}$

$$\left\|L^{-1}\left(\lambda\right)\right\|\leq\left\|L_0^{-1}\left(\lambda\right)\right\|\left\|\left(E+L_1\left(\lambda\right)L_0^{-1}\left(\lambda\right)\right)^{-1}\right\|\leq\frac{b_0\left(\varepsilon\right)}{1-\alpha\left(\varepsilon\right)}=const.$$

The theorem is proved.

Now, let's prove the following theorem.

Theorem 2. *Let the conditions of theorem 1 be fulfilled. Then the operator bundle $L\left(\lambda\right)$ has only a discrete spectrum with a unique limiting point at infinity.*

If $C\in\sigma_\rho\left(0<\rho<\infty\right)$, then $L^{-1}\left(\lambda\right)$ is represented in the form of ratio of entire functions of order ρ and of minimal type at order ρ.

Proof. Obviously,

$$L\left(\lambda\right)=E+\lambda^4C^4+\sum_{j=1}^{3}\lambda^jB_jC_j+B_0=\left(E+B_0\right)+\lambda^4C^4+\sum_{j=1}^{3}\lambda^jB_jC^j.$$

Since $\alpha\left(\varepsilon\right)<1$, hence it follows that $\left\|B_0\right\|<b_0^{-1}\left(\varepsilon\right)<1$. Thus, the operator $E+B_0$ is invertible and bounded in H. Then

$$L\left(\lambda\right)=\left(E+\sum_{j=1}^{3}\lambda^jB_jC^j\left(E+B_0\right)^{-1}+\lambda^4C^4\left(E+B_0\right)^{-1}\right)\left(E+B_0\right).$$

Let

$$Q(\lambda)=\sum_{j=1}^{3}\lambda^{j}B_{j}C^{j}(E+B_{0})^{-1}+\lambda^{4}C^{4}(E+B_{0})^{-1}.$$

Then for $\lambda\in\mathbf{C}$ the operator $Q(\lambda)$ is a completely invertible operator in H, and

$$L(\lambda)=(E+Q(\lambda))(E+B_{0}).$$

Since $E+Q(0)=E$ is invertible in H, then by the Keldysh lemma, the operator function $E+Q(\lambda)$ has a discrete spectrum with a limit point at infinity. From the representation

$$L^{-1}(\lambda)=(E+B_{0})^{-1}(E+Q(\lambda))^{-1},$$

we get that $L(\lambda)$ also has a discrete spectrum with a unique limit point at infinity.

If $C\in\sigma_{\rho}$, then the operators $B_{j}C^{j}(E+B_{0})^{-1}\in\sigma_{\rho/j},\quad j=\overline{1,3},\ \ C^{4}\in\sigma_{\rho/j}$.

Then by the Keldysh lemma, $L^{-1}(\lambda)$ is represented in the form of ratio of two entire functions of order

$$\max_{j=\overline{1,4}}\left(j\cdot\frac{\rho}{j}\right)=\rho.$$

and of minimal type at order p.

The theorem is proved.

Now prove a theorem on four-fold completeness of the system of eigen and associated vectors.

Theorem 3. *Let the conditions of theorem 1 be fulfilled, and* $A^{-1}\in\sigma_{\rho}\,(0<\rho\leq 2)$. *Then the system of eigen and associated vectors is four-fold complete in* H.

Proof. Assume the contrary. If the system of eigen and associated vectors of the bundle $L(\lambda)$ is not a four-fold complete system in H, then there exist the vectors $f_{j}\left(j=\overline{0,3}\right)$ even one of which is not zero, the vector-function

$$R(\lambda)=\left(L^{-1}\left(\overline{\lambda}\right)\right)^{*}\sum_{j=0}^{3}\lambda^{j}f_{j}$$

is an entire function [1]. Since on the rays $\lambda\in\Gamma_{k}$, $k=\overline{0,3}\quad L^{-1}(\lambda)$ exists and it holds the estimation $\left\|L^{-1}(\lambda)\right\|\leq const$, then by the Fragmen –Lindeloff theorem the operator bundle $\left\|L^{-1}(\lambda)\right\|\leq const$ for all λ from the complex plane, since $0<\rho\leq 2$ and the angle between the neighboring rays Γ_{k} equals $\frac{\pi}{4}$. Therefore,

$$\|R(\lambda)\|\leq const\,|\lambda|^{2},\quad \lambda\in\mathbf{C}.$$

Hence we have $R(\lambda)=g_{0}+\lambda g_{1}+\lambda^{2}g_{2}+\lambda^{3}g_{3}$. Then $\sum_{j=0}^{3}\lambda^{j}f_{j}=L^{*}\left(\overline{\lambda}\right)\sum_{q=0}^{3}\lambda^{q}g_{q}$. Comparing the coefficients in front of λ^{7}, we get $C^{*4}g_{3}=0$, i.e. $g_{3}=0$. Similarly we have that all $g_{j}=0,\ j=0,1,2$. Thus, $R(\lambda)\equiv 0$. Hence it follows that $f_{0}=f_{1}=f_{2}=f_{3}=0$. This contradiction proves the theorem.

The theorem is proved.

Using the results of the Keldysh paper [1], the following theorem is easily proved.

Theorem 4. *Let the conditions of theorem 3 be fulfilled, and the operators T_j, be completely continuous in H, then the system of eigen and associated vectors of the bundle*

$$M(\lambda) = E + \lambda^4 C^4 + \sum_{j=0}^{3} \lambda^j (B_j + T_j) C^j$$

is four-fold complete in H.

References.

[1]. Keldysh M.V. *On completeness of eigen functions of some classes of not self-adjoint operators.* UMN, 1971, vol. 26, No 4, pp. 15-41 (Russian).

[2]. Allahverdiyev J.E. *On completeness of the system of associated vectors close to normal ones.* DAN SSSR, 1957, vol. 115, No 2, pp. 207-210 (Russian).

[3]. Radzievskii G.V. *A problem on completeness of the root vectors in spectral theory of operator-functions.* // Uspekhi matem. Nauk, 1982, vol. 37, No 2, pp. 81-145 (Russian).

[4]. Gasymov M.G. *To theory of polynomial operator bundles of operators.* DAN SSSR, 1971, vol. 199, No 4, pp. 747-750 (Russian).

[5]. Mirzoyev S.S., Karaaslan M.D. *On well-posed solvability of a boundary value problem for second order operator-differential equations.* // Vestnik Bakinskogo Universiteta. ser. fiz. mat. Nauk, 2013, No 2, pp. 21-28 (Russian).

[6]. Sultanova E.B. *On a spectrum of a class of quadratic operator bundle.* Izvestia Pedagogicheskogo Universiteta. 2013, No 1, pp. 11-15 (Russian).

Elnara B. Sultanova
Baku State University,
23, Z. Khalilov str., AZ 1148, Baku, Azerbaijan
Tel.: (99412) 539 47 20 (off.).

Received October 03, 2013; Revised December 12, 2013.