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ON ONE NON-LOCAL BOUNDARY VALUE PROBLEM FOR A THIRD ORDER OPERATOR-DIFFERENTIAL EQUATION IN HILBERT SPACE

Abstract

In the paper, a non-local boundary value problem for a third order operator-differential equation is considered. The equation and boundary conditions were perturbed by some operators. The sufficient conditions providing well-defined solvability of the problem under consideration are obtained. All these conditions are expressed in the terms of the coefficients of the equation and the operator participating in the boundary conditions.

Let H be a separable Hilbert space, A be a positive-definite self-adjoint operator in H. Denote by H_γ a scale of Hilbert spaces generated by the operator A, i.e.

$$H_\gamma = D\left(A^\gamma\right), \quad (x,y)_\gamma = \left(A^\gamma x, A^\gamma y\right), \gamma \geq 0, \quad (H_0 = H).$$

Denote by $L_2\left(R_+; H\right)$ Hilbert space of all vector-functions $f\left(t\right)$ determined almost everywhere in R_+, with the values in H, with the norm

$$\|f\|_{L_2(R_+;H)} = \left(\int\limits_0^{+\infty} \|f\left(t\right)\|^2 dt\right)^{1/2} < \infty.$$

Further, following the monograph [1], introduce the Hilbert space $W_2^3\left(R_+; H\right) = \left\{u : u''' \in L_2\left(R_+; H\right), \quad A^3 u \in L_2\left(R_+; H\right)\right\}$

$$\|u\|_{W_2^3(R_+;H)} = \left(\left\|u'''\right\|^2_{L_2(R_+;H)} + \left\|A^3 u\right\|^2_{L_2(R_+;H)}\right)^{1/2}.$$

Here and in the sequel, the derivatives are understood in the sense of theory of distributions [1]. Denote by $L\left(X;Y\right)$ Banach space of bounded operators, and assume that the operator $K \in L\left(W_2^3\left(R_+; H\right), H_{1/2}\right)$. Determine the following subspace of the space $W_2^3\left(R_+; H\right)$:

$$W_{2,K}^3\left(R_+; H\right) = \left\{u : u \in W_2^3\left(R_+; H\right), \ u\left(0\right) = 0, u''\left(0\right) = Ku\right\},$$

where $K \in L\left(W_2^3\left(R_+; H\right), H_{1/2}\right)$. From the traces theorem [1], for $u \in W_2^3\left(R_+; H\right)$, $u^{(j)}\left(0\right) \in H_{3-j-1/2}$ $(j = 0,1,2)$ therefore the space $W_{2,K}^3\left(R_+; H\right)$ was well-defined.

Consider in H the following boundary value problem

$$P\left(d/dt\right) u\left(t\right) = u'''\left(t\right) - A^3 u\left(t\right) + \sum_{j=0}^{3} A_{3-j} u^{(j)}\left(t\right) = f\left(t\right), \quad t \in R_+, \qquad (1)$$

$$u\left(0\right) = 0, \quad u''\left(0\right) = Ku, \qquad (2)$$

where $f(t)$, $u(t)$ are vector functions with the values in H, and the operator coefficients satisfy the conditions:

1) A is a positive-definite self-adjoint operator;

2) $B_j = A_j A^{-j} \in L(H;H)$, $j = 1,2,3$;

3) $K \in L\left(W_2^3(R_+;H), H_{1/2}\right)$, moreover $\kappa = \|K\|_{W_2^3(R_+;H)\to H_{1/2}}$.

Definition 1. *If for $f(t) \in L_2(R_+;H)$ there exists a vector-function $u(t) \in W_2^3(R_+;H)$ satisfying equation (1) almost everywhere, then $u(t)$ is called a regular solution of equation (1).*

Definition 2. *If for any $f(t) \in L_2(R_+;H)$ there exists a regular solution of equation (1) that satisfies boundary conditions in the sense of convergence*

$$\lim_{t\to+0}\|u(t)\|_{5/2} = 0, \qquad \lim_{t\to+0}\left\|u''(t) - Ku\right\|_{1/2} = 0$$

and it holds the estimation

$$\|u\|_{W_2^3(R_+;H)} \leq const\,\|f\|_{L_2(R_+;H)},$$

then we say that problem (1), (2) is regularly solvable.

In the present paper, we'll find sufficient conditions on the coefficients of the boundary value problem, that provide regular solvability of problem (1), (2). Similar problems were investigated in [2.3].

First of all we'll consider the regular solvability of the problem

$$P_0(d/dt)u = u''' - A_u^3 = f(t), \tag{3}$$

$$u(0) = 0, \quad u''(0) = Ku. \tag{4}$$

To this end we introduce the following operators

$$P_0 u = P_0(d/dt)u = u''' - A^3 u,$$

$$P_1 u = P_1(d/dt)u = \sum_{j=0}^{3} A_{3-j} u^{(j)}(t), \quad Pu = P_0 u + P_1 u, \quad u \in W_{2,K}^3(R_+;H).$$

It holds

Theorem 1. *Let conditions 1) and 3) be fulfilled, and $\kappa > 1$. Then problem (3), (4) is regularly solvable.*

Proof. Since the general solution of the equation $P_0(d/dt)u(t) = 0$ from the space $W_2^3(R_+;H)$ is of the form

$$u_0(t) = e^{\omega_1 tA}x + e^{\omega_2 tA}y, x, y \in H_{5/2},\ \omega_1 = -\frac{1}{2}\left(1 + i\sqrt{3}\right),\ \ \omega_2 = -\frac{1}{2}\left(1 - i\sqrt{3}\right),$$

then from condition (2) with respect x and y we get the system of equations

$$x + y = 0, \quad \omega_1^2 A^2 x + \omega_2^2 A^2 y = K\left(e^{\omega_1 tA}x + e^{\omega_2 tA}y\right),$$

or with respect to x get the equation

$$\left(\omega_1^2 - \omega_2^2\right)x = A^{-2}K\left(e^{\omega_1 tA}x - e^{\omega_2 tA}x\right),$$

or

$$x-\frac{i}{\sqrt{3}}A^{-2}K\left(e^{\omega_1 tA}x-e^{\omega_2 tA}x\right)=0. \tag{5}$$

Define in $H_{5/2}$ the operator

$$Qx=\frac{i}{\sqrt{3}}A^{-2}K\left(e^{\omega_1 tA}x-e^{\omega_2 tA}x\right),\quad x\in H_{5/2}. \tag{6}$$

Then, it is obvious that for $x\in H_{5/2}$ it holds the inequality

$$\|Qx\|_{5/2}\le\frac{1}{\sqrt{3}}\left\|A^{1/2}K\left(e^{\omega_1 tA}x-e^{\omega_2 tA}x\right)\right\|\le$$

$$\le\frac{\kappa}{\sqrt{3}}\left\|e^{\omega_1 tA}x-e^{\omega_2 tA}x\right\|_{W_2^3(R_+;H)}. \tag{7}$$

On the other hand, for $x\in H_{5/2}$

$$\left\|e^{\omega_1 tA}x-e^{\omega_2 tA}x\right\|^2_{W_2^3(R_+;H)}=\left\|\omega_1^3A^3e^{\omega_1 tA}x-\omega_2^3A^3e^{\omega_2 tA}x\right\|^2_{L_2(R_+;H)}+$$

$$+\left\|A^3e^{\omega_1 tA}x-A^3e^{\omega_2 tA}x\right\|^2_{L_2(R_+;H)}=2\left\|A^3e^{\omega_1 tA}x-A^3e^{\omega_2 tA}x\right\|^2_{L_2(R_+;H)}=$$

$$=2\left(\left\|A^3e^{\omega_1 tA}x\right\|^2_{L_2(R_+;H)}+\left\|A^3e^{\omega_1 tA}x\right\|^2_{L_2(R_+;H)}\right)-$$

$$-2\operatorname{Re}\left(A^3e^{\omega_1 tA}x,e^{\omega_2 tA}x\right)_{L_2(R_+;H)}. \tag{8}$$

Let $A^{5/2}x=z$. Then from the spectral expansion of the operator A it follows that

$$\left\|A^3e^{\omega_1 tA}x\right\|^2_{L_2(R_+;H)}+\left\|A^{1/2}e^{\omega_1 tA}z\right\|^2_{L_2(R_+;H)}=\int_0^\infty\left(\int_{\mu_0}^\infty\mu e^{2\operatorname{Re}\omega_1 tA}\left(dE_\mu z,z\right)\right)dt=$$

$$=\int_{\mu_0}^\infty\mu\left(dE_\mu z,z\right)\int_{\mu_0}^\infty e^{t\mu}dt=\int_{\mu_0}^\infty d\left(E_\mu z,z\right)=\|z\|^2=\left\|A^{5/2}x\right\|^2=\|x\|^2_{5/2}. \tag{9}$$

Similarly we have

$$\left\|A^3e^{\omega_2 tA}x\right\|^2=\|x\|^2_{5/2}. \tag{10}$$

In the same way we get:

$$\operatorname{Re}\left(A^3e^{\omega_1 tA}x,A^3e^{\omega_2 tA}x\right)_{L_2(R_+;H)}=\operatorname{Re}\left(A^{1/2}e^{\omega_1 tA}z,A^{1/2}e^{\omega_2 tA}z\right)_{L_2(R_+;H)}=$$

$$=\operatorname{Re}\left(Ae^{(\omega_1+\overline{\omega}_2)tA}z,z\right)_{L_2(R_+;H)}=\operatorname{Re}\left(Ae^{2\operatorname{Re}\omega_1 tA}z,z\right)dz=$$

$$=\operatorname{Re}\left(\int_0^\infty\int_{\mu_0}^\infty\mu e^{2\omega_1 t\mu}dE_\mu z,z\right)=\operatorname{Re}\int_{\mu_0}^\infty\mu\left(\int_0^\infty e^{2\omega_1 t\mu}dt\right)\left(dE_\mu z,z\right)=$$

$$= \operatorname{Re}\left(-\frac{1}{2\omega_1}\int_{\mu_0}^{\infty}(dE_\mu z, z)\right) = -\operatorname{Re}\frac{1}{2\omega_1}\|z\|^2 = \frac{1}{4}\|z\|^2 = \frac{1}{4}\|x\|_{5/2}^2 . \quad (11)$$

Then from equality (8) allowing for equalities (9)-(11) we get

$$\left\|e^{\omega_1 tA}x - e^{\omega_2 tA}x\right\|_{W_2^3(R_+;H)}^2 \leq 3\|x\|_{5/2}^2 . \quad (12)$$

Therefore, from (7) it follows that $\|Qx\|_{5/2} \leq \kappa\|x\|_{5/2}$. Since $\kappa < 1$, then the operator $E - Q$ is invertible in $H_{5/2}$ and from equality (5) it follows that $x = 0$. Then $y = -x = 0$, consequently, $u_0(t) = 0$. Now prove that problem (3), (4) for any $f(t) \in L_2(R_+;H)$ has a regular solution. It is easy to see that the vector-function

$$u_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(i^3\varsigma^3 - A^3\right)^{-1}\int_0^{+\infty} f(s)\, e^{s(t-s)}d\varsigma ds$$

belongs to the space $W_2^3(R;H)$ and satisfies the equation $P_0(d/dt)\,u_1(t) = f(t)$ for $t \in R_+$ almost everywhere [2]. Denote the contraction of the vector-function $u_1(t)$ on $[0,\infty)$ by $\varsigma(t)$. Then $\varsigma(t) \in W_2^3(R_+;H)$ and $\varsigma^{(j)}(0) \in H_{3-j-1/2}$, $j = 0,1,2$. Now we look for the regular solution of problem (3), (4) in the form

$$u(t) = \varsigma(t) + e^{\omega_1 tA}x + e^{\omega_2 tA}y, \quad x, y \in H_{5/2}.$$

From condition (2) it follows that

$$\begin{cases} x + y = -\varsigma(0) \\ \varsigma''(0) + \omega_1^2 A^2 x + \omega_2^2 A^2 y = K(\varsigma(t)) + K\left(e^{\omega_1 tA}x + e^{\omega_2 tA}y\right). \end{cases}$$

Then $y = -\varsigma(0) - x$ and with respect to x we get the equation

$$\left(\omega_1^2 - \omega_2^2\right)x = -A^{-2}\varsigma''(0) + A^{-2}K(\varsigma(t)) + K\left(e^{\omega_1 tA}x - e^{\omega_2 tA}x - e^{-\omega_2 tA}\varsigma(0)\right),$$

or

$$x - \frac{i}{\sqrt{3}}K\left(e^{\omega_1 tA}x - e^{\omega_2 tA}x\right) =$$

$$= \frac{i}{\sqrt{3}}\left(A^{-2}\varsigma''(0) + A^{-2}K(\varsigma(t)) - A^{-2}K\left(e^{\omega_2 tA}\varsigma(0)\right)\right) \equiv \psi,$$

where $\psi \in H_{5/2}$. Thus $(E - Q)\,x = \psi$, $x = (E - Q)^{-1}\psi \in H_{5/2}$, and $y = -\varsigma(0) - (E - Q)^{-1}\psi \in H_{5/2}$. Consequently, $u(t)$ is a regular solution of problem (3), (4). On the other hand, for $u \in W_{2,K}^3(R_+;H)$ it holds the inequality

$$\|P_0 u\| = \left\|u''' - A^3 u\right\|_{L_2(R_+;H)} \leq \sqrt{2}\,\|u\|_{W_2^3(R_+;H)} .$$

Therefore, by the Banach theorem on the inverse operator, $P_0^{-1} : L_2(R_+;H) \to W_{2,K}^3(R_+;H)$ exists and is bounded. Then

$$\|u\|_{W_2^3(R_+;H)} \leq const\,\|f\|_{L_2(R_+;H)} .$$

The theorem is proved.

Corollary 1. *Let condition 1) be fulfilled. Then the problem*

$$P_0\left(d/dt\right)u - u''' - A^3u = f\left(t\right), \tag{13}$$

$$u\left(0\right) = 0, \quad u'''\left(0\right) = 0 \tag{14}$$

has a unique regular solution $\varsigma_0\left(t\right)$.

The proof follows from theorem 1 for $\kappa = 0\,(K = 0)$.

Now we use the method of [4] and prove some preliminarily statements that we'll need further.

Lemma 1. *The regular solution of problem (13), (14)* $\varsigma\left(t\right)$ *satisfies the following conditions*

$$\|\varsigma_0\|_{W_2^3(R_+;H)} \leq \frac{2}{\sqrt{3}}\|P_0\varsigma_0\|_{L_2(R_+;H)}, \tag{15}$$

$$\left\|A^2\varsigma_0'\right\|_{L_2(R_+;H)} \leq \frac{1}{\sqrt[3]{2}}\|P_0\varsigma_0\|_{L_2(R_+;H)}, \tag{16}$$

$$\left\|A'\varsigma_0''\right\|_{L_2(R_+;H)} \leq \frac{2}{\sqrt[3]{2}}\|P_0\varsigma_0\|^2_{L_2(R_+;H)}, \tag{17}$$

Proof. Taking into attention $\varsigma\left(0\right) = \varsigma''\left(0\right) = 0$, after integration by parts we get

$$\|f\|^2_{L_2(R_+;H)} = \|P_0\varsigma_0\|^2_{L_2(R_+;H)} = \left\|-\varsigma_0''' - A^3\varsigma_0\right\|^2 =$$

$$= \|\varsigma\|^2_{W_2^3} - 2\operatorname{Re}\left(\varsigma''', \varsigma\right)_{L_2(R_+;H)} = \|\varsigma\|^2_{W_2^3} - 2\left\|\varsigma'\left(0\right)\right\|^2_{3/2}. \tag{18}$$

At first consider the polynomial for $\beta \in (0,1)$

$$P\left(\lambda;\beta:A\right) = (1-\beta)\left(-\lambda^6E + A^6\right) = \varphi\left(\lambda;\beta:A\right)\cdot\varphi\left(-\lambda;\beta:A\right)$$

where $\varphi\left(\lambda;\beta:A\right) = \sqrt{1-\beta}\left(\lambda^3E + 2\lambda^2A + 2\lambda^2A^2 + A^3\right)$.

Obviously

$$\varphi\left(\lambda;\beta:A\right) = \sqrt{1-\beta}\left(\lambda E + A\right)\left(\lambda E - \omega_1 A\right)\left(\lambda - \omega_2 A\right),$$

$$\omega_1 = \overline{\omega_2} = -\frac{1}{2}\left(1 + \sqrt{3}i\right).$$

On the other hand, for all $\varsigma \in W_2^3\left(R_+;H\right)$ we have

$$\|\varphi\left(\lambda;\beta:A\right)\|^2_{L_2(R_+;H)} = (1-\beta)\left\|\varsigma''' + 2A\varsigma'' + 2A^2\varsigma' + A^3\varsigma\right\|^2_{L_2(R_+;H)} =$$

$$= (1-\beta)\left\|\varsigma'''\right\|^2_{L_2(R_+;H)} + 4\left\|A\varsigma''\right\|^2_{L_2(R_+;H)} + 4\left\|A^2\varsigma'\right\|^2_{L_2(R_+;H)} + \left\|A^3\varsigma\right\|^2_{L_2(R_+;H)} +$$

$$+4\operatorname{Re}\left(\varsigma''', A\varsigma''\right)_{L_2(R_+;H)} + 4\operatorname{Re}\left(\varsigma''', A^2\varsigma'\right)_{L_2(R_+;H)} + 2\operatorname{Re}\left(\varsigma''', A^2\varsigma\right)_{L_2(R_+;H)} +$$

$$+8\operatorname{Re}\left(A\varsigma'', A^2\varsigma'\right)_{L_2(R_+;H)} + 4\operatorname{Re}\left(A\varsigma'', A^3\varsigma\right)_{L_2(R_+;H)} +$$

$$+4\operatorname{Re}\left(A^2\varsigma', A^3\varsigma\right)_{L_2(R_+;H)}. \tag{19}$$

Taking into attention $\varsigma\left(0\right) = 0$, $\varsigma''\left(0\right) = 0$ by means of integration we have:

$$4\operatorname{Re}\left(\varsigma'', A\varsigma''\right) = -2\left\|\varsigma''\left(0\right)\right\|^2_{H_{1/2}} = 0, \tag{20}$$

$$4\,\mathrm{Re}\left(\varsigma''', A^2\varsigma'\right) = -4\,\mathrm{Re}\left(A^{1/2}\varsigma''(0), A^{3/2}\varsigma'(0)\right) -$$

$$-4\left\|A\varsigma''\right\|^2_{L_2(R_+;H)} = -4\left\|A\varsigma''\right\|^2_{L_2(R_+;H)} \tag{21}$$

$$8\,\mathrm{Re}\left(A\varsigma'', A^2\varsigma'\right) = -4\left\|\varsigma'(0)\right\|^2_{3/2} \tag{22}$$

$$4\,\mathrm{Re}\left(A\varsigma'', A^2\varsigma\right) = 4\,\mathrm{Re}\left(A^{3/2}\varsigma'(0), A^{5/2}\varsigma(0)\right) -$$

$$-4\left\|A\varsigma'\right\|^2_{L_2(R_+;H)} = -4\left\|A\varsigma'\right\|^2_{L_2(R_+;H)} \tag{23}$$

$$4\,\mathrm{Re}\left(A^2\varsigma', A^3\varsigma\right)_{L_2(R_+;H)} = -2\left\|\varsigma(0)\right\|^2_{5/2} = 0. \tag{24}$$

Taking into account equalities (20)-(24) in (19), we get

$$\left\|\varphi(d/dt:\beta:A)\,\varsigma\right\|^2_{L_2(R_+;H)} =$$

$$= \sqrt{1-\beta}\left(\left\|u'''\right\|^2_{L_2(R_+;H)} + \left\|A^3u\right\|^2_{L_2(R_+;H)} - 4\left\|\varsigma'(0)\,u'''\right\|^2_{H_{3/2}}\right).$$

Then from equality (19) we get

$$\left\|\varphi(d/dt:\beta:A)\,\varsigma\right\|^2 + 4(1-\beta)\left\|\varsigma'(0)\right\|^2_{L_2(R_+;H)} = (1-\beta)\left\|\varsigma\right\|^2_{W_2^3(R_+;H)} =$$

$$= \left\|\varsigma\right\|^2_{W_2^3(R_+;H)} - \beta\left\|\varsigma\right\|^2_{W_2^3(R_+;H)} =$$

$$= \left[\left(\left\|\varsigma\right\|^2_{W_2^3(R_+;H)} - \left\|\varsigma'(0)\right\|^2_{H_{3/2}}\right) - \beta\left\|\varsigma\right\|^2_{W_2^3(R_+;H)}\right] + \left\|\varsigma'(0)\right\|^2_{H_{3/2}}, \tag{25}$$

or

$$\left\|\varphi(d/dt:\beta:A)\,\varsigma\right\|^2 + (3-4\beta)\left\|\varsigma'(0)\right\|^2_{H_{3/2}} = \left\|P_0\varsigma\right\|^2_{L_2(R_+;\beta)} - \beta\left\|\varsigma\right\|^2_{W_2^3(R_+;H)}.$$

Then from the results of the paper [4] it follows that $\|\varsigma\|^2_{W_2^3(R_+;H)} \le \frac{4}{3}\|P_0\varsigma\|^2_{L_2(R_+;H)}$, and the inequality is precise. Thus, inequality (15) is true. Now prove the remaining inequalities.

Further, by the method of [4], it is proved that the operator bundle $P_1(\lambda;\beta:A) = (i\lambda)^6 E + A^6 - \beta(i\lambda)^2 A^4$ for $\beta \in \left[0, 3/2^{3/2}\right)$ has no spectrum on the imaginary axis and is represented in the form

$$P_1(\lambda;\beta:A) = \phi_1(\lambda;\beta:A)\,\phi_2(\lambda;\beta:A), \tag{26}$$

where

$$\phi_1(\lambda;\beta:A) = \lambda^3 E + a_1(\beta)\,\lambda^2 A + a_2(\beta)\,\lambda A^2 + A^3 =$$

$$= (\lambda E - \eta_1(\beta)\,A)(\lambda E - \eta_2(\beta)\,A)(\lambda E - \eta_3(\beta)\,A),$$

and $\alpha_j(\beta) > 0$, $\mathrm{Re}\,\eta_j(\beta) < 0$. From equality (26) it follows that the operator coefficients satisfy the conditions

$$\alpha_1^2 - 2\alpha_2 = -\beta, \quad \alpha_2^2 = 2\alpha_1. \tag{27}$$

On the other hand, using these relations, after integration by parts, we easily get the formula

$$\left\|\varphi(d/dt:\beta:A)\,\varsigma\right\|^2_{L_2(R_+;H)} + (\alpha_1\alpha_2 - 2)\left\|\varsigma'(0)\right\|^2_{3/2} =$$

$$= \|P_0\varsigma\|^2_{L_2(R_=;H)} - \beta \left\|A^2\varsigma'\right\|_{L_2(R_=;H)}.$$

In [4] it is proved that if the equation $\alpha_1(\beta)\alpha_2(\beta) - 2 = 0$ has a solution from the interval $\left(0; 3/2^{3/2}\right)$ equal to β_0, then it holds the precise inequality

$$\left\|A\varsigma''\right\|^2_{L_2(R_+;H)} \le \frac{1}{\beta_0^2}\|P_0\varsigma\|_{L_2(R_+;H)}.$$

By solving the equations $\alpha_1\alpha_2 = 2$ together with equations from (27), we get $\beta_0^2 = \sqrt[3]{4}$, i.e.

$$\left\|A\varsigma_0''\right\|_{L_2(R_+;H)} \le \frac{1}{\sqrt[3]{2}}\|P_0\varsigma\|_{L_2(R_+;H)}.$$

Similarly, it is proved that

$$\left\|A^2\varsigma_0'\right\|_{L_2(R_+;H)} \le \frac{1}{\sqrt[3]{2}}\|P_0\varsigma\|_{L_2(R_+;H)}.$$

The lemma is proved.

Theorem 2. *It holds the following inequalities for any $u \in W^3_{2,K}(R_+;H)$*

$$\left\|A^3u\right\|_{L_2(R_+;H)} \le \frac{2}{\sqrt{3}}\left(1 + \frac{\sqrt{3}\kappa}{1-\kappa}\right)\|P_0u\|_{L_2(R_+;H)} \tag{28}$$

$$\left\|u'''\right\|_{L_2(R_+;H)} \le \frac{2}{\sqrt{3}}\left(1 + \frac{\sqrt{3}\kappa}{1-\kappa}\right)\|P_0u\|_{L_2(R_+;H)} \tag{29}$$

$$\left\|A^2u'\right\|_{L_2(R_+;H)} \le \left(\frac{1}{\sqrt[3]{2}} + \frac{\kappa}{1-\kappa}\right)\|P_0u\|_{L_2(R_+;H)} \tag{30}$$

$$\left\|Au'\right\|_{L_2(R_+;H)} \le \left(\frac{1}{\sqrt[3]{2}} + \frac{2\kappa}{1-\kappa}\right)\|P_0u\|_{L_2(R_+;H)}. \tag{31}$$

Proof. Let $u \in W^3_{2,K}(R_+;H)$. Represent u in the form

$$u(t) = \varsigma_0(t) + e^{\omega_1 tA}x + e^{\omega_2 tA}y,$$

where $\varsigma_0(t)$ is the solution of problem (11), (12), $x, y \in H_{5/2}$ are unknown vectors. Then we get $y = -x$ and $u(t) = \varsigma_0(t) + e^{\omega_1 tA}x - e^{\omega_2 tA}x$.

Hence it follows that

$$\|u(t)\|_{W_2^3(R_+;H)} \le \|\varsigma_0(t)\|_{W_2^3(R_+;H)} + \left\|e^{\omega_1 tA}x - e^{\omega_2 tA}\right\|_{W_2^3(R_+;H)}.$$

Since $P_0u = P_0\varsigma = f$, then using inequality (15), (16) and taking into account $x = (E-Q)^{-1}K\varsigma_0(t)$, we get

$$\|u\|_{W_2^3(R_+;H)} \le \frac{2}{\sqrt{3}}\|P_0u\|_{L_2(R_+;H)} + \sqrt{3}\left\|(E-Q)^{-1}\right\|\kappa\|\varsigma_0\|_{W_2^3(R_+;H)} \le$$

$$\le \left(\frac{2}{\sqrt{3}} + \frac{\sqrt{3}\kappa}{1-\kappa}\cdot\frac{2}{\sqrt{3}}\right)\|P_0u\|_{L_2(R_+;H)} = \frac{2}{\sqrt{3}}\left(1 + \frac{\sqrt{3}\kappa}{1-\kappa}\right)\|P_0u\|_{L_2(R_+;H)}.$$

Hence it follows that

$$\left\|A^3u\right\|_{L_2(R_+;H)} \le \frac{2}{\sqrt{3}}\left(1+\frac{\sqrt{3}\kappa}{1-\kappa}\right)\|P_0u\|_{L_2(R_+;H)}$$

and

$$\left\|u'''\right\|_{L_2(R_+;H)} \le \frac{2}{\sqrt{3}}\left(1+\frac{\sqrt{3}\kappa}{1-\kappa}\right)\|P_0u\|_{L_2(R_+;H)}.$$

On the other hand,

$$\left\|Au''\right\|_{L_2(R_+;H)} \le \left\|A\varsigma_0''\right\|_{L_2(R_+;H)} + \left\|\omega_1^2A^3e^{\omega_1tA} - \omega_2^2A^3e^{\omega_2tA}x\right\|_{L_2(R_+;H)} =$$

$$= \left\|A\varsigma_0''\right\| + \left\|\omega_2A^3e^{\omega_1tA}x - \omega_1A^3e^{\omega_1tA}x\right\|_{L_2(R_+;H)}.$$

Here again using the spectral expansion of the operator A and representing $z = A^{5/2}$, we get

$$\left\|\omega_2A^{1/2}e^{\omega_1tA}z - \omega_1A^{1/2}e^{\omega_2tA}z\right\|^2 = \left\|A^{1/2}e^{\omega_1tA}z\right\|^2 + \left\|A^{1/2}e^{\omega_2tA}z\right\|^2 -$$

$$-2\operatorname{Re}\left(A\omega_2^2e^{2\omega_1tA}z,z\right)_{L_2(R_+;H)} = \|z\|^2 + \|z\|^2 - 2\operatorname{Re}\int_{\mu_0}^{\infty}\omega_2^2\mu\,(dEz,z)\times$$

$$\times\int_0^t e^{-2\omega_1t\mu}dt = 2\|z\|^2 + 2\operatorname{Re}\omega_2^2/2^{\omega_1}\|z\|^2 =$$

$$= 2\|z\|^2 + 2\operatorname{Re}\frac{\omega_2^3}{\omega_1\omega_2}\|z\|^2 = 3\|z\|^2 = 3\|x\|_{5/2}^2.$$

Therefore

$$\left\|Au''\right\|_{L_2(R_+;H)} \le \left\|A\varsigma_0''\right\|_{L_2(R_+;H)} + \sqrt{3}\left\|(E+Q)^{-1}K\varsigma_0\right\| =$$

$$= \left\|A\varsigma_0''\right\| + \sqrt{3}\frac{\kappa}{1-\kappa}\|\varsigma_0\|_{W_2^3(R_+;H)} =$$

$$= \left\|A\varsigma_0''\right\| + \frac{2\kappa}{1-\kappa}\|P_0\varsigma\|_{L_2(R_+;H)} = \left(\frac{1}{\sqrt[3]{2}} + \frac{2\kappa}{1-\kappa}\right)\|P_0u\|_{L_2(R_+;H)}.$$

Similarly we have

$$\left\|A^2u'\right\|_{L_2(R_+;H)} \le \left\|A^2\varsigma_0'\right\|_{L_2(R_+;H)} + \left\|\omega_1A^3e^{\omega_1tA}x - \omega_2A^3e^{\omega_2tA}x\right\|_{L_2(R_+;H)}.$$

Again assuming $A^{5/2}x = z$ we get

$$\left\|\omega_1A^{1/2}e^{\omega_1tA}z - \omega_2e^{\omega_1tA}z\right\|^2 = \left\|A^{1/2}e^{\omega_1tA}z\right\|^2 + \left\|A^{1/2}e^{\omega_2tA}z\right\|^2 -$$

$$-2\operatorname{Re}\left(\omega_1^2Ae^{2\operatorname{Re}\omega_1tA}z,z\right) = 2\|z\|^2 - 2\operatorname{Re}\omega_1^2\int_{\mu_0}^{\infty}\mu\,(dE_\mu z,z)\times$$

$$\times\int_0^\infty e^{2\omega_1 t}Adt = 2\left\|z\right\|^2 + 2\operatorname{Re}\frac{\omega_1^2}{2\omega_1}\left\|z\right\|^2 =$$

$$= 2\left\|z\right\|^2 + 2\operatorname{Re}\omega_1\left\|z\right\|^2 = 2\left\|z\right\|^2 - \frac{1}{2}\left\|z\right\|^2 = \frac{3}{2}\left\|z\right\|^2 = \frac{3}{2}\left\|x\right\|^2_{5/2}.$$

Then

$$\left\|A^2u'\right\|_{L_2(R_+;H)} \le \frac{1}{\sqrt[3]{2}}\left\|P_0u\right\| + \frac{\sqrt{3}}{2}\cdot\frac{\kappa}{1-\kappa}\left\|\varsigma_0\right\|_{W_2^3(R_+;H)} \le$$

$$\le \left(\frac{1}{\sqrt[3]{2}} + \frac{\sqrt{3}}{2}\cdot\frac{\kappa}{1-\kappa}\cdot\frac{2}{\sqrt{3}}\right)\left\|P_0u\right\|_{L_2(R_+;H)} =$$

$$= \left(\frac{1}{\sqrt[3]{2}} + \frac{\kappa}{1-\kappa}\right)\left\|P_0u\right\|_{L_2(R_+;H)}.$$

The theorem is proved.

Now prove the main theorem.

Theorem 3. *Let conditions 1) – 3) be fulfilled, $\kappa < 1$ and*

$$\alpha(\kappa) = \sum_{j=0}^{3} c_j(\kappa)\left\|B_{3-j}\right\| < 1,$$

where $c_0(\kappa) = c_3(\kappa) = \frac{2}{\sqrt{3}}\left(1 + \frac{\sqrt{3}\kappa}{1-\kappa}\right)$, $c_1(\kappa) = \frac{1}{\sqrt[3]{2}} + \frac{\kappa}{1-\kappa}$, $c_2(\kappa) = \frac{1}{\sqrt[3]{2}} + \frac{2\kappa}{1-\kappa}$.

Then problem (1), (2) is regularly solvable.

Proof. Write problem (1), (2) in the form of the equation $Pu = P_0u + P_1u = f$, where $f \in L_2(R_+;H)$, $u \in W^3_{2,K}(R_+;H)$. After substitution of $\nu = P_0u$ we get the equation $\nu + P_1P_0^{-1}\nu = f$ in the space $L_2(R_+;H)$. Using theorem 3, we get

$$\left\|P_1P_0^{-1}\nu\right\|_{L_2(R_+;H)} = \left\|P_1u\right\|_{L_2(R_+;H)} \le \sum_{j=0}^{3}\left\|B_{3-j}\right\|\left\|A^{3-j}u^{(j)}\right\|_{L_2(R_+;H)} \le$$

$$\le \left(\sum_{j=0}^{3} c_j(\kappa)\left\|B_{3-j}\right\|\right)\left\|P_0u\right\|_{L_2(R_+;H)} =$$

$$= \alpha(\kappa)\left\|P_0u\right\|_{L_2(R_+;H)} = \alpha(\kappa)\left\|\nu\right\|_{L_2(R_+;H)}.$$

Since $\alpha(\kappa) < 1$, then $E + P_1P_0^{-1}$ is invertible in $L_2(R_+;H)$ and

$$u = P_0^{-1}\left(E + P_1P_0^{-1}\right)^{-1}f.$$

Hence we get $\|u\|_{W_2^2(R_+;H)} \le const\,\|f\|_{L_2(R_+;H)}$.

The theorem is proved.

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