MATHEMATICS

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ASYMPTOTICAL PROPERTIES OF THE RENEWAL MATRIX FOR SOME CLASS OF INFINITE-DIMENSIONAL RENEWAL EQUATIONS

Abstract

Conditions on the tolimit matrix of measures and on the limit matrix of full masses of measures are found. Existence of normalized multiplier ρ^{ε} such, that on a time scale t/ρ^{ε} the asymptotic of difference of the renewal matrix (associated with family of dependency upon a small parameter of matrix-valued measures with the block-resoluble infinite-dimensional limit matrix of full masses of measures) on the finite interval is nontrivial is proved.

1. Introduction

In this paper we will consider the transitional phenomena for the multidimentional renewal equation with the block-resoluble infinite-dimensional limit matrix of full masses of measures . Equations of such type arise up, for example, at the analysis of the stochastic systems, which assume the asymptotical integration of state space, in particular,of the high reliable systems, and also of some classes of the queuing systems in the conditions of small probability of requirement loss. Problems about integration of phase space of difficult systems considered Korolyuk V.S. and Turbin A.F. (see [1]). Task for the renewal matrix of irreducible matrix is considered by Shurenkov V.M. in [2]. Problem for the finite-dimensional case with resoluble block-diagonal matrix of full masses of measures was considered Nishchenko I.I. in [3]. In this work opinion for the finite-dimensional case is extended to the infinite-dimensional case. Conditions on matrix-valued measure $F(dt)$), at which exists a time scale, for which difference of the renewal matrix has nontrivial limit are finded.

2. Main part

It will be our task to investigate the asymptotical properties for a family of matrix-valued renewal functions $H^{\varepsilon}(t) = \sum\limits_{n=0}^{\infty} F^{\varepsilon(*n)}(t)$ as $\varepsilon \to 0$, $t \to \infty$. The principal assumption in relation to $F^{\varepsilon}(dt)$ there will be that it weakly converges to the measure $F(dt)$ as $\varepsilon \to 0$. Its matrix of full masses of measures $F[0,\infty)$ is infinite-dimensional resoluble of block-diagonal type.

During all exposition we assume that if all elements of some matrix $D(x) = [D_{ij}(x)]$ own definite functional property, the matrix $D(x)$ owns this property too.

Matrices with elements $|D_{ij}|$ and $\int D_{ij}(x)dx$ are denoted $|D|$ and $\int D(x)dx$ accordingly.

Let $F^{\varepsilon}(dt)$ is a family of infinite-dimensional matrices, the elements of which are finite nonnegative measures concentrated on $[0,\infty)$ and depend on a small parameter $\varepsilon > 0$. For each ε matrix $F^{\varepsilon}(dt)$ is irreducible, that is, for arbitrary subset $I \in E = \{1, 2, 3, \ldots\}$ of indexes set can be indicated such $i \in I$, $j \notin I$ that $F^{\varepsilon}_{ij}(dt) > 0$. In addition, we will consider, that such condition are true:
1) the matrix-valued measure $F^{\varepsilon}(dt)$ weakly converges as $\varepsilon \to 0$ to the matrix-valued measure $F(dt)$, that is

$$\int_0^\infty g(t)F^{\varepsilon}(dt) \underset{\varepsilon\to 0}{\longrightarrow} \int_0^\infty g(t)F(dt) \tag{2.1}$$

for the arbitrary continuous bounded function $g(t)$;
2) the family $F^{\varepsilon}(t)$ is asymptotically and uniformly in ε separated from a zero, that is, for some $\alpha > 0$, $\beta < 1$

$$\overline{\lim_{\varepsilon\to 0}} \sup_i \sum_{j=1}^{\infty} F^{\varepsilon}_{ij}(\alpha) \le \beta; \tag{2.2}$$

3) for arbitrary ε

$$\sup_i \sum_{j=1}^{\infty} F^{\varepsilon}_{ij}[0,\infty) \le 1; \tag{2.3}$$

4) the limits matrix of full masses of measures $F \equiv F[0,\infty)$ is stochastic

$$\sum_{j=1}^{\infty} F_{ij}[0,\infty) = 1 \tag{2.4}$$

and resoluble of the block-diagonal type: $F = diag\{F^1, F^2, F^3, \ldots\}$. Therefore the indexes set $E = \{1, 2, 3, \ldots\}$ may to give as association of mutually exclusive sets $E_1, E_2, E_3 \ldots$ such, that $F_{ij} = \delta_{sk}F^s_{ij}$, when $i \in E_s$, $j \in E_k$, where δ_{sk} is Kroneker character. Suppose, that each of matrices F^i, $i = \overline{1,\infty}$ along the diagonal of matrix F is irreducible. We will consider also, that dimension of matrices F^s is not greater than d_0.

Assume still, that such condition holds

$$\sup_i \sum_{j=1}^{\infty} a^{\varepsilon}_{ij} < \infty,$$

where

$$a^{\varepsilon}_{ij} = \int_0^\infty y F^{\varepsilon}_{ij}(dy) < \infty \tag{2.5}$$

and let

$$\inf_i \sum_{j=1}^{\infty} a^{\varepsilon}_{ij} \ge a_0 > 0. \tag{2.6}$$

From the condition (2.4) and resolubility of matrix F it follows, that each with matrices F^s along its diagonal is stochastic. Therefore 1 is maximal eigenvalue with corresponding right eigenvector $\vec{1}^{(s)}$. Since the matrix F^s is irreducible and nonnegative, then there exists unique positive left eigenvector $\vec{p}^{(s)}$ with corresponding eigenvalue 1 such, that $(\vec{p}^{(s)}, \vec{1}^{(s)}) = 1$. The matrix F is stochastic. Thus unit is its maximal eigenvalue with corresponding right and left eigenvectors $\vec{1}$, $\vec{p} = (\vec{p}^{(1)}, \vec{p}^{(2)}, \ldots)$. The vector $\vec{p}$ is positive.

From irreducible of the matrix F^s it follows existence such integer n_0, that $(F^s_{ij})^{n_0} > 0$ (ij element of matrix $(F^s)^{n_0}$). Thus $n_0 \le m_0 \le d_0$, where m_0 – degree of minimal polynomial of matrix F^s, d_0 – maximal dimension of matrices F^s. We will assume that $\forall\ s\ \forall\ i,j \in E_s$ there exists such integer $n_0 \le m_0$, for which $(F^s_{ij})^{n_0} \ge \delta_0 > 0$.

In addition, we suppose, that displacement of tolimit matrix of full masses of measures is little, that is, for $i \in E_s$ we have

$$\sum_{j\in E_s} (F_{ij} - F^{\varepsilon}_{ij}) \le \delta_s(\varepsilon),$$

where $\sum\limits_{s=1}^{\infty} \delta_s(\varepsilon) < \infty$. For $i \in E_s$ also holds condition

$$\sum_{j\notin E_s} F^{\varepsilon}_{ij} \le \delta^2_s(\varepsilon). \qquad (2.7)$$

Let $H^{\varepsilon}(t)$ be the renewal matrix associated with $F^{\varepsilon}(t)$, that is

$$H^{\varepsilon}(t) = \sum_{n=0}^{\infty} F^{\varepsilon(*n)}(t),$$

where

$$F^{\varepsilon(*0)}(t) = I,$$

$$F^{\varepsilon(*n)}(t) = F^{\varepsilon}(t) * F^{\varepsilon(*(n-1))}(t) = \int_0^t F^{\varepsilon}(dy) F^{\varepsilon(*(n-1))}(t-y).$$

Here I – unit matrix. Two next lemmas assert resriction of the renewal matrix norm on every finite interval $[0,T]$ and uniform in ε and t resriction of difference norm of the renewal matrix on finite intervals $[t, t+s]$.

Proof of next lemmas coincides with proof of the proper results Nishchenko [3] in the finite case. Therefore we do not give it.

Lemma 1. *If a condition (2.2) holds, then*

$$\sup_i \sum_{j=1}^{\infty} H^{\varepsilon}_{ij}[0,T] < \infty$$

for each $\varepsilon > 0$ *and for arbitrary* $T < \infty$*.*

Lemma 2. *If conditions (2.1) -(2.4) hold, then for arbitrary* $s \geq 0$

$$\lim_{\varepsilon\to 0}\sup_t \sup_i \sum_{j=1}^{\infty} H_{ij}^{\varepsilon}[t, t+s] \leq As + B \quad ,$$

where A, B *– some finite constants.*

Our next purpose – finding of the asymptotical behaviour of the renewal matrix difference $H^{\varepsilon}[t, t+s]$ as $\varepsilon \to 0$, $t \to \infty$. We will do a few remarks.

Remind that matrix $F(dt)$ is called latticed with step h if there exists such $h > 0$, that each of measures $F_{ij}(dt)$, $i, j \in E$, may to concentrate on the displaced grate with the same step h, that is, on the set of kind $\{c_{ij} + nh,\ n = 0, \pm 1, \pm 2, \ldots\}$, thus $c_{ij} + c_{jk} - c_{ki}$ is the element of initial grate for arbitrary i, j, $k \in E$. If there such h does not exist, then the matrix $F(dt)$ is called unlatticed.

Assume that matrices $F^{\varepsilon}(dt)$ are unlatticed.

Theorem 1. *If conditions (2.1) -(1.7) hold and each of matrices* $F^s(dt)$ *is unlatticed, then there exist nonnull matrix and normalized multiplier* $\rho^{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} 0$ *such, that as* $i \in E_s$, $j \in E_k$

$$\lim_{\varepsilon\to 0} H_{ij}^{\varepsilon}\ \left[\frac{t}{\rho^{\varepsilon}},\ \frac{t}{\rho^{\varepsilon}} + u\right] = u \cdot q_{sk}(t) \cdot \frac{p_j^{(k)}}{\pi_k},$$

where $q_{sk}(t) = [e^{tC}]_{sk}$, $\pi_k = \sum\limits_{I,j\in E_k} p_I^{(k)} \cdot \int\limits_0^{\infty} tF_{ij}(dt)$.

Proof. It is known, that elements of the renewal matrix $H^{\varepsilon}(t)$ is satisfied multidimensional renewal equation

$$H_{ij}^{\varepsilon}(t) = \delta_{ij} + \sum_{m=1}^{\infty} \int_0^t F_{im}^{\varepsilon}(du) H_{mj}^{\varepsilon}(t-u).$$

At the first by analogy with work [2] transform the irreducible matrix renewal equation into scalar. After that replace d_s equations for a block E_s on one equation.

Fix on one index in each with E_s: $w_1 \in E_1$, $w_2 \in E_2$,..., $w_r \in E_r$, ...and define the sequence of functions $H_{ij}^{\varepsilon(n)}(t)$, $n = 0, 1, \ldots$ by such correlations

$$H_{ij}^{\varepsilon(0)}(t) = \delta_{ij} + \sum_{s=1}^{\infty} F_{iw_s}^{\varepsilon} * H_{w_s j}^{\varepsilon}(t)$$

$$H_{ij}^{\varepsilon(n)}(t) = H_{ij}^{\varepsilon(0)}(t) + \sum_{m\notin D} F_{im}^{\varepsilon} * H_{mj}^{\varepsilon(n-1)}(t)$$

where $D = \{w_1, w_2, \ldots, w_r, \ldots\}$ –set of the fixed indexes. From construction we see that

$$H_{ij}^{\varepsilon(n+1)}(t) \geq H_{ij}^{\varepsilon(n)}(t), \quad H_{ij}^{\varepsilon(n)}(t) \leq H_{ij}^{\varepsilon}(t) \quad \forall n \geq 0$$

and

$$H^{\varepsilon}_{ij}(t) - H^{\varepsilon(n)}_{ij}(t) = \sum_{m_1 \notin D} F^{\varepsilon}_{im_1} * \left(H^{\varepsilon}_{m_1 j}(t) - H^{\varepsilon(n-1)}_{m_1 j}(t) \right) = \cdots =$$

$$= \sum_{m_1 \notin D \ldots, m_n \notin D} F^{\varepsilon}_{im_1} * \cdots * F^{\varepsilon}_{m_{n-1} m_n} * \left(H^{\varepsilon}_{m_n j}(t) - H^{\varepsilon(0)}_{m_n j}(t) \right) =$$

$$= \sum_{m_1 \notin D \ldots, l \notin D} F^{\varepsilon}_{im_1} * \cdots * F^{\varepsilon}_{m_n l} * H^{\varepsilon}_{lj}(t) \leq ||F^{\varepsilon(*n)} * H^{\varepsilon}(t)|| \underset{n\to\infty}{\longrightarrow} 0$$

in accordance with convergence of row $\sum\limits_{n=0}^{\infty} F^{\varepsilon(*n)}(t)$.

Consequently, the sequence of functions $H^{\varepsilon(n)}_{ij}(t)$ approaches the function $H^{\varepsilon}_{ij}(t)$ as $n \to \infty$.

We will enter still the sequence of monotonous in t functions as follows

$$L^{\varepsilon(0)}_{ij}(t) = F^{\varepsilon}_{ij}(t),$$

$$L^{\varepsilon(n)}_{ij}(t) = F^{\varepsilon}_{ij}(t) + \sum_{m \notin D} F^{\varepsilon}_{im} * L^{\varepsilon(n-1)}_{mj}(t).$$

By induction we receive such correlation

$$H^{\varepsilon(n)}_{ij}(t) = \delta_{ij} + \sum_{m \notin D} L^{\varepsilon(n-1)}_{im}(t) \cdot \delta_{mj} + \sum_{s=1}^{\infty} L^{\varepsilon(n)}_{iw_s} * H^{\varepsilon}_{w_s j}(t), \tag{2.8}$$

in which put, that $L^{\varepsilon(n-1)}_{ij}(t) = 0$ as $n = 0$.

Since $L^{\varepsilon(n)}_{ij}(t) \leq H^{\varepsilon}_{ij}(t)$ for arbitrary n and the sequence $L^{\varepsilon(n)}_{ij}(t)$ is nondecreasing in n, there exist limits $L^{\varepsilon}_{ij}(t) = \lim\limits_{n\to\infty} L^{\varepsilon(n)}_{ij}(t)$. Letting n to infinity in (2.8), we will get such equality

$$H^{\varepsilon}_{ij}(t) = \delta_{ij} + \sum_{m \notin D} L^{\varepsilon}_{im}(t) \cdot \delta_{mj} + \sum_{n=1}^{\infty} L^{\varepsilon}_{iw_n} * H^{\varepsilon}_{w_n j}(t). \tag{2.9}$$

Then for elements $H^{\varepsilon}_{w_s w_k}(t)\,(s,k \in \mathbb{N})$ of the matrix $H^{\varepsilon}(t)$ we have the renewal equation

$$H^{\varepsilon}_{w_s w_k}(t) = \delta_{sk} + \sum_{n=1}^{\infty} L^{\varepsilon}_{w_s w_n} * H^{\varepsilon}_{w_n w_k}(t) \tag{2.10}$$

It means that the matrix $\widetilde{H}^{\varepsilon}(t) = [H^{\varepsilon}_{w_s w_k}(t)]_{s,k\in\mathbb{N}}$ is the renewal matrix for matrix $\widetilde{L}^{\varepsilon}(t) = [L^{\varepsilon}_{w_s w_k}(t)]_{s,k\in\mathbb{N}}$. For elements $H^{\varepsilon}_{w_s j}(t)$ $(s \in \mathbb{N},\ j \notin D)$ we get the renewal equation

$$H^{\varepsilon}_{w_s j}(t) = L^{\varepsilon}_{w_s j}(t) + \sum_{n=1}^{\infty} L^{\varepsilon}_{w_s w_n} * H^{\varepsilon}_{w_n j}(t). \tag{2.11}$$

The solution of this equation is equal to

$$H^{\varepsilon}_{w_s j}(t) = \sum_{n=1}^{\infty} \int_0^t H^{\varepsilon}_{w_s w_n}(du) L^{\varepsilon}_{w_n j}(t-u). \tag{2.12}$$

Relations (2.9), (2.10), (2.12) show that the asymptotical properties of the matrix $H^\varepsilon(t)$ may to study by the matrices $\widetilde{H}^\varepsilon(t)$ and $L^\varepsilon(t)$.

Show now, that the function $L^\varepsilon(t)$ weakly converges to some function $L(t)$ as $\varepsilon \to 0$, thus the matrix $\widetilde{L}(\infty) = [L_{w_s w_k}(\infty)]_{s,k\in\mathbb{N}}$ is unit.

We will mark some properties of functions $L^\varepsilon_{ij}(t)$. At first, by induction we prove, that

$$\sum_{m\notin D} F^\varepsilon_{im} * L^{\varepsilon(n)}_{mj}(t) = \sum_{m\notin D} L^{\varepsilon(n)}_{im} * F^\varepsilon_{mj}(t) \quad n = 0, 1, \dots \tag{2.13}$$

Consequently, the functions $L^\varepsilon_{ij}(t) = \lim\limits_{n\to\infty} L^{\varepsilon(n)}_{ij}(t)$ satisfy such two equations

$$L^\varepsilon_{ij}(t) = F^\varepsilon_{ij}(t) + \sum_{m\notin D} F^\varepsilon_{im} * L^\varepsilon_{mj}(t)$$

$$L^\varepsilon_{ij}(t) = F^\varepsilon_{ij}(t) + \sum_{m\notin D} L^\varepsilon_{im} * F^\varepsilon_{mj}(t). \tag{2.14}$$

Summarize both parts of the last equation in all $j \in E$. Taking into account condition (2.3), we will get

$$\sum_{j=1}^{\infty} L^\varepsilon_{ij}(t) \le 1 + \sum_{m\notin D} L^\varepsilon_{im}(t),$$

whence

$$\sum_{s=1}^{\infty} L^\varepsilon_{iw_s}(t) \le 1 \quad \forall\, I \in E. \tag{2.15}$$

For proof of uniform boundary in ε and t of the functions $L^\varepsilon_{ij}(t)$ as $j \notin D$, use such estimation (which is proved by induction)

$$\sum_{j\notin D} \sum_{k=1}^{\infty} L^\varepsilon_{ij} * F^{\varepsilon(*n)}_{jw_k}(t) \le n, \quad k \in \mathbb{N}, \quad n \ge 1. \tag{2.16}$$

Since a sum $\sum\limits_{j\notin D} L^\varepsilon_{ij} * F^{\varepsilon(*n)}_{jw_k}(t)$ as a function in t is nondecreasing and uniformly bounded in t, there exist finite limit of this function as $t \to \infty$. Letting t to infinity in (2.16), we get inequality

$$\sum_{k=1}^{\infty} \sum_{j\notin D} L^\varepsilon_{ij}(\infty) \cdot (F^\varepsilon_{jw_k})^n \le n.$$

Assume for definiteness, that $j \in E_s \backslash w_s$. Since a matrix F^s is irreducible, there exists a such integer $n_0 = n_0(j)$, that $(F_{jw_s})^{n_0} \equiv \delta_j \ge \delta_0 > 0$. Thus $n_0 \le m_0 \le d_0$, where m_0 – degree of minimum polynomial of the matrix F^s, d_0 – maximal dimension of matrices F_s. From that $(F^\varepsilon_{jw_s})^{n_0} \underset{\varepsilon\to 0}{\longrightarrow} (F_{jw_s})^{n_0}$ it follows existence of such $\varepsilon_0 > 0$, that $(F^\varepsilon_{jw_s})^{n_0} \ge \frac{\delta_j}{2}$ for all $\varepsilon < \varepsilon_0$. Functions $L^\varepsilon_{ij}(t)$ do not decrease in t, and, consequently, for all enough small ε inequalities hold

$$\sum_{j\notin D} \frac{\delta_0}{2} L^\varepsilon_{ij}(t) \le \sum_{j\notin D} \frac{\delta_j}{2} L^\varepsilon_{ij}(t) \le \sum_{j\notin D} \frac{\delta_j}{2} L^\varepsilon_{ij}(\infty) \le \sum_{j\notin D} L^\varepsilon_{ij}(\infty) \cdot (F^\varepsilon_{jw_k})^{n_0(j)} \le$$

$$\leq \sum_{n=1}^{d_0} \sum_{j \notin D} \sum_{k=1}^{\infty} L_{ij}^{\varepsilon}(\infty) \cdot (F_{j\omega_k}^{\varepsilon})^n \leq \frac{(d_0+1)d_0}{\delta_0}.$$

Therefore

$$\sup_i \sum_{j \in E} L_{ij}^{\varepsilon}(t) \leq \frac{(d_0+1)d_0}{\delta_0} + 1 = R, \tag{2.17}$$

where R – some constant, that does not depend on ε and t.

Thus, we proved, that the matrices family $L^{\varepsilon}(t)$ is uniformly bounded in ε and t. By Helly theorem from it may to choose weakly convergence subsequence. Without loss of generality, we will suppose, that $L_{ij}^{\varepsilon}(t) \to L_{ij}(t)$ weakly as $\varepsilon \to 0$.

From monotonicity of the function $L_{ij}^{\varepsilon}(t)$ in t it follows, that this convergence is uniform in every continuous point of the function $L_{ij}(t)$, that is

$$\lim_{c \to 0} \overline{\lim_{\varepsilon \to 0}} \sup_{|h| \leq c} \left(L_{ij}^{\varepsilon}(t+h) - L_{ij}(t) \right) = 0.$$

In addition, monotony and uniform bounded in t of functions $L_{ij}^{\varepsilon}(t)$ guarantee existence of finite limits

$$L_{ij} = \lim_{t \to \infty} L_{ij}(t) \quad \text{and} \quad L_{ij}^{\varepsilon} = \lim_{t \to \infty} L_{ij}^{\varepsilon}(t).$$

From resolution of matrix F it follows resolution of matrix L, that is $L_{ij} = 0$ as $i \in E_s$, $j \in E_k$, $s \neq k$. For finding L_{ij} as i, $j \in E_s$ $(s \in \mathbb{N})$ we have such resolution

$$L_{ij} = F_{ij} + \sum_{m \in E_s \setminus w_s} F_{im} \cdot L_{mj},$$

$$L_{ij} = F_{ij} + \sum_{m \in E_s \setminus w_s} L_{im} \cdot F_{mj}. \tag{2.18}$$

If first of these equations multiply by $p_I^{(s)}$ and summarize in $i \in E_s$ and in the second equation to conduct adding in $j \in E_s$, we find, that

$$L_{iw_s} = 1, \; L_{w_s j} = \frac{p_j^{(s)}}{p_{w_s}^{(s)}} \quad \forall i, j \in E_s.$$

Examine, that this solution is unique. Assume, that L_{ij} and $\widetilde{L}_{ij}$ – two solutions of these equations and let $Z_{ij} = L_{ij} - \widetilde{L}_{ij}$. Then

$$Z_{ij} = \sum_{m \in E_s \setminus w_s} F_{im} \cdot Z_{mj}.$$

From here

$$\sum_i p_i^{(s)} \cdot Z_{ij} = \sum_{m \in E_s \setminus w_s} p_m^{(s)} \cdot Z_{mj} \quad \text{and means} \quad Z_{w_s j} = 0.$$

Analogously make, that $Z_{iw_s} = 0$.

Obvious type of constants L_{ij} as $i \notin D$ and $j \notin D$ to us isn't need. Notice, that from monotony $L^{\varepsilon}_{ij}(t)$ in t and from existence of finite limits $L^{\varepsilon}_{ij} = \lim\limits_{t\to\infty} L^{\varepsilon}_{ij}(t)$ it follows, that as arbitrary fixed $s \geq 0$

$$\lim_{\varepsilon\to 0} \left[L^{\varepsilon}_{ij}(t_\varepsilon + s) - L^{\varepsilon}_{ij}(t_\varepsilon)\right] = 0 \tag{2.19}$$

for an arbitrary sequence t_ε such, that $t_\varepsilon \uparrow \infty$ as $\varepsilon \to 0$.

Apply to the renewal equation(2.10) the known results of the renewal theory. For this show finiteness of quantity $\sup\limits_{\varepsilon} \int\limits_0^\infty tL^{\varepsilon}_{w_s w_k}(dt)$.

Applying Laplace transform to equation

$$L^{\varepsilon}_{w_s j}(t) = F^{\varepsilon}_{w_s j}(t) + \sum_{m\notin D} L^{\varepsilon}_{w_s m} * F^{\varepsilon}_{mj}(t)$$

we get such equality

$$\widehat{L}^{\varepsilon}_{w_s j}(\lambda) = \widehat{F}^{\varepsilon}_{w_s j}(\lambda) + \sum_{m\notin D} \widehat{L}^{\varepsilon}_{w_s m}(\lambda) \cdot \widehat{F}^{\varepsilon}_{mj}(\lambda), \tag{2.20}$$

where

$$\widehat{L}^{\varepsilon}_{ij}(\lambda) = \int\limits_0^\infty e^{\lambda t} L^{\varepsilon}_{ij}(dt) \;\; \widehat{F}^{\varepsilon}_{ij}(\lambda) = \int\limits_0^\infty e^{\lambda t} F^{\varepsilon}_{ij}(dt) \;\; \lambda \leq 0.$$

Remark, that

$$\widehat{L}^{\varepsilon}_{ij}(0) = L^{\varepsilon}_{ij}(\infty) \leq R \quad \widehat{F}^{\varepsilon}_{ij}(0) = F^{\varepsilon}_{ij}(\infty)$$

$$(\widehat{L^{\varepsilon}}_{ij})'(0) = \int\limits_0^\infty tL^{\varepsilon}_{ij}(dt) \equiv m^{\varepsilon}_{ij} \quad (\widehat{F^{\varepsilon}}_{ij})'(0) = \int\limits_0^\infty tF^{\varepsilon}_{ij}(dt) \equiv a^{\varepsilon}_{ij}$$

$$\sum_{j\in E} \widehat{F}^{\varepsilon}_{ij}(\lambda) \leq \sum_{j\in E} \widehat{F}^{\varepsilon}_{ij}(0) = \sum_{j\in E} F^{\varepsilon}_{ij}(\infty) \leq 1.$$

Summing both parts of equality

$$\frac{\widehat{L}^{\varepsilon}_{w_s j}(0) - \widehat{L}^{\varepsilon}_{w_s j}(\lambda)}{-\lambda} = \frac{\widehat{F}^{\varepsilon}_{w_s j}(0) - \widehat{F}^{\varepsilon}_{w_s j}(\lambda)}{-\lambda} +$$

$$+\sum_{m\notin D} \frac{\widehat{L}^{\varepsilon}_{w_s m}(0) - \widehat{L}^{\varepsilon}_{w_s m}(\lambda)}{-\lambda} \cdot \widehat{F}^{\varepsilon}_{mj}(\lambda) + \sum_{m\notin D} \widehat{L}^{\varepsilon}_{w_s m}(0) \cdot \frac{\widehat{F}^{\varepsilon}_{mj}(0) - \widehat{F}^{\varepsilon}_{mj}(\lambda)}{-\lambda} \tag{2.21}$$

in all $j \in E$, we get

$$\sup_{\varepsilon} \sum_{n=1}^{\infty} \frac{\widehat{L}^{\varepsilon}_{w_s w_n}(0) - \widehat{L}^{\varepsilon}_{w_s w_n}(\lambda)}{-\lambda} \leq (R+1) \sup_{\varepsilon} \sum_{j\in E} \frac{\widehat{F}^{\varepsilon}_{w_s j}(0) - \widehat{F}^{\varepsilon}_{w_s j}(\lambda)}{-\lambda}.$$

From existence of limit as $\lambda\to 0$ of the right part of inequality it follows existence of limit of the left its part. Limit transition as $\lambda\to 0$ gives us the inequality

$$\sup_{\varepsilon}\sum_{n=1}^{\infty} m^{\varepsilon}_{w_s w_n}\le (R+1)\sup_{\varepsilon}\sum_{j\in E} a^{\varepsilon}_{w_s j}<\infty$$

in accordance with a condition (2.5). Then from correlation

$$m^{\varepsilon}_{w_s w_k}=a^{\varepsilon}_{w_s w_k}+\sum_{n\notin D} m^{\varepsilon}_{w_s n}F^{\varepsilon}_{nj}+\sum_{n\notin D} L^{\varepsilon}_{w_s n}a^{\varepsilon}_{n w_k}$$

it follows, that $\sup\limits_{\varepsilon} m^{\varepsilon}_{w_s j}<\infty$ for all $j\in E$.

Passing in (2.21) to limit, when $\lambda\to 0$, $\varepsilon\to 0$, we get

$$m_{w_s j}=\sum_{I\in E_s} L_{w_s i}\cdot a_{ij}+\sum_{I\in E_s\backslash w_s} m_{w_s i}F_{ij}.$$

Adding in $j\in E_s$ converts this correlation into such equality

$$\sum_{j\in E_s} m_{w_s j}=\sum_{I,j\in E_s}\frac{p^{(s)}_I}{p^{(s)}_{w_s}}\cdot a_{ij}+\sum_{I\in E_s\backslash w_s} m_{w_s i}$$

which we find from, that

$$m_s=m_{w_s w_s}=\sum_{I,j\in E_s}\frac{p^{(s)}_I}{p^{(s)}_{w_s}}\cdot a_{ij}.$$

Examine implementation of uniform integrality condition of measures family $[L^{\varepsilon}_{w_s w_k}(dt)]_{s,k\in\mathbb{N}}$. Denote

$$\overline{L}^{\varepsilon}(t,\lambda)=\int\limits_t^{\infty} e^{\lambda u}L^{\varepsilon}(du),\quad \overline{F}^{\varepsilon}(t,\lambda)=\int\limits_t^{\infty} e^{\lambda u}F^{\varepsilon}(du),\quad \lambda<0.$$

From equation

$$L^{\varepsilon}_{w_s j}(t)=F^{\varepsilon}_{w_s j}(t)+\sum_{m\notin D} L^{\varepsilon}_{w_s m}*F^{\varepsilon}_{mj}(t)$$

we prove a next equality

$$\overline{L}^{\varepsilon}_{w_s j}(t,\lambda)=\overline{F}^{\varepsilon}_{w_s j}(t,\lambda)+$$

$$+\sum_{m\notin D}\overline{L}^{\varepsilon}_{w_s m}(t,\lambda)\cdot\widehat{F}^{\varepsilon}_{mj}(\lambda)-\sum_{m\notin D}\int\limits_t^{\infty} e^{\lambda u}L^{\varepsilon}_{w_s m}(du)\cdot\overline{F}^{\varepsilon}_{mj}(t-u,\lambda),$$

Since

$$\int\limits_t^{\infty} e^{\lambda u}L^{\varepsilon}(du)\cdot\overline{F}^{\varepsilon}(t-u,\lambda)-\int\limits_t^{\infty} L^{\varepsilon}(du)\cdot\overline{F}^{\varepsilon}(t-u,0)\le 0,$$

for a difference $[\overline{L}^{\varepsilon}_{w_sj}(t,0)-\overline{L}^{\varepsilon}_{w_sj}(t,\lambda)]$ we have such inequality

$$\overline{L}^{\varepsilon}_{w_sj}(t,0)-\overline{L}^{\varepsilon}_{w_sj}(t,\lambda)\le \overline{F}^{\varepsilon}_{w_sj}(t,0)-\overline{F}^{\varepsilon}_{w_sj}(t,\lambda)+$$

$$+\sum_{m\notin D}[\overline{L}^{\varepsilon}_{w_sm}(t,0)-\overline{L}^{\varepsilon}_{w_sm}(t,\lambda)]\cdot \widehat{F}^{\varepsilon}_{mj}(\lambda)+\sum_{m\notin D}\overline{L}^{\varepsilon}_{w_sm}(t,0)\cdot[\widehat{F}^{\varepsilon}_{mj}(0)-\widehat{F}^{\varepsilon}_{mj}(\lambda)].$$

Since

$$\overline{L}^{\varepsilon}(t,0)=\int\limits_t^{\infty}L^{\varepsilon}(du),\quad \overline{F}^{\varepsilon\,'}_{\lambda}(t,0)=\int\limits_t^{\infty}uF^{\varepsilon}(du),\quad (\widehat{F^{\varepsilon}}_{ij})'(0)=a^{\varepsilon}_{ij},$$

then summing both parts of the last inequality in all $j\in E$, dividing it on $-\lambda$ and letting λ to the zero, we get

$$\sup_{\varepsilon}\sum_{n=1}^{\infty}\int\limits_t^{\infty}uL^{\varepsilon}_{w_sw_n}(du)=\sup_{\varepsilon}\sum_{n=1}^{\infty}\lim_{\lambda\to 0}\frac{\overline{L}^{\varepsilon}_{w_sw_n}(t,0)-\overline{L}^{\varepsilon}_{w_sw_n}(t,\lambda)}{-\lambda}\le$$

$$\le\sup_{\varepsilon}\left[\sum_{j\in E}\int\limits_t^{\infty}uF^{\varepsilon}_{w_sj}(du)+\sum_{m\notin D}\sum_{j\in E}\int\limits_t^{\infty}L^{\varepsilon}_{w_sm}(du)\cdot a^{\varepsilon}_{mj}\right]\underset{t\to\infty}{\longrightarrow}0.$$

The first element heads for a zero by (2.5). From uniform bounded of series $\sum\limits_{j=1}^{\infty}a^{\varepsilon}_{ij}$, $\sum\limits_{j=1}^{\infty}L^{\varepsilon}_{ij}(t)$ it follows, that second element let to zero too. It is possible to show, that quantities $L_{w_sw_s}(t)$, $s\in\mathbb{N}$, are unlatticed.

Define a parameter ρ^{ε} and matrix C, which is in a theorem. Put

$$\rho^{\varepsilon}=\sum_{s=1}^{\infty}\rho^{\varepsilon}_s,\quad\text{where}\quad \rho^{\varepsilon}_s=\frac{1-L^{\varepsilon}_{w_sw_s}(\infty)}{m_s},\quad s\in\mathbb{N};$$

$$C^{\varepsilon}_{ss}=-\frac{\rho^{\varepsilon}_s}{\rho^{\varepsilon}};\quad C^{\varepsilon}_{sk}=\frac{L^{\varepsilon}_{w_sw_k}(\infty)}{\rho^{\varepsilon}\,m_s},\quad s\ne k.$$

For each $i\notin D$, $j\in E$ equation (2.14) may to write in a kind

$$(1-F^{\varepsilon}_{ii})L^{\varepsilon}_{ij}-\sum_{m\notin D,m\ne I}F^{\varepsilon}_{im}L^{\varepsilon}_{mj}=F^{\varepsilon}_{ij},$$

that is vector $L^{\varepsilon(j)}_{E\backslash D}=(L^{\varepsilon}_{mj})_{m\notin D}$ satisfy infinite-dimensional linear algebraic equation

$$(I-F^{\varepsilon}_{E\backslash D})L^{\varepsilon(j)}_{E\backslash D}=F^{\varepsilon(j)}_{E\backslash D}.$$

For this equation may to write the weak equation, accepting $L^{\varepsilon}_{mj}=0,m\notin E_s$

$$(I-F^{\varepsilon}_{E_s\backslash\omega_s})L^{*\varepsilon(j)}_{E_s\backslash\omega_s}=F^{\varepsilon(j)}_{E_s\backslash\omega_s}.$$

The solution difference of weak and infinity systems is estimated as follows (see [7])

$$\max_{m\in E_s\backslash\omega_s}|L^{*\varepsilon}_{mj}-L^{\varepsilon}_{mj}|\leq\frac{\max\limits_{I\in E_s\backslash\omega_s}\sum\limits_{j\notin E_s}F^{\varepsilon}_{ij}}{1-\max\limits_{I\in E_s\backslash\omega_s}\sum\limits_{j\in E_s\backslash\omega_s}F^{\varepsilon}_{ij}}\leq\delta_s(\varepsilon).$$

For $L_{ij}, i\in E_s\setminus\omega_s, j\in E_s$ we have equation

$$(I-F_{E_s\backslash\omega_s})L^{(j)}_{E_s\backslash\omega_s}=F^{(j)}_{E_s\backslash\omega_s}.$$

The solution of the weak system difference from the solution of system for $L^{(j)}_{E_s\backslash\omega_s}$ no more, than on $d_0\delta_s(\varepsilon)$. Consequently, we come to conclusion, that

$$L_{i\omega_s}-L^{\varepsilon}_{i\omega_s}\leq(d_0+1)\delta_s(\varepsilon)\frac{1-L^{\varepsilon}_{\omega_s\omega_s}}{m_s}=$$

$$=\frac{1}{m_s}\Big(F_{\omega_s\omega_s}-F^{\varepsilon}_{\omega_s\omega_s}+\sum_{m\in E_s\backslash\omega_s}(F_{im}L_{mj}-F^{\varepsilon}_{im}L^{\varepsilon}_{mj})+\sum_{m\notin D,m\notin E_s}F^{\varepsilon}_{im}L^{\varepsilon}_{mj}\Big)\leq$$
$$\leq\frac{1}{m_s}\Big(\sum_{m\in E_s}(F_{im}-F^{\varepsilon}_{im})+\sum_{m\in E_s\backslash\omega_s}(L_{im}-L^{\varepsilon}_{im})+\sum_{m\notin E_s}F^{\varepsilon}_{im}\Big)\leq K\delta_s(\varepsilon).$$

From here, taking into account bounded $\sum\limits_{s=1}^{\infty}\delta_s(\varepsilon)<\infty$, we have

$$\rho^{\varepsilon}=\sum_{s=1}^{\infty}\frac{1-L^{\varepsilon}_{\omega_s\omega_s}(\infty)}{m_s}<\infty.$$

As $\rho^{\varepsilon}_s=\frac{1-L^{\varepsilon}_{\omega_s\omega_s}(\infty)}{m_s}\underset{\varepsilon\to0}{\longrightarrow}0$, for arbitrary s we get

$$\rho^{\varepsilon}\underset{\varepsilon\to0}{\longrightarrow}0.$$

Directly from construction it follows, that

$$|C^{\varepsilon}_{ss}|\leq1,\quad C^{\varepsilon}_{sk}\geq0\ \text{ at }\ s\neq k.$$

As, consonant to (2.15), $\sum\limits_{k=1}^{\infty}L^{\varepsilon}_{w_sw_k}\leq1$, then

$$\sum_{k\neq s}C^{\varepsilon}_{sk}=\frac{\sum\limits_{k\neq s}L^{\varepsilon}_{w_sw_k}(\infty)}{\rho^{\varepsilon}\,m_s}\leq\frac{1-L^{\varepsilon}_{w_sw_s}(\infty)}{\rho^{\varepsilon}\,m_s}=\frac{\rho^{\varepsilon}_s}{\rho^{\varepsilon}}=-C^{\varepsilon}_{ss}\leq1.$$

Show that there exists limit $C=\lim\limits_{\varepsilon\to0}C^{\varepsilon}$. Fix arbitrary two positive numbers $a,\ b,\ b>a$. Since $L^{\varepsilon}(t)$ as function from t is nondecreasing, then

$$\frac{1-L^{\varepsilon}_{w_sw_s}\left(\frac{b}{\rho^{\varepsilon}}\right)}{m_s\rho^{\varepsilon}}\leq\frac{1-L^{\varepsilon}_{w_sw_s}\left(\frac{a}{\rho^{\varepsilon}}\right)}{m_s\rho^{\varepsilon}}.$$

From here we get inequality

$$\overline{\lim_{\varepsilon\to 0}}\frac{1-L^{\varepsilon}_{w_sw_s}\left(\frac{b}{\rho^{\varepsilon}}\right)}{m_s\rho^{\varepsilon}}\le \underline{\lim}_{\varepsilon\to 0}\frac{1-L^{\varepsilon}_{w_sw_s}\left(\frac{a}{\rho^{\varepsilon}}\right)}{m_s\rho^{\varepsilon}}$$

which we rewrite as follows

$$\overline{\lim_{\varepsilon\to 0}}\left[\frac{1-L^{\varepsilon}_{w_sw_s}(\infty)}{m_s\rho^{\varepsilon}}+\frac{L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}\left(\frac{b}{\rho^{\varepsilon}}\right)}{m_s\rho^{\varepsilon}}\right]\le$$
$$\le \underline{\lim}_{\varepsilon\to 0}\left[\frac{1-L^{\varepsilon}_{w_sw_s}(\infty)}{m_s\rho^{\varepsilon}}+\frac{L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}\left(\frac{a}{\rho^{\varepsilon}}\right)}{m_s\rho^{\varepsilon}}\right]$$

or

$$\overline{\lim_{\varepsilon\to 0}}\left[-C^{\varepsilon}_{ss}+\frac{L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}\left(\frac{b}{\rho^{\varepsilon}}\right)}{m_s\rho^{\varepsilon}}\right]\le$$
$$\le \underline{\lim}_{\varepsilon\to 0}\left[-C^{\varepsilon}_{ss}+\frac{L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}\left(\frac{a}{\rho^{\varepsilon}}\right)}{m_s\rho^{\varepsilon}}\right]. \qquad (2.22)$$

From a condition

$$\sup_{\varepsilon}\int_t^{\infty}uL^{\varepsilon}_{w_sw_s}(du)\underset{t\to\infty}{\longrightarrow}0,$$

which was set higher, and nonnegativity of function $[L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}(u)]$ it follows, that

$$\sup_{\varepsilon}t\cdot[L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}(t)]\le\sup_{\varepsilon}\Big(t\cdot[L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}(t)]+$$
$$+\int_t^{\infty}[L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}(u)]du\Big)=\sup_{\varepsilon}\int_t^{\infty}uL^{\varepsilon}_{w_sw_s}(du)\underset{t\to\infty}{\longrightarrow}0.$$

Therefore for an arbitrary sequence $u^{\varepsilon}\underset{\varepsilon\to 0}{\longrightarrow}\infty$

$$\sup_{\varepsilon}\ u^{\varepsilon}\cdot[L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}(u^{\varepsilon})]\underset{\varepsilon\to 0}{\longrightarrow}0.$$

Putting $u^{\varepsilon}=\frac{t}{\rho^{\varepsilon}}$, where t – arbitrary fixed number, we get

$$\sup_{\varepsilon}\ \frac{1}{\rho^{\varepsilon}}\cdot\left[L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}\left(\frac{t}{\rho^{\varepsilon}}\right)\right]\underset{\varepsilon\to 0}{\longrightarrow}0$$

and

$$\overline{\lim_{\varepsilon}}\ \frac{1}{\rho^{\varepsilon}}\cdot\left[L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}\left(\frac{b}{\rho^{\varepsilon}}\right)\right]=0,$$

$$0\le\underline{\lim}_{\varepsilon}\ \frac{1}{\rho^{\varepsilon}}\cdot\left[L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}\left(\frac{a}{\rho^{\varepsilon}}\right)\right]\le\overline{\lim_{\varepsilon}}\ \frac{1}{\rho^{\varepsilon}}\cdot\left[L^{\varepsilon}_{w_sw_s}(\infty)-L^{\varepsilon}_{w_sw_s}\left(\frac{a}{\rho^{\varepsilon}}\right)\right]=0.$$

Thus, with (2.22) we get inequality

$$\overline{\lim_{\varepsilon\to 0}}\,C^{\varepsilon}_{ss}\le\underline{\lim}_{\varepsilon\to 0}\,C^{\varepsilon}_{ss}$$

which is possible only on condition of equality $\overline{\lim\limits_{\varepsilon\to 0}}\,C^{\varepsilon}_{ss}=\underline{\lim\limits_{\varepsilon\to 0}}\,C^{\varepsilon}_{ss}$. of upper and lower limits.

Analogously we derive equality $\overline{\lim\limits_{\varepsilon\to 0}}\,C^{\varepsilon}_{sk}=\underline{\lim\limits_{\varepsilon\to 0}}\,C^{\varepsilon}_{sk}$ of upper and lower limits C^{ε}_{sk} for arbitrary $k\neq s$.

From it and from uniform restriction in ε of elements of matrix C^{ε} it follows, that indeed there exists finite limit $C=\lim\limits_{\varepsilon\to 0}C^{\varepsilon}$. Therefore such image takes place

$$L^{\varepsilon}_{w_s w_k}(\infty)=\delta_{sk}+\rho^{\varepsilon}m_s C_{sk}+o(\rho^{\varepsilon}),\quad \varepsilon\to 0.$$

Using the result with [4], we can assert that

$$H^{\varepsilon}_{w_s w_k}\left(\frac{t}{\rho^{\varepsilon}}+y\right)-H^{\varepsilon}_{w_s w_k}\left(\frac{t}{\rho^{\varepsilon}}\right)\underset{\varepsilon\to 0}{\longrightarrow}y\cdot q_{sk}(t)\cdot\frac{1}{m_k},$$

where $q_{sk}(t)\,-\,(s,k)$ element of matrix $[e^{tC}]$.

Farther, with (2.12) we find

$$H^{\varepsilon}_{w_s j}\left(\frac{t}{\rho^{\varepsilon}}+u\right)-H^{\varepsilon}_{w_s j}\left(\frac{t}{\rho^{\varepsilon}}\right)=$$

$$=\sum_{m=1}^{\infty}\left[\int\limits_{0}^{\frac{t}{\rho^{\varepsilon}}+y}H^{\varepsilon}_{w_s w_m}(du)L^{\varepsilon}_{w_m j}\left(\frac{t}{\rho^{\varepsilon}}+y-u\right)-\int\limits_{0}^{\frac{t}{\rho^{\varepsilon}}}H^{\varepsilon}_{w_s w_m}(du)L^{\varepsilon}_{w_m j}\left(\frac{t}{\rho^{\varepsilon}}-u\right)\right].$$

Integrating by parts, we get

$$H^{\varepsilon}_{w_s j}\left(\frac{t}{\rho^{\varepsilon}}+u\right)-H^{\varepsilon}_{w_s j}\left(\frac{t}{\rho^{\varepsilon}}\right)=$$

$$=\sum_{m=1}^{\infty}\int\limits_{0}^{\frac{t}{\rho^{\varepsilon}}}\left[H^{\varepsilon}_{w_s w_m}\left(\frac{t}{\rho^{\varepsilon}}+y-u\right)-H^{\varepsilon}_{w_s w_m}\left(\frac{t}{\rho^{\varepsilon}}-u\right)\right]L^{\varepsilon}_{w_m j}(du)+$$

$$+\sum_{m=1}^{\infty}\int\limits_{\frac{t}{\rho^{\varepsilon}}}^{\frac{t}{\rho^{\varepsilon}+y}}H^{\varepsilon}_{w_s w_m}\left(\frac{t}{\rho^{\varepsilon}}+y-u\right)L^{\varepsilon}_{w_m j}(du)-$$

$$-\sum_{m=1}^{\infty}\left[L^{\varepsilon}_{w_m j}\left(\frac{t}{\rho^{\varepsilon}}+y\right)-L^{\varepsilon}_{w_m j}\left(\frac{t}{\rho^{\varepsilon}}\right)\right].$$

Find limit of right part of the last equality as $\varepsilon\to 0$. We have

$$\sum_{m=1}^{\infty}\int\limits_{\frac{t}{\rho^{\varepsilon}}}^{\frac{t}{\rho^{\varepsilon}+y}}H^{\varepsilon}_{w_s w_m}\left(\frac{t}{\rho^{\varepsilon}}+y-u\right)L^{\varepsilon}_{w_m j}(du)-\sum_{m=1}^{\infty}\left[L^{\varepsilon}_{w_m j}\left(\frac{t}{\rho^{\varepsilon}}+y\right)-L^{\varepsilon}_{w_m j}\left(\frac{t}{\rho^{\varepsilon}}\right)\right]\leq$$

$$\leq\sum_{m=1}^{\infty}\left(H^{\varepsilon}_{w_s w_m}(y)-I\right)\left[L^{\varepsilon}_{w_m j}\left(\frac{t}{\rho^{\varepsilon}}+y\right)-L^{\varepsilon}_{w_m j}\left(\frac{t}{\rho^{\varepsilon}}\right)\right]\leq$$

$$\leq \sup_m \Big[L^\varepsilon_{\omega_m j}\Big(\frac{t}{\rho^\varepsilon}+y\Big) - L^\varepsilon_{\omega_m j}\Big(\frac{t}{\rho^\varepsilon}\Big)\Big] \sum_{m=1}^{\infty} (H^\varepsilon_{\omega_s\omega_m}(y) - I) \underset{\varepsilon\to 0}{\longrightarrow} 0$$

in accordance with lemma 1 and remark (2.19). Fix some $T > 0$ and, applying lemma 2, estimate an integral

$$\sup_\varepsilon \sum_{m=1}^{\infty} \int_T^{\frac{t}{\rho^\varepsilon}} \left[H^\varepsilon_{w_s w_m}\left(\frac{t}{\rho^\varepsilon}+y-u\right) - H^\varepsilon_{w_s w_m}\left(\frac{t}{\rho^\varepsilon}-u\right)\right] L^\varepsilon_{w_m j}(du) \leq$$

$$\leq [Ay+B] \sup_{\varepsilon,m} \int_T^{\infty} L^\varepsilon_{w_m j}(du) \leq [Ay+B] \sup_{\varepsilon,m} [L^\varepsilon_{w_m j}(\infty) - L^\varepsilon_{w_m j}(T)].$$

By the choice enough large T expression $\sup\limits_{\varepsilon,m}[L^\varepsilon_{w_m j}(\infty) - L^\varepsilon_{w_m j}(T)]$ may to do as enough small with properties of matrix $L^\varepsilon(t)$.

Finally, as at first

$$H^\varepsilon_{w_s w_k}\left(\frac{t}{\rho^\varepsilon}+y-u\right) - H^\varepsilon_{w_s w_k}\left(\frac{t}{\rho^\varepsilon}-u\right) \underset{\varepsilon\to 0}{\longrightarrow} y\cdot q_{sk}(t)\cdot \frac{1}{m_k}$$

uniformly in $u \in [0,T]$; secondly, in accordance with lemma 2

$$\overline{\lim_{\varepsilon\to 0}} \sup_{0\leq u\leq T} \left[H^\varepsilon_{w_s w_k}\left(\frac{t}{\rho^\varepsilon}+y-u\right) - H^\varepsilon_{w_s w_k}\left(\frac{t}{\rho^\varepsilon}-u\right)\right] \leq Ay+B < \infty$$

and measure $L^\varepsilon_{w_k j}(du)$ weakly convergences to the measure $L_{w_k j}(du)$, on the basis of lemma 2 of the work [5] we find, that as $j \in E_k$

$$\int_0^T \Big[H^\varepsilon_{w_s w_k}\left(\frac{t}{\rho^\varepsilon}+y\right) - H^\varepsilon_{w_s w_k}\left(\frac{t}{\rho^\varepsilon}\right)\Big] L^\varepsilon_{w_k j}(du) \underset{\varepsilon\to 0}{\longrightarrow} y\cdot q_{sk}(t)\cdot \frac{1}{m_k} L_{w_k j}(T).$$

Thus, for $j \in E_k$ we get finally

$$H^\varepsilon_{w_s j}\left(\frac{t}{\rho^\varepsilon}+y\right) - H^\varepsilon_{w_s j}\left(\frac{t}{\rho^\varepsilon}\right) \underset{\varepsilon\to 0}{\longrightarrow} y\cdot q_{sk}(t)\cdot \frac{1}{m_k} L_{w_k j}. \qquad (2.23)$$

From correlation (2.9) for $i \in E_s$, $j \in E_k$ $(s,k \in \mathbb{N})$ we will have

$$H^\varepsilon_{ij}\left(\frac{t}{\rho^\varepsilon}+y\right) - H^\varepsilon_{ij}\left(\frac{t}{\rho^\varepsilon}\right) = \sum_{m\notin D} \left[L^\varepsilon_{im}\left(\frac{t}{\rho^\varepsilon}+y\right) - L^\varepsilon_{ij}\left(\frac{t}{\rho^\varepsilon}\right)\right]\cdot \delta_{mj} +$$

$$+ \sum_{m=1}^{\infty} \int_0^{\frac{t}{\rho^\varepsilon+y}} L^\varepsilon_{iw_m}(du) \left[H^\varepsilon_{w_m j}\left(\frac{t}{\rho^\varepsilon}+y-u\right) - H^\varepsilon_{w_m j}\left(\frac{t}{\rho^\varepsilon}-u\right)\right].$$

Using (2.19) and the same reasoning, that and as proving (2.23) we can assert that

$$H^\varepsilon_{ij}\left(\frac{t}{\rho^\varepsilon}+y\right) - H^\varepsilon_{ij}\left(\frac{t}{\rho^\varepsilon}\right) \underset{\varepsilon\to 0}{\longrightarrow} y\cdot L_{iw_s}\cdot q_{sk}(t)\cdot \frac{1}{m_k}\cdot L_{w_k j}$$

as $i \in E_s$, $j \in E_k$. For completion of proving remind that

$$L_{iw_s} = 1, \quad L_{w_k j} = \frac{p_j^{(k)}}{p_{w_k}^{(k)}}, \quad \text{and} \quad m_k \cdot p_{w_k}^{(k)} = \sum_{i,j \in E_k} p_i^{(k)} a_{ij} = \pi_k.$$

Theorem is proved.

3. Conclusions

Conditions on the tolimit matrix of measures and on the limit matrix of full masses of measures are found. Existence of normalized multiplier ρ^ε such, that on a time scale t/ρ^ε the asymptotic of difference of the renewal matrix (associated with family of dependency upon a small parameter of matrix-valued measures with the block-resoluble infinite-dimensional limit matrix of full masses of measures) on the finite interval is nontrivial is proved.

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Received February 20, 2013; Revised May 23, 2013.