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CONSTRUCTIVE METHOD FOR THE SOLUTION OF SINGULAR INTEGRAL EQUATIONS WITH HILBERT NUCLEUS IN HOLDER SPACES

Abstract

The constructive method for the solution of linear singular integral equations with Hilbert nucleus in Holder spaces was suggested and substantiated. In the developed constructive method, the singular operator is approximated by the operators preserving the main features of this operator, and this enables to get more accurate estimations from the point of view of convergence rate, than the earlier used methods. Furthermore, this method requires less calculating expenditures since it allows to find the approximate solution in the explicit way (but not at separate points), and the coefficients of appropriate systems of linear algebraic equations are easily calculated.

Let $H_\alpha \equiv H_\alpha([0,2\pi])$ be a space of 2π -periodic Holder continuous functions with the exponent $\alpha\,(0<\alpha\le 1)$ and with the norm $\|\varphi\|_\alpha = \|\varphi\|_\infty + h(\varphi;\alpha)$, where $\|\varphi\|_\infty = \max\limits_{t\in[0;2\pi]} |\varphi(t)|$,

$$H(\varphi;\alpha) \equiv \sup\left\{ |\varphi(t_1) - \varphi(t_2)| \,/\, |t_1 - t_2|^\alpha : t_1, t_2 \in [0,2\pi],\ t_1 \ne t_2 \right\}.$$

Consider in $H_\alpha\,(0<\alpha\le 1)$ the singular integral operator (SIO)

$$(R\varphi)(t) = a(t)\varphi(t) + b(t)(S\varphi)(t) + (\mathcal{K}\varphi)(t),$$

where

$$(S\varphi)(t) = \frac{1}{2\pi}\int\limits_0^{2\pi} ctg\frac{t-\tau}{2}\varphi(\tau)\,d\tau,$$

$$(\mathcal{K}\varphi)(t) = \frac{1}{2\pi}\int\limits_0^{2\pi} K(t,\tau)\varphi(\tau)\,d\tau, a(t), b(t), K(t,\tau)$$

are the known 2π-periodic Holder continuous functions with the exponent $\alpha\,(0<\alpha\le 1)$, and $a^2(t) + b^2(t) \ne 0$ for all $t \in [0;2;\pi)$.

Constructive methods for the solution of singular integral equations (SIE)

$$(R\varphi)(t) = f(t), \tag{1}$$

whose theory was stated in monographs [1-4] have found wide application in aerodynamics, theory of elasticity, electrodynamics and other applied fields [3;5], and a series of papers were devoted to their construction.

In the constructive method that we have developed, the singular part of the operator R is approximated by the operators preserving the main features of a singular operator (see [15, theorem 1]), that enables to get more accurate estimations from the point of view of convergence rate, than the earlier used methods. Furthermore, this method requires less calculating expenditures since it allows to find approximate solutions in the obvious way (but not at separate points), and the coefficients of appropriate system of linear algebraic equations () are easily calculated.

In the paper, the operator R is approximated by the sequence of operators of the form

$$(R_n\varphi)(t) = \sum_{k=0}^{2n-1} \alpha_k^{(n)}(t)\,\varphi\left(t + \frac{\pi k}{n}\right),$$

where $\alpha_k^{(n)}(t)$ are 2π -periodic Holder continuous functions expressed by the given functions, $k = \overline{0, 2n-1}$, $n \in N$, and it is proved that from the invertibility of the operator R in the space $H_\beta\,(0 < \beta < \alpha)$ it follows the invertibility of operator R_n (for rather large n) in this space, and the estimation of approximate solution of SIE (1) in the space $H_{\beta'}\left(0 < \beta^1 \le \beta\right)$ is given. Notice that in this method, the finding of the inverse operator R_n^{-1} is equivalent to consideration of the equation

$$\sum_{k=0}^{2n-1} \alpha_k^{(n)}(t)\,\varphi\left(t + \frac{\pi k}{n}\right) = f(t)$$

at the points $t + \pi m/n$, $m = \overline{0, 2n-1}$ since the solution of the obtained s.l.a.e.

$$\sum_{k=0}^{2n-1} \alpha_k^{(n)}\left(t + \frac{\pi m}{n}\right)\varphi\left(t + \frac{\pi(k+m)}{n}\right) = f\left(t + \frac{\pi m}{n}\right), \quad m = \overline{0, 2n-1},$$

with respect to $\left(\varphi(t), \varphi\left(t + \frac{\pi}{n}\right), ..., \varphi\left(t + \frac{(2n-1)\pi}{n}\right)\right)$ reduces to finding of the function $\varphi(t)$.

For SIE with Cauchy and Hilbert nucleus in the space L_2 such a method was developed and substantiated in [14] and [15], respectively.

It is known [1] that the operators S and $\mathcal{K}$ act from the space H_α to itself, where $0 < \alpha < 1$ in the case of the operator S, and $0 < \alpha \le 1$ in the case of the operator $\mathcal{K}$.

Let's consider the sequence of the operators

$$(S_n\varphi)(t) = \frac{1}{n}\sum_{k=0}^{n-1} ctg\left(-\frac{\pi(2k+1)}{2n}\right)\varphi\left(t + \frac{\pi(2k+1)}{n}\right), \quad n = 2, 3, ...$$

From the inequality $ctg\theta \le \pi/2\theta\ \ (\theta \in [0;\pi/2))$ it follows that for any $\varphi \in H_\alpha\,(0 < \alpha \le 1)$

$$\|S_n\varphi\|_\alpha \le \frac{\|\varphi\|_\alpha}{n}\sum_{k=0}^{n-1}\left|ctg\frac{\pi(2k+1)}{2n}\right| \le 2\,\|\varphi\|_\alpha \cdot [\ln n + c_0],$$

where c_0 is Euler's constant. This shows that the operators S_n, $n = 2, 3, ...$, also act from the space H_α to itself and the following inequality is fulfilled:

$$\|S_n\|_{H_\alpha \to H_\alpha} \le 2\ln n + 2c_0. \tag{2}$$

Let $E_n(\varphi)$ be the best uniform approximation of the function $\varphi \in C([0,2\pi])$ by trigonometric polynomials T_n at most of order n, $n \in N$. The polynomial $T_n^*(t)$ for which $E_n(\varphi) = \|\varphi - T_n^*\|_\infty$, is called a polynomial of the best uniform approximation of the function $\varphi(t)$. It is known (see [16;17]) that for each function $\varphi \in C([0,2\pi])$ the polynomial of the best uniform approximation exists and is unique.

Lemma 1. *Let $\varphi \in H_\alpha$, and $T_n^*(t)$ be a polynomial of the best uniform approximation of the function $\varphi(t)$ of order $n \in N$. Then for $0 < \beta \le \alpha < 1$*

$$\|\varphi - T_n^*\|_\beta \le \frac{c_1}{n^{\alpha-\beta}} \cdot H(\varphi;\alpha), \tag{3}$$

where c_1 is a constant independent of n and the function $\varphi(t)$ (everywhere in the sequel, by c, $k = 1,2,...$, we'll denote the constants independent of n and the function $\varphi(t)$).

Proof. For each fixed $n \in N$ in the case $|t_1 - t_2| \ge 1/n$ from the inequality (see [18], and also [16;17]) $E_n(\varphi) \le 1/12\omega(\varphi;1/n) \le 1/12n^\alpha H(\varphi;\alpha)$ where $\omega(\varphi;\delta) = \sup\{|\varphi(t_1) - \varphi(t_2)| : |t_1 - t_2| \le \delta\}$ $(\delta \ge 0)$ we have

$$\frac{|\varphi(t_1) - T_n^*(t_1) - \varphi(t_2) + T_n^*(t_2)|}{|t_1 - t_2|^\beta} \le 2\|\varphi - T_n^*\|_\infty \cdot n^\beta =$$

$$= 2n^\beta E_n(\varphi) \le \frac{1}{6n^{\alpha-\beta}} \cdot H(\varphi;\alpha),$$

and in the case $|t_1 - t_2| < 1/n$ from the inequality (see [19; §4, corollary 3,1], $\omega(T_n^*;\delta) \le c_2\omega(\varphi;\delta)$ we get

$$\frac{|\varphi(t_1) - T_n^*(t_1) - \varphi(t_2) + T_n^*(t_2)|}{|t_1 - t_2|^\beta} \le \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\beta} +$$

$$+ \frac{|T_n^*(t_1) - T_n^*(t_2)|}{|t_1 - t_2|^\beta} \le \frac{1 + c_2}{n^{\alpha-\beta}} \cdot H(\varphi;\alpha),$$

Hence the required estimation (3) follows. The lemma is proved.

Theorem 1. *For any function $\varphi \in H_\alpha\,(0 < \alpha \le 1)$ the sequence of functions $\{S_n\varphi(t)\}$ converges to the function $(S\varphi)(t)$ in the space $H_\alpha\,(0 < \beta < \alpha \le 1)$, and the following estimation is valid*

$$\|S\varphi - S_n\varphi\|_\beta \le \frac{c_3 + c_4 \ln n}{n^{\alpha-\beta}} \cdot H(\varphi;\alpha). \tag{4}$$

Proof. Since for any trigonometric polynomial T_{n-1} of order no more than $n-1$, the following equality (see [15]) $(S_nT_{n-1})(t) = (S_nT_{n-1})(t)$ is valid, then

$$(S\varphi)(t) - (S_n\varphi)(t) = S\left(\varphi - T_{n-1}^*\right)(t) - S_n\left(\varphi - T_{n-1}^*\right)(t),$$

where $T_{n-1}^*(t)$ is a polynomial of the best uniform approximation of the function $\varphi(t)$ of order $n-1$.

Hence, from inequality (2) and lemma 1 it follows the estimation

$$\|S\varphi - S_n\varphi\|_\beta \le \|S\|_{H_\beta \to H_\beta} \left\|\varphi - T_{n-1}^*\right\|_\beta +$$

$$+\left\|S_n\right\|_{H_\beta\to H_\beta}\left\|\varphi-T_{n-1}^*\right\|_\beta\le\frac{c_3+c_4\ln n}{n^{\alpha-\beta}}\cdot H\left(\varphi;\alpha\right).$$

Theorem 1 is proved.

Let the function $P\left(t,\tau\right)$ be 2π-periodic and Holder continuous with the exponent $\alpha\left(0<\alpha\le1\right)$ with respect to both arguments. Denote

$$H\left(P;\alpha\right)=\inf\left\{M:\left|P\left(t_1,\tau_1\right)-P\left(t_2,\tau_2\right)\right|\le M\left(\left|t_1-t_2\right|^\alpha+\left|\tau_1-\tau_2\right|^\alpha\right)\right\}.$$

Consider the integral

$$F\left(t\right)=\frac{1}{2\pi}\int\limits_0^{2\pi}P\left(t,\tau\right)d\tau$$

and the sequence of functions

$$F_n\left(t\right)=\frac{1}{2n}\sum_{k=0}^{2n-1}P\left(t,t+\frac{\pi k}{n}\right),\quad n=1,2,....$$

Lemma 2. *For any $0<\beta\le\alpha$ it is valid the estimation*

$$\left\|F-F_n\right\|_\beta\le\frac{c_5}{n^{\alpha-\beta}}\cdot H\left(P;\alpha\right).$$

Proof. We have

$$\left|F\left(t\right)-F_n\left(t\right)\right|=\frac{1}{2\pi}\left|\sum_{k=0}^{2n-1}\int\limits_{t+\frac{\pi k}{n}}^{t+\frac{\pi(k+1)}{n}}\left[P\left(t,\tau\right)-P\left(t,t+\frac{\pi k}{n}\right)\right]d\tau\right|\le$$

$$\le\frac{1}{2\pi}\sum_{k=0}^{2n-1}\int\limits_{t+\frac{\pi k}{n}}^{t+\frac{\pi(k+1)}{n}}\left|\tau-t-\frac{\pi k}{n}\right|^\alpha\cdot H\left(P;\alpha\right)d\tau\le$$

$$\le\frac{1}{2\pi}\sum_{k=0}^{2n-1}\int\limits_{t+\frac{\pi k}{n}}^{t+\frac{\pi(k+1)}{n}}\left(\frac{\pi}{n}\right)^\alpha H\left(P;\alpha\right)d\tau=\left(\frac{\pi}{n}\right)^\alpha H\left(P;\alpha\right)$$

hence the following estimation follows

$$\left\|F-F_n\right\|_\infty\le\left(\pi/n\right)^\alpha H\left(P;\alpha\right). \tag{5}$$

If $\left|t_1-t_2\right|\ge1/n$, then from (5)

$$\frac{\left|F\left(t_1\right)-F_n\left(t_1\right)-F\left(t_2\right)+F_n\left(t_2\right)\right|}{\left|t_1-t_2\right|^\beta}\le2\left\|F-F_n\right\|_\infty\cdot n^\beta\le\frac{2\pi^\alpha}{n^{\alpha-\beta}}\cdot H\left(P;\alpha\right),$$

and if $\left|t_1-t_2\right|<\frac{1}{n}$, then taking into account the estimates $\left|F\left(t_1\right)-F\left(t_2\right)\right|\le\left|t_1-t_2\right|^\alpha H\left(P;\alpha\right),$

$$\left|F_n\left(t_1\right)-F_n\left(t_2\right)\right|\le2\left|t_1-t_2\right|^\alpha H\left(P;\alpha\right),$$

we get

$$\frac{|F(t_1)-F_n(t_1)-F(t_2)+F_n(t_2)|}{|t_1-t_2|^{\beta}} \le \frac{|F(t_1)-F(t_2)|}{|t_1-t_2|^{\beta}} + \frac{|F_n(t_1)-F_n(t_2)|}{|t_1-t_2|^{\beta}} \le$$

$$\le 3\,|t_1-t_2|^{\alpha-\beta} H(P;\alpha) < \frac{3}{n^{\alpha-\beta}} \cdot H(P;\alpha)$$

Lemma 2 is proved.

Consider the sequence of the operators

$$(\mathcal{K}_n\varphi)(t) = \frac{1}{2n}\sum_{k=0}^{2n-1} K\left(t, t+\frac{\pi k}{n}\right)\varphi\left(t+\frac{\pi k}{n}\right), \quad n = 1,2,....$$

Similarly, we can see that the operators $\mathfrak{K}_n$, $n = 1,2,...$, act from the space $H_\alpha\,(0<\alpha\le 1)$ to itself. Assuming $P(t,\tau) = K(t,\tau)\varphi(\tau)$, from lemma 2 it holds

Theorem 2. *For any $\varphi \in H_\alpha\,(0<\alpha\le 1)$ the sequence of functions $\{\mathcal{K}_n(\varphi)\}$ converges to the function $(\mathcal{K}_n\varphi)(t)$ in the space $H_\alpha\,(0<\beta<\alpha)$, and the following estimation is valid:*

$$\|\mathcal{K}\varphi - \mathcal{K}_n\varphi\|_\beta \le \frac{c_5}{n^{\alpha-\beta}}\left[\|\varphi\|_\infty H(K;\alpha) + \|K\|_\infty \cdot H(\varphi;\alpha)\right].$$

Lemma 3. *Let* $(M\varphi)(t) = \frac{1}{2\pi}\int\limits_0^{2\pi} M(t,\tau)\varphi(\tau)\,d\tau$,

$(M^*\varphi)(t) = \frac{1}{2\pi}\int\limits_0^{2\pi} M^*(t,\tau)\varphi(\tau)\,d\tau$

$$M(M^*\varphi)(t) = \frac{1}{2\pi}\int\limits_0^{2\pi}\left[\frac{1}{2\pi}\int\limits_0^{2\pi} M(t,\tau')\,M^*(\tau',\tau)\,d\tau\right]\varphi(\tau)\,d\tau =$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi}\widetilde{M}(t,\tau)\varphi(\tau)\,d\tau,$$

where $M(t,\tau)$, $M^*(t,\tau)$ are 2π - periodic and Holder continuous functions with the exponent $\alpha\,(0<\alpha\le 1)$. Then for any $0<\beta\le\alpha$ it holds the inequality

$$\|(MM^*)_n - M_nM_n^*\|_\beta \le \frac{c_5}{n^{\alpha-\beta}}\left[H(M;\alpha)\cdot\|M^*\|_\infty \cdot H(M^*;\alpha)\cdot\|M\|_\infty\right]. \qquad (6)$$

Proof. For any $\varphi \in H_\beta$ we have

$$\lfloor (MM^*)_n - M_nM_n^* \rfloor \varphi(t) =$$

$$= \frac{1}{2\pi}\sum_{k=0}^{2n-1}\left[\frac{1}{2\pi}\int\limits_0^{2\pi} M(t,\tau')\,M^*\left(\tau', t+\frac{\pi k}{n}\right)d\tau' - \right.$$

$$\left. -\frac{1}{2n}\sum_{m=0}^{2n-1} M\left(t, t+\frac{\pi m}{n}\right)M^*\left(t+\frac{\pi m}{n}, t+\frac{\pi k}{n}\right)\right]\varphi\left(t+\frac{\pi k}{n}\right). \qquad (7)$$

Assuming $P(t,\tau) = M(t,\tau)\cdot M^*\left(\tau, t+\frac{\pi k}{n}\right)$, from lemma 2 we get the inequality

$$\left\| \frac{1}{2\pi}\int_0^{2\pi} M(t,\tau')\, M^*\left(\tau', t+\frac{\pi k}{n}\right) d\tau' - \right.$$

$$\left. -\frac{1}{2n}\sum_{m=0}^{2n-1} M\left(t, t+\frac{\pi m}{n}\right) M^*\left(t+\frac{\pi m}{n}, t+\frac{\pi k}{n}\right)\right\|_\beta \le$$

$$\le \frac{c_5}{n^{\alpha-\beta}} H(P;\alpha) \le \frac{c_5}{n^{\alpha-\beta}}\left[H(M;\alpha)\cdot\|M^*\|_\infty + H(M^*;\alpha)\cdot\|M\|_\infty\right].$$

Hence, allowing for equality (7), we get inequality (6). Lemma 3 is proved.

Lemma 4. *Let $K(t,\tau)$ be 2π periodic, Holder continuous function with the exponent $\alpha\,(0<\alpha\le 1)$, and there exist the inverse operator $(I+\mathcal{K})^{-1}$ in space $H_\beta\,(0<\beta<\alpha)$. Then for large values of n, the operators $I+\mathcal{K}_n$ are also investible in space H_β and for any $f\in H_\beta$ it holds the estimation*

$$\left\|(I+\mathcal{K}_n)^{-1} f - (I+\mathcal{K})^{-1} f\right\|_{\beta'} \le \frac{c_6}{n^{\beta-\beta'}}\|f\|_{\beta'} \quad \left(0<\beta'\le\beta\right). \tag{8}$$

Proof. From Fredholm's theory of equations it is known that $(I+\mathcal{K}_n)^{-1} = I+\mathcal{K}^*$, where $(\mathcal{K}^*\varphi)(t) = \frac{1}{2\pi}\int_0^{2\pi} K^*(t,\tau)\varphi(\tau)\,d\tau$, $K^*(t,\tau)$ is an 2π periodic Holder continuous function with the exponent α. Then $(I+\mathcal{K}_n^*)(I+\mathcal{K}_n) = I+\mathcal{K}_n^*+\mathcal{K}_n+\mathcal{K}_n^*\mathcal{K}_n = I+\delta_n$ where $\delta_n\,(\mathcal{K}^*\mathcal{K}) - \mathcal{K}_n^*\mathcal{K}_n$. According to lemma 3 we have

$$\|\delta_n\|_\beta \le \frac{c_5}{n^{\alpha-\beta}}\left[H(K;\alpha)\cdot\|K^*\|_\infty + H(K^*;\alpha)\cdot\|K\|_\infty\right]. \tag{9}$$

Consequently, for large values of n the operators $I+\delta_n$ and therefore the operators $I+\mathcal{K}_n$ are also investible in H_β and $(I+\mathcal{K}_n)^{-1} = (I+\delta_n)^{-1}(I+\mathcal{K}_n^*)$.

Estimation (8) follows from the equality

$$(I+\mathcal{K}_n)^{-1} f - (I+\mathcal{K}_n)^{-1} f = (I+\delta_n)^{-1}(I+\mathcal{K}_n^*) f - (I+\mathcal{K}^*) f =$$

$$= \left\lfloor (I+\delta_n)^{-1} - I\right\rfloor (I+\mathcal{K}_n^*) f + (\mathcal{K}_n^* - \mathcal{K}^*) f,$$

by inequality (9) and theorem 2. Lemma 4 is proved.

In addition to the sequence of operators $\mathcal{K}_n$ we'll need the sequence of operators of the form

$$\left(\widetilde{\mathcal{K}}_n^{(even)}\varphi\right)(t) = \frac{1}{n}\sum_{k=0}^{n-1} K\left(t, t+\frac{2\pi k}{n}\right)\varphi\left(t+\frac{2\pi k}{n}\right),$$

$$\left(\mathcal{K}_n^{(odd)}\varphi\right)(t) = \frac{1}{n}\sum_{k=0}^{n-1} K\left(t, t+\frac{\pi(2k+1)}{n}\right)\varphi\left(t+\frac{2(k+1)}{n}\right), \quad n=1,2,....$$

It is easy to see that the analogies of theorem 2 and 3 are valid for these operators as well.

Lemma 5. *Let $K(t,\tau)$ and $\widetilde{K}(t,\tau)$ be 2π-periodic, Holder continuous functions with the exponent $\alpha\,(0<\alpha\leq 1)$, and there exist the inverse operators $\left(I\pm\mathcal{K}+\widetilde{\mathcal{K}}\right)^{-1}$ in the space $H_\beta\,(0<\beta<\alpha)$. Then for large values of n operators $I+K_n^{(odd)}+\widetilde{\mathcal{K}}_n^{(even)}$ are also invertible in space H_β, and for any the $f\in H$ it holds the estimation*

$$\left\|\left(I+\mathcal{K}_n^{(odd)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}f-\left(I+\mathcal{K}+\widetilde{\mathcal{K}}\right)^{-1}f\right\|_{\beta'}\leq$$

$$\leq\frac{c_1}{n^{\beta-\beta'}}\|f\|_{\beta'}\left(0<\beta'\leq\beta\right). \tag{10}$$

Proof. From theory of Fredholm equations it is known that

$$\left(I-\mathcal{K}+\widetilde{\mathcal{K}}\right)^{-1}=I+\mathcal{K}^*,\quad \left[\left(I-\mathcal{K}+\widetilde{\mathcal{K}}\right)^{-1}\left(I+\mathcal{K}+\widetilde{\mathcal{K}}\right)\right]^{-1}=I+\mathcal{K}^{**},$$

where $(\mathcal{K}^*\varphi)(t)=\frac{1}{2\pi}\int\limits_0^{2\pi}\mathcal{K}^*(t,\tau)\,\varphi(\tau)\,d\tau$, $(\mathcal{K}^{**}\varphi)(t)=\frac{1}{2\pi}\int\limits_0^{2\pi}\mathcal{K}^{**}(t,\tau)\,\varphi(\tau)\,d\tau$, $\mathcal{K}^*(t,\tau)$, $\mathcal{K}^{**}(t,\tau)$ are 2π periodic and Holder continuous functions with the exponent α. Then according to lemma 4 for large values of n the operators $I+\mathcal{K}_n^{(odd)}+\widetilde{\mathcal{K}}_n^{(even)}$ are also invertible and

$$\left(I-\mathcal{K}_n^{(even)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}=\left(I+\delta_n^{(even)}\right)^{-1}\left(I+\mathcal{K}_n^{*(even)}\right),$$

where $\delta_n^{(even)}=\left(\mathcal{K}^*\left(\widetilde{\mathcal{K}}-\mathcal{K}\right)\right)_n^{(even)}-\mathcal{K}_n^{*(even)}\left(\widetilde{\mathcal{K}}-\mathcal{K}\right)_n^{(even)}$. From the equalities

$$I+2\left(I-\mathcal{K}+\widetilde{\mathcal{K}}\right)^{-1}\mathcal{K}=\left(I-\mathcal{K}+\widetilde{\mathcal{K}}\right)^{-1}\left(I+\mathcal{K}+\widetilde{\mathcal{K}}\right),$$

$$1+2\left(I-\mathcal{K}_n^{(even)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}\mathcal{K}_n=$$

$$=\left(I-\mathcal{K}_n^{(even)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}\left(I-\mathcal{K}_n^{(even)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}$$

it follows that

$$\left(I+\mathcal{K}_n^{**}\right)\left(I-\mathcal{K}_n^{(even)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}\left(I+\mathcal{K}_n^{(odd)}+\widetilde{\mathcal{K}}_n^{(even)}\right)=$$

$$=\left(I+\mathcal{K}_n^{**}\right)\left[I+2\left(I+\delta_n^{(even)}\right)^{-1}\left(I+\mathcal{K}_n^{(even)}\right)\mathcal{K}_n\right]=I+\delta_n',$$

where

$$\delta_n'=2\left(I+\mathcal{K}_n^{**}\right)\left[\left(I+\delta_n^{(even)}\right)^{-1}-I\right]\left(I+\mathcal{K}_n^{*(even)}\right)\mathcal{K}_n+2\left(I+\mathcal{K}_n^{**}\right)\times$$

$$\times\left[\mathcal{K}_n^{*(even)}\mathcal{K}_n-\left(\mathcal{K}^*\mathcal{K}\right)_n\right]+$$

$$+2\left[\mathcal{K}_n^{**}\left(\mathcal{K}_n+\left(\mathcal{K}^*\mathcal{K}\right)_n\right)-\left(\mathcal{K}^{**}\left(\mathcal{K}+\mathcal{K}^*\mathcal{K}\right)\right)_n\right]$$

and according to lemma 3

$$\left\|\delta'_n\right\|_\beta \le \frac{c_8}{n^{\alpha-\beta}}.$$

Consequently, for large values of n the operators $I+\delta'_n$ and therefore the operators $I+\mathcal{K}_n^{(odd)}+\widetilde{\mathcal{K}}_n^{(even)}$ are invertible in H_β, and

$$\left(I+\mathcal{K}_n^{(odd)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}=\left(I+\delta'_n\right)^{-1}+\left(I+\mathcal{K}_n^{**}\right)\left(I-\mathcal{K}_n^{(even)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}.$$

Estimation (10) follows from the equality

$$\left(I+\mathcal{K}_n^{(odd)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}f-\left(I+\mathcal{K}+\widetilde{\mathcal{K}}\right)^{-1}f=$$

$$=\left[\left(I+\delta'_n\right)^{-1}-I\right]\left(I+\mathcal{K}_n^{**}\right)\left(I-\mathcal{K}_n^{(even)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}f+$$

$$+\left(\mathcal{K}_n^{**}-\mathcal{K}^{**}\right)\left(I-\mathcal{K}_n^{(even)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}f+\left(I+\mathcal{K}^{**}\right)\times$$

$$\times\left[\left(I-\mathcal{K}_n^{(even)}+\widetilde{\mathcal{K}}_n^{(even)}\right)^{-1}f-\left(I-\mathcal{K}+\widetilde{\mathcal{K}}_n\right)^{-1}f\right].$$

Lemma 5 is proved.

Lemma 6. *Let $M(t,\tau)$ be a 2π -periodic, Holder continuous function with the exponent $\alpha\,(0<\alpha\le 1)$. Then for any $0<\beta<\alpha$ it holds the estimation*

$$\left\|\frac{1}{2\pi}\int_0^{2\pi} ctg\frac{t-\tau}{2}M(t,\tau)\,d\tau-\frac{1}{n}\sum_{k=0}^{n-1}ctg\left(-\frac{\pi(2k+1)}{2n}\right)M\left(t,t+\frac{\pi(2k+1)}{n}\right)\right\|_\beta\le$$

$$\le\frac{c_9+c_{10}\ln n}{n^{\alpha-\beta}}\cdot H(M;\alpha). \qquad (11)$$

Proof. Since for a constant $c: S(c)=S_n(c)=c$, then

$$\frac{1}{2\pi}\int_0^{2\pi} ctg\frac{t-\tau}{2}M(t,\tau)\,d\tau-$$

$$-\frac{1}{n}\sum_{k=0}^{n-1}ctg\left(-\frac{\pi(2k+1)}{2n}\right)M\left(t,t+\frac{\pi(2k+1)}{n}\right)=$$

$$=\frac{1}{2\pi}\int_0^{2\pi} ctg\frac{t-\tau}{2}\left[M(t,\tau)-M(t,t)\right]d\tau-$$

$$-\frac{1}{n}\sum_{k=0}^{n-1}ctg\left(-\frac{\pi(2k+1)}{2n}\right)\left[M\left(t,t+\frac{\pi(2k+1)}{n}\right)-M(t,t)\right]=$$

$$=N(t)-N_n^{odd}(t),$$

where $N(t)=\frac{1}{2\pi}\int\limits_0^{2\pi}N(t,\tau)\,d\tau$, $N(t,\tau)=ctg\frac{t-\tau}{2}\left[M(t,\tau)-M(t,t)\right]$.

From the estimations

$$\left|\int_{t-\frac{\pi}{n}}^{t+\frac{\pi}{n}}N(t,\tau)\,d\tau\right|\le\int_{t-\frac{\pi}{n}}^{t+\frac{\pi}{n}}\frac{\pi}{2}\left|\frac{2}{t-\tau}\right|\cdot H(M;\alpha)\cdot|\tau-t|^\alpha\,d\tau=\frac{c_{11}}{n^\alpha}\cdot H(M;\alpha),$$

$$\left|\frac{1}{2\pi}\int\limits_{t+\frac{\pi}{n}}^{t+2\pi-\frac{\pi}{n}} N(t,\tau)\,d\tau-\frac{1}{n}\sum_{k=0}^{n-2}N\left(t,t+\frac{\pi(2k+1)}{n}\right)\right|\le$$

$$\le\frac{1}{2\pi}\sum_{k=0}^{n-2}\int\limits_{t+\frac{\pi(2k+1)}{n}}^{t+\frac{\pi(2k+3)}{n}}\left|N(t,\tau)-N\left(t,t+\frac{\pi(2k+1)}{n}\right)\right|d\tau\le$$

$$\le\frac{1}{2\pi}\sum_{k=0}^{n-2}\int\limits_{t+\frac{\pi(2k+1)}{n}}^{t+\frac{\pi(2k+3)}{n}}\left\{\left|ctg\frac{t-\tau}{2}\right|\left|M(t,\tau)-M\left(t,t+\frac{\pi(2k+1)}{n}\right)\right|+\right.$$

$$\left.+\left|ctg\frac{t-\tau}{2}-ctg\left(-\frac{\pi(2k+1)}{n}\right)\right|\times M\left(t,t+\frac{\pi(2k+1)}{n}\right)-M(t,t)\right\}d\tau\le$$

$$\le\frac{1}{2\pi}\sum_{k=0}^{n-2}\int\limits_{0}^{\frac{2\pi}{n}}\left\{\left|ctg\left(-\frac{\pi(2k+1)}{n}-\tau'\right)\right|\cdot H(M;\alpha)\cdot\left|\tau'\right|^{\alpha}+\right.$$

$$\left.+\left|\frac{\sin\tau'}{\sin\frac{\pi(2k+1)}{n}\sin\left(\frac{\pi(2k+1)}{n}+\tau'\right)}\right|\cdot H(M;\alpha)\left(\frac{\pi(2k+1)}{n}\right)^{\alpha}\right\}d\tau'\le$$

$$\le\frac{c_{12}+c_{13}\ln n}{n^{\alpha}}\cdot H(M;\alpha),$$

$$\left|\frac{1}{n}N\left(t,t+\frac{\pi(2n-1)}{n}\right)\right|=\left|\frac{1}{n}ctg\frac{\pi}{n}\right|\times$$

$$\times\left|M\left(t,t-\frac{\pi}{n}\right)-M(t,t)\right|\le\frac{\pi^{\alpha-1}}{n^{\alpha}}\cdot H(M;\alpha).$$

It holds the inequality

$$\left\|N-N_n^{(odd)}\right\|_{\infty}\le\frac{c_{14}+c_{15}\ln n}{n^{\alpha}}\cdot H(M;\alpha). \tag{12}$$

If $|t_1-t_2|\ge 1/n$, then from (12) we get

$$\frac{\left|N(t_1)-N_n^{(odd)}(t_1)-N(t_2)+N_n^{(odd)}(t_2)\right|}{|t_1-t_2|^{\beta}}\le$$

$$\le 2\left\|N-N_n^{(odd)}\right\|_{\infty}\cdot\frac{1}{n^{\beta}}\le\frac{2c_{14}+2c_{15}\ln n}{n^{\alpha-\beta}}\cdot H(M;\alpha).$$

Consider the case $|t_1-t_2|\ge 1/n$. Let $\delta=2|t_1-t_2|$. Then

$$N(t_1)-N(t_2)=\frac{1}{2\pi}\int\limits_{t_1-2\delta}^{t_1+2\delta}N(t_1,\tau)\,d\tau-\frac{1}{2\pi}\int\limits_{t_1-2\delta}^{t_1+2\delta}N(t_2,\tau)\,d\tau+$$

$$+\frac{1}{2\pi}\int\limits_{t_1+2\delta}^{t_1+2\pi-2\delta}ctg\frac{t_1-\tau}{2}\left[M(t_1,\tau)-M(t_2,\tau)\right]d\tau+$$

$$+\frac{1}{2\pi}\int\limits_{t_1+2\delta}^{t_1+2\pi-2\delta}ctg\frac{t_1-\tau}{2}\left[M(t_2,t_2)-M(t_1,t_1)\right]d\tau+$$

$$+\frac{1}{2\pi}\int\limits_{t_1+2\delta}^{t_1+2\pi-2\delta}\left[ctg\frac{t_1-\tau}{2}-ctg\frac{t_2-\tau}{2}\right]\times$$

$$\times\left[M\left(t_1,\tau\right)-M\left(t_2,\tau\right)\right]d\tau=J_1+J_2+J_3+J_4+J_5.$$

Estimate the integrals J_m, $m=\overline{1,5}$. We have

$$|J_1|\le\frac{1}{2\pi}\int\limits_{t_1-2\delta}^{t_1+2\delta}\left|ctg\frac{t-\tau}{2}\right|\cdot H\left(M;\alpha\right)\cdot|\tau-t|^{\alpha}\le$$

$$\le c_{16}\cdot H\left(M;\alpha\right)\cdot|t_1-t_2|^{\alpha}.$$

The integral J_2 is estimated in the same way. Further,

$$|J_3|\le\frac{1}{2\pi}\int\limits_{t_1+2\delta}^{t_1+2\pi-2\delta}\left|ctg\frac{t_1-\tau}{2}\right|\cdot H\left(M;\alpha\right)d\tau\,|t_1-t_2|^{\alpha}\le$$

$$\le\left(c_{17}+c_{18}\ln|t_1-t_2|\right)\cdot H\left(M;\alpha\right)\cdot|t_1-t_2|^{\alpha},$$

$$J_4=\frac{1}{2\pi}\int\limits_{t_1+2\delta}^{t_1+2\pi-2\delta}ctg\frac{t_1-\tau}{2}d\tau\cdot\left[M\left(t_2,t_2\right)-M\left(t_1,t_1\right)\right]=0,$$

$$|J_5|\le\frac{1}{2\pi}\int\limits_{t_1+2\delta}^{t_1+2\pi-2\delta}\left|\frac{\sin\frac{t_1-t_2}{2}}{\sin\frac{t_1-\tau}{2}\sin\frac{t_2-\tau}{2}}\right|\cdot H\left(M;\alpha\right)\cdot|\tau-t_2|^{\alpha}d\tau\le$$

$$\le c_{19}\cdot H\left(M;\alpha\right)\cdot|t_1-t_2|^{\alpha}.$$

The estimations obtained for the integrals J_m, $m=\overline{1,5}$, in totality give the inequality

$$|N\left(t_1\right)-N\left(t_2\right)|\le\left(c_{20}+c_{21}\ln|t_1-t_2|\right)\cdot|t_1-t_2|^{\alpha}\cdot H\left(M;\alpha\right). \qquad (13)$$

For the difference $\left|N_n^{(odd)}\left(t_1\right)-N_n^{(odd)}\left(t_2\right)\right|$ we have

$$\left|N_n^{(odd)}\left(t_1\right)-N_n^{(odd)}\left(t_2\right)\right|\le\frac{1}{n}\sum_{k=0}^{n-1}\left|ctg\left(-\frac{\pi\left(2k+1\right)}{2n}\right)\right|\times$$

$$\times\left|M\left(t_1,t_1+\frac{\pi\left(2k+1\right)}{n}\right)-M\left(t_2,t_2+\frac{\pi\left(2k+1\right)}{n}\right)\right|\le$$

$$\le\frac{1}{n}\sum_{k=0}^{n-1}\left|ctg\left(-\frac{\pi\left(2k+1\right)}{2n}\right)\right|\cdot H\left(M;\alpha\right)2\cdot|t_1-t_2|^{\alpha}\le$$

$$\le\left(c_{22}+c_{23}\ln n\right)\cdot|t_1-t_2|^{\alpha}\cdot H\left(M;\alpha\right). \qquad (14)$$

From inequalities (13) and (14) we get that in the case $|t_1-t_2|\ge 1/n$ the following estimation is valid:

$$\frac{\left|N\left(t_1\right)-N_n^{(odd)}\left(t_1\right)-\left|N\left(t_2\right)+N_n^{(odd)}\left(t_2\right)\right|\right|}{|t_1-t_2|^{\beta}}\le$$

$$\leq \frac{c_{24}+c_{25}\ln n}{n^{\alpha-\beta}}\cdot H(M;\alpha).$$

Inequality (11) follows from the obtained estimations. Lemma 6 is proved.

Lemma 7. *Let* $(M\varphi)(t)=\frac{1}{2\pi}\int\limits_0^{2\pi} M(t,\tau)\varphi(\tau)\,d\tau$,

$$(MS)\varphi(t)=\frac{1}{2\pi}\int\limits_0^{2\pi}\left[\frac{1}{2\pi}\int\limits_0^{2\pi} ctg\frac{t-\tau'}{2}M(t,\tau)\,d\tau\right]\varphi(\tau')\,d\tau'=$$

$$=\frac{1}{2\pi}\int\limits_0^{2\pi}\widetilde{M}(t,\tau')\varphi(\tau')\,d\tau',$$

where $M(t,\tau)$ *is a* 2π *-periodic and Holder continuous function with the exponent* $\alpha\,(0<\alpha\leq 1)$. *Then for any* $0<\alpha$ *the following inequalities hold:*

$$\|M_nS_n-(MS)_n\|_\beta\leq\frac{c_{26}+c_{27}\ln n}{n^{\alpha-\beta}}\cdot H(M;\alpha), \tag{15}$$

$$\|S_nM_n-(MS)_n\|_\beta\leq\frac{c_{28}+c_{29}\ln n}{n^{\alpha-\beta}}\cdot H(M;\alpha). \tag{16}$$

Proof. For any $\varphi\in H_\beta$ we have

$$[M_nS_n-(MS)_n]\varphi(t)=$$

$$=\frac{1}{2n}\sum_{m=0}^{2n-1}M\left(t,t+\frac{\pi m}{n}\right)\frac{1}{n}\sum_{p=0}^{n-1}ctg\left(-\frac{\pi(2p+1)}{2n}\right)\varphi\left(t+\frac{\pi(m+2p+1)}{n}\right)-$$

$$-\frac{1}{2n}\sum_{k=0}^{2n-1}\widetilde{M}\left(t,t+\frac{\pi k}{n}\right)\varphi\left(t+\frac{\pi k}{n}\right)=$$

$$=\frac{1}{2n}\sum_{k=0}^{2n-1}\left[\frac{1}{2n}\sum_{p=0}^{n-1}ctg\left(-\frac{\pi(2p+1)}{2n}\right)M\left(t,t+\frac{\pi(k-2p-1)}{n}\right)-\right.$$

$$\left.-\frac{1}{2\pi}\int\limits_0^{2\pi}ctg\frac{\tau-t}{2}M\left(t,\tau+\frac{\pi k}{n}\right)d\tau\right]\cdot\varphi\left(t+\frac{\pi k}{n}\right). \tag{17}$$

Applying lemma 6 to the function $M\left(t,\tau+\frac{\pi k}{n}\right)$, $k=\overline{0,2n-1}$, from equality (17) we get inequality (15). Inequality (16) is proved in the same way. Lemma 7 is proved.

Theorem 3. *Let the functions* $a(t)$, $b(t)$ *be continuously-differentiable, the derivatives* $a'(t)$, $b'(t)$ *belong to the class* H_α, $a^2(t)+b^2(t)\neq 0$ *for all* $t\in[0;2\pi]$, *and let the operator* R *be invertible in the space* $H_\beta\,(0<\beta<\alpha)$. *Then for large values of* n *the operators*

$$(R_n\varphi)(t)=a(t)\varphi(t)+b(t)(S_n\varphi)(t)+(K_n\varphi)(t)$$

are also invertible in the space H_β *and for any* $f \in \beta$ *it holds the estimation:*

$$\left\|R_n^{-1} f - R^{-1} f\right\|_{\beta'} \le \frac{c_{30} + c_{31}\ln n}{n^{\beta-\beta'}} \cdot \|f\|_\beta \quad \left(0 < \beta' \le \beta\right). \tag{18}$$

Proof. Since for any $t \in [0; 2\pi]$ the inequality $a^2(t) + b^2(t) \neq 0$ is fulfilled, then $r_0 = \min\limits_{t\in[0;2\pi]} |b(t) + ia(t)| > 0$. Denote $\delta_0 = \left(\frac{r_0}{2\|a\|_\infty + 1}\right) \cdot i$. Then for any $t \in [0; 2\pi]$

$$|b(t) + \delta_0 a(t)| \ge |b(t) + ia(t)| - |(\delta_0 - i)\, a(t)| \ge r_0 - \frac{r_0}{2} = \frac{r_0}{2} > 0,$$

and consequently,

$$R(I + \delta_0 S) = [b + \delta_0 a]\left\{a_1 I + S + a_2 J + \widetilde{K} + \delta_0 \widetilde{K} S\right\},$$

$$R_n(I + \delta_0 S) = [b + \delta_0 a]\left\{a_1 I + S_n + a_2 J_n^{(even)} + \widetilde{K}_n + \delta_0 \widetilde{K}_n S_n\right\},$$

where $(J\varphi)(t) = \frac{1}{2\pi}\int\limits_0^{2\pi} \varphi(\tau)\, d\tau$, $a_1(t) = \frac{a(t) - \delta_0 b(t)}{a(t) + \delta_0 b(t)}$, $a_2(t) = \frac{\delta_0 b(t)}{b(t) + \delta_0 a(t)}$,

$$\left(\widetilde{K}\varphi\right)(t) = \frac{1}{2\pi}\int\limits_0^{2\pi} \widetilde{K}(t,\tau)\,\varphi(\tau)\, d\tau, \quad \widetilde{K}(t,\tau) = \frac{K(t,\tau)}{a(t) + \delta_0 b(t)}.$$

Hence it follows that

$$(a_1 I - S)\left[\frac{1}{b + \delta_0 a} R(I + \delta_0 S)\right] = a_3 I + A + (a_1 a_2 - 1)\, J - S a_2 J + a_1 \widetilde{K} +$$

$$+ \delta a_1 \widetilde{K} S - S\widetilde{K} - \delta_0 S\widetilde{K} S, \tag{19}$$

$$(a_1 I - S_n)\left[\frac{1}{b + \delta_0 a} R_n(I + \delta_0 S_n)\right] =$$

$$= a_3 I + A_n^{(odd)} + (a_1 a_2 - 1)\, J_n^{(even)} - S_n a_2 J_n^{(even)} +$$

$$+ a_1 \widetilde{K}_n + \delta_0 a_1 \widetilde{K}_n S_n - S_n \widetilde{K}_n - \delta_0 S_n \widetilde{K}_n S_n, \tag{20}$$

where $a_3(t) = a_1^2(t) + 1 \neq 0$ for all $t \in [0; 2\pi]$,

$$A(\varphi)(t) = \frac{1}{2\pi}\int\limits_0^{2\pi} ctg\frac{t - \tau}{2}\left[a_1(t) - a_1(\tau)\right]\varphi(\tau)\, d\tau.$$

From equalities (19) and (20), according to lemma 5 it follows that for large values of n the operators standing at the right sides of these equalities are invertible, and therefore the operators R_n are also invertible for large values of n. By attracting lemmas 3 and 7, estimation (18) is proved similar to estimation (10). Theorem 3 is proved.

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