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AN OPTIMAL SPEED PROBLEM FOR A BAR OSCILLATION EQUATION WITH DISTRIBUTED CONTROL

Abstract

In the paper, an optimal speed problem for a bar oscillation equation with distributed control is considered. For the optimal control the expression in the form of a series is obtained, and for determining the speed time an effective algorithm is suggested.

There exist oscillating and wave processes whose mathematical description is reduced to a mixed problem for a fourth order hyperbolic equation. A number of problems on oscillations of bars, tuning fork, stability analysis of rotating shafts, study of vessel vibrations reduce to such equations. Therefore, it is natural to state different optimal control problems for such equations [1,2,3].

Problem statement. Consider a speed optimal-control problem of a process described by the bar oscillations equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}=\nu\left(x,t\right), \quad (x,t)\in Q=\{0<x<l;\ \ 0<t<T\}, \tag{1}$$

with boundary

$$u\left(0,t\right)=u\left(l,t\right)=0, \quad \frac{\partial^2 u\left(o,t\right)}{\partial x^2}=\frac{\partial^2 u\left(l,t\right)}{\partial x^2}=0, \quad 0\le t\le T, \tag{2}$$

and initial conditions

$$u\left(x,0\right)=u_0\left(x\right), \quad \frac{\partial u\left(x,0\right)}{\partial t}=u_1\left(x\right), \quad 0\le x\le l, \tag{3}$$

where $u\left(x,t\right)$ is the process state, $\nu\left(x,t\right)$ is a control function from $L_2\left(Q\right)$. In place of admissible controls we take the ball $\|\nu\|_{L_2(Q)}\le R$, where $R>0$ is the given number. Suppose that $u_0\in W_{2,0}^2\left(0,l\right)$, $u_1\in L_2\left(0,l\right)$. Under these conditions, for each fixed admissible control ν problem (1)-(3) has a unique generalized solution $u\left(x,t\right)$ from $C\left(0,T;W_{2,0}^2\left(0,l\right),L_2\left(0,l\right)\right)$ [4]. Hence it follows that $u\in C\left(0,T;W_{2,0}^2\left(0,l\right)\right)$, $\frac{\partial u}{\partial t}\in C\left(0,T;L_2\left(0,l\right)\right)$.

The following optimal control problem is stated: to find the admissible control $\nu\left(x,t\right)$ from the ball $\|\nu\|_{L_2(Q)}\le R$ that takes system (1), (2) from the given initial state (3) is to the final zero state for the least time T [5,6], i.e. for minimal time the following condition is fulfilled:

$$u\left(x,t\right)|_{t=T}=0, \quad \left.\frac{\partial u\left(x,t\right)}{\partial t}\right|_{t=T}=0. \tag{4}$$

Problem solution. By the Fourier method we can represent the solution of problem (1)-(3) for the given admissible control $\nu(x,t)$ in the form

$$u(x,t)=\sum_{k=1}^{\infty}\left(u_{0k}\cos\left(\frac{\kappa\pi}{l}\right)^{2}t+\left(\frac{l}{k\pi}\right)^{2}u_{1k}\sin\left(\frac{k\pi}{l}\right)^{2}t\right)X_k(x)+$$

$$+\sum_{k=1}^{\infty}\left(\frac{l}{k\pi}\right)^{2}\int_0^t\left(\int_0^l\nu(\xi,\tau)X_k(\xi)\,d\xi\right)\sin\left(\frac{k\pi}{l}\right)^{2}(t-\tau)\,d\tau X_k(x), \qquad (5)$$

where $\lambda_k=\left(\frac{k\pi}{l}\right)^4$, $X_k(x)=\sin\frac{k\pi}{l}x$ are eigen values and orthogonal eigen functions of the spectral problem

$$X^{IV}(x)-\lambda X(x)=0,\quad X(0)=X(l)=X''(0)=X''(l)=0,$$

u_{0k} and u_{1k} are Fourier coefficients of the functions $u_0(x)$ and $u_1(x)$, respectively in the system of functions $X_k(x)$.

Introduce the denotation

$$a_{1k}=-\left(u_{0k}\cos\left(\frac{k\pi}{l}\right)^{2}T+\left(\frac{l}{k\pi}\right)^{2}u_{1k}\sin\left(\frac{k\pi}{l}\right)^{2}T\right),$$

$$a_{2k}=\left(\frac{k\pi}{l}\right)^{2}u_{0k}\sin\left(\frac{k\pi}{l}\right)^{2}T-u_{1k}\cos\left(\frac{k\pi}{l}\right)^{2}T,\quad k=1,2,...$$

From conditions (4), according to (5), we have:

$$a_i(x)=\int_0^T\int_0^l\nu(\xi,\tau)K_i(\xi,\tau,x,T)\,d\xi d\tau \qquad (i=1,2),$$

where

$$a_i(x)=\sum_{k=1}^{\infty}a_{ik}\sin\frac{k\pi}{l}x \quad (i=1,2), \qquad (6)$$

$$K_1(\xi,\tau,x,T)=\sum_{k=1}^{\infty}\left(\frac{l}{k\pi}\right)^{2}\sin\left(\frac{k\pi}{l}\right)^{2}(T-\tau)\sin\frac{k\pi}{l}x\sin\frac{k\pi}{l}\xi, \qquad (7)$$

$$K_2(\xi,\tau,x,T)=\sum_{k=1}^{\infty}\cos\left(\frac{k\pi}{l}\right)^{2}(T-\tau)\sin\frac{k\pi}{l}x\sin\frac{k\pi}{l}\xi. \qquad (8)$$

For solving the stated optimal speed problem we use the results obtained in [6] (chapter IX, § 2).

It is known [6] that an optimal control problem is represented in the form

$$\nu(\xi,\tau)=\|\nu\|^2\int_0^l\sum_{i=1}^{2}\varphi_i(x)K_i(\xi,\tau,xT)\,dx, \qquad (9)$$

and the least positive root T of the equation

$$\int_0^l\int_0^T\left[\int_0^l\sum_{i=1}^{2}\varphi_i(x)\,K_i(\xi,\tau,x,T)\,dx\right]^2 d\xi d\tau=\frac{1}{R^2} \tag{10}$$

gives the speed time, and the unknown functions $\varphi_i(x)\quad(i=1,2)$are determined from the system of equations

$$\|\nu\|^2\int_0^l\sum_{j=1}^{2}\varphi_j(x)\int_0^l\int_0^T K_j(\xi,\tau,x,T)\,K_p(\xi,\tau,y,T)\,d\xi d\tau dx=\alpha_p(y)\ \ (p=1,2)\,. \tag{11}$$

Introduce the function

$$\psi_j(x)=\|\nu\|^2\,\varphi_j(x)\quad(j=1,2) \tag{12}$$

and substitute (12) into expression (9)-(11). We look for the function $\psi_j(x)\quad(j=1,2)$ in the form of expansion of the Fourier series

$$\psi_j(x)=\sum_{i=1}^{\infty}\psi_{ji}\sin\frac{i\pi}{l}x\qquad(j=1,2)\,, \tag{13}$$

where

$$\psi_{ji}=\frac{2}{l}\int_0^l\psi_j(x)\sin\frac{i\pi}{l}xdx,\quad i=1,2,...;\ \ j=1,2.$$

Using formula (6)-(8) and (12), we represent system (11) in the form

$$\int_0^l\psi_1(x)\int_0^l\int_0^T\left[\sum_{i=1}^{\infty}\left(\frac{l}{i\pi}\right)^2\sin\left(\frac{i\pi}{l}\right)^2(T-\tau)\sin\frac{i\pi}{l}\xi\sin\frac{i\pi}{l}x\times\right.$$

$$\left.\times\sum_{k=1}^{\infty}\left(\frac{l}{k\pi}\right)^2\sin\left(\frac{k\pi}{l}\right)^2(T-\tau)\sin\frac{k\pi}{l}\xi\sin\frac{k\pi}{l}y\right]d\xi d\tau dx+$$

$$+\int_0^l\psi_2(x)\int_0^l\int_0^T\left[\sum_{i=1}^{\infty}\cos\left(\frac{i\pi}{l}\right)^2(T-\tau)\sin\frac{i\pi}{l}\xi\sin\frac{i\pi}{l}x\times\right.$$

$$\left.\times\sum_{k=1}^{\infty}\left(\frac{l}{k\pi}\right)^2\sin\left(\frac{k\pi}{l}\right)^2(T-\tau)\sin\frac{k\pi}{l}\xi\sin\frac{k\pi}{l}y\right]d\xi d\tau dx=$$

$$=\sum_{k=1}^{\infty}\alpha_{1k}\sin\frac{k\pi}{l}y,$$

$$\int_0^l\psi_1(x)\int_0^l\int_0^T\left[\sum_{i=1}^{\infty}\left(\frac{l}{i\pi}\right)^2\sin\left(\frac{i\pi}{l}\right)^2(T-\tau)\sin\frac{i\pi}{l}\xi\sin\frac{i\pi}{l}x\times\right.$$

$$\left.\times\sum_{k=1}^{\infty}\cos\left(\frac{k\pi}{l}\right)^{2}(T-\tau)\sin\frac{k\pi}{l}\xi\sin\frac{k\pi}{l}y\right]d\xi d\tau dx+$$

$$+\int_{0}^{l}\psi_{2}(x)\int_{0}^{l}\int_{0}^{T}\left[\sum_{i=1}^{\infty}\cos\left(\frac{i\pi}{l}\right)^{2}(T-\tau)\sin\frac{i\pi}{l}\xi\sin\frac{i\pi}{l}x\times\right.$$

$$\left.\times\sum_{k=1}^{\infty}\cos\left(\frac{k\pi}{l}\right)^{2}(T-\tau)\sin\frac{k\pi}{l}\xi\sin\frac{k\pi}{l}y\right]d\xi d\tau dx=\sum_{k=1}^{\infty}\alpha_{2k}\sin\frac{k\pi}{l}y.$$

Using the orthogonality of $X_k(x)$ in $L_2(0,l)$, we get the following system for finding the coefficients ψ_{1i} and ψ_{2i} $(i=1,2,...)$

$$\psi_{1i}\left(\frac{T}{2}-\frac{1}{4\sqrt{\lambda_i}}\sin 2\sqrt{\lambda_i}T\right)+\psi_{2i}\left(\frac{1}{4}-\frac{1}{4}\cos 2\sqrt{\lambda_i}T\right)=\frac{4}{l^2}\lambda_i\alpha_{1i}, \tag{14}$$

$$\psi_{1i}\left(\frac{1}{4}-\frac{1}{4}\cos 2\sqrt{\lambda_i}T\right)+\psi_{2i}\left(\frac{T\lambda_i}{2}+\frac{\sqrt{\lambda_i}}{4}\sin 2\sqrt{\lambda_i}T\right)=\frac{4}{l^2}\lambda_i\alpha_{2i}\quad i=1,2,\dots .$$

For each i the chief determinant of this system

$$\Delta_i=\begin{vmatrix}\frac{T}{2}-\frac{1}{4\sqrt{\lambda_i}}\sin 2\sqrt{\lambda_i}T & \frac{1}{4}-\frac{1}{4}\cos 2\sqrt{\lambda_i}T\\ \frac{1}{4}-\frac{1}{4}\cos 2\sqrt{\lambda_i}T & \frac{T}{2}\lambda_i+\frac{\sqrt{\lambda_i}}{4}\sin 2\sqrt{\lambda_i}T\end{vmatrix}=$$

$$=\frac{T^2}{4}\lambda_i+\frac{T\sqrt{\lambda_i}}{8}\sin 2\sqrt{\lambda_i}T-\frac{T\sqrt{\lambda_i}}{8}\sin 2\sqrt{\lambda_i}T-$$

$$-\frac{1}{16}\sin^2 2\sqrt{\lambda_i}T-\frac{1}{16}+\frac{1}{8}\cos 2\sqrt{\lambda_i}T-$$

$$-\frac{1}{16}\cos^2 2\sqrt{\lambda_i}T=\frac{T^2}{4}\lambda_i-\frac{1}{8}+\frac{1}{8}\cos 2\sqrt{\lambda_i}T,$$

and auxiliary determinants

$$\Delta_{1i}=\begin{vmatrix}\frac{4}{l^2}\lambda_i\alpha_{1i} & \frac{1}{4}-\frac{1}{4}\cos 2\sqrt{\lambda_i}T\\ \frac{4}{l^2}\lambda_i\alpha_{2i} & \frac{T}{2}\lambda_i+\frac{\sqrt{\lambda_i}}{4}\sin 2\sqrt{\lambda_i}T\end{vmatrix}=$$

$$=\frac{2T\alpha_{1i}\lambda_i^2}{l^2}+\frac{\lambda_i\sqrt{\lambda_i}\alpha_{1i}\sin 2\sqrt{\lambda_i}T}{l^2}-\frac{\lambda_i\alpha_{2i}}{l^2}+\frac{\lambda_i\alpha_{2i}\cos 2\sqrt{\lambda_i}T}{l^2},$$

$$\Delta_{2i}=\begin{vmatrix}\frac{T}{2}-\frac{1}{4\sqrt{\lambda_i}}\sin 2\sqrt{\lambda_i}T & \frac{4}{l^2}\lambda_i\alpha_{1i}\\ \frac{1}{4}-\frac{1}{4}\cos 2\sqrt{\lambda_i}T & \frac{4}{l^2}\lambda_i\alpha_{2i}\end{vmatrix}=$$

$$=\frac{2T\lambda_i\alpha_{2i}}{l^2}-\frac{\sqrt{\lambda_i}\alpha_{2i}\sin 2\sqrt{\lambda_i}T}{l^2}-\frac{\lambda_i\alpha_{1i}}{l^2}+\frac{\lambda_i\alpha_{1i}\cos 2\sqrt{\lambda_i}T}{l^2}.$$

Then from the system of equations (14) it follows that

$$\psi_{1i}^{0}=\frac{\Delta_{1i}}{\Delta_i}=\frac{2T\alpha_{1i}\lambda_i^2+\lambda_i\sqrt{\lambda_i}\alpha_{1i}\sin 2\sqrt{\lambda_i}T-\lambda_i\alpha_{2i}-\lambda_i\alpha_{2i}\cos 2\sqrt{\lambda_i}T}{l^2\left(\frac{T^2}{4}\lambda_i-\frac{1}{8}+\frac{1}{8}\cos 2\sqrt{\lambda_i}T\right)},$$

$$\psi_{2i}^{0}=\frac{\Delta_{2i}}{\Delta_{i}}=\frac{2T\lambda_{i}\alpha_{2i}-\sqrt{\lambda_{i}}\alpha_{2i}\sin 2\sqrt{\lambda_{i}}T-\lambda_{i}\alpha_{1i}+\lambda_{i}\alpha_{1i}\cos 2\sqrt{\lambda_{i}}T}{l^{2}\left(\frac{T^{2}}{4}\lambda_{i}-\frac{1}{8}+\frac{1}{8}\cos 2\sqrt{\lambda_{i}}T\right)}, \quad i=1,2,...$$

or

$$\psi_{1i}^{0}=\frac{16T\alpha_{1i}\lambda_{i}^{2}+8\lambda_{i}\sqrt{\lambda_{i}}\alpha_{1i}\sin 2\sqrt{\lambda_{i}}T-8\lambda_{i}\alpha_{2i}\left(1-\cos 2\sqrt{\lambda_{i}}T\right)}{l^{2}\left(2T^{2}\lambda_{i}-\left(1-\cos 2\sqrt{\lambda_{i}}T\right)\right)}, \tag{15}$$

$$\psi_{2i}^{0}=\frac{16T\lambda_{i}\alpha_{2i}-8\sqrt{\lambda_{i}}\alpha_{2i}\sin 2\sqrt{\lambda_{i}}T-8\lambda_{i}\alpha_{1i}\left(1-\cos 2\sqrt{\lambda_{i}}T\right)}{l^{2}\left(2T^{2}\lambda_{i}-\left(1-\cos 2\sqrt{\lambda_{i}}T\right)\right)}, \quad i=1,2,...$$

Substituting the found values of ψ_{1i}^{0}, ψ_{2i}^{0} in (13) and take account (9) and (12), we get the optimal control

$$\nu^{0}(\xi,\tau)=\sum_{i=1}^{\infty}\sum_{j=1}^{2}\int_{0}^{l}\psi_{ji}^{0}\sin\frac{i\pi}{l}xK_{j}(\xi,\tau,xT)\,dx.$$

From formula (10) it follows that the least positive root T of the equation

$$\frac{l}{2}\sum_{i=1}^{\infty}\left[\left(\psi_{1i}^{0}\right)^{2}\frac{1}{\lambda li}\left(\frac{T}{2}-\frac{1}{4\sqrt{\lambda_{i}}}\sin 2\sqrt{\lambda_{i}}T\right)+2\psi_{1i}^{0}\psi_{2i}^{0}\frac{1}{\sqrt{\lambda_{i}}}\left(\frac{1}{4}-\frac{1}{4}\cos 2\sqrt{\lambda_{i}}T\right)+\right.$$

$$\left.+\left(\psi_{2i}^{0}\right)^{2}\left(\frac{T}{2}+\frac{1}{4\sqrt{\lambda_{i}}}\sin 2\sqrt{\lambda_{i}}T\right)\right]=R^{2} \tag{16}$$

gives the optimal speed time. Thus we proved

Theorem. *Let the following conditions be fulfilled:* $u_{0}\in W_{2,0}^{2}(0,l)$, $u_{1}\in L_{2}(0,l)$. *Then the solution of the optimal speed problem in the ball* $\|\nu\|_{L_{2}(Q)}\leq R$ *for mixed problem (1)-(3) is represented in the form*

$$\nu^{0}(x,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{2}\int_{0}^{l}\psi_{ji}^{0}\sin\frac{i\pi}{l}\xi K_{j}(x,t,\xi,T)\,d\xi,$$

where ψ_{ji}^{0} *is determined by formula (15), and the least positive root of equation (16) gives the optimal speed time.*

For finding the least positive root T of equation (16) we can suggest the following algorithm [6].

Write the system

$$\int_{0}^{l}\sum_{i=1}^{2}\psi_{i}(x)K_{ij}(x,y,T)\,dx=\alpha_{j}(y), \quad j=1,2, \tag{17}$$

$$\nu(\xi,\tau)=\int_{0}^{l}\sum_{i=1}^{2}\psi_{i}(x)K_{i}(\xi,\tau,x,T)\,dx,$$

$$\|\nu\|=\left(\int_{0}^{l}\int_{0}^{T}\nu^{2}(x,t)\,dxdt\right)^{1/2}=\frac{1}{F(\varphi)}\leq R,$$

where

$$K_{ij}(x,y,T)=\int_0^l\int_0^T K_i(\xi,\tau,x,T)\,K_j(\xi,\tau,y,T)\,d\xi d\tau,$$

$$F(\varphi)=\left\{\int_0^l\int_0^T\left[\int_0^l\sum_{i=1}^2\varphi_i(x)\,K_i(\xi,t,x,T)\,dx\right]^2 d\xi dt\right\}^{\frac{1}{2}}.$$

We give the first approximation $T=T_1$.

By solving system (17), we find $\psi_i(x)\quad(i=1,2)$.

We calculate $\|\nu\|$, verify the inequality $\|\nu\|\le R$. If $\|\nu\|<R$, then in the next approximation we take $T_2=T_1-\Delta T_1$, where $\Delta T_1>0$. For $\|\nu\|>R$ we assume $T_2=T_1+\Delta T_1$, $\Delta T_1>0$. The process is repeated until we find the least time T_0 at which the condition $F(\varphi)=\frac{1}{R}$ is fulfilled, and for $T=T_0-\Delta T_1$, where ΔT_1 is an infinite small positive value, this condition violates, i.e. $\|\nu\|>R$.

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