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ON SPECTRAL THEORY OF QUADRATIC OPERATOR PENCILS

Abstract

We obtain sufficient conditions on the coefficients of quadratic operator pencils, the main part of which contains a normal operator with its eigen and adjoint vectors doubly complete in the sense of M.V. Keldysh in Hilbert space. Note that the main part of the investigated pencil contains a normal operator.

In the paper we investigate a problem on double completeness of the system of eigen and adjoint vectors of the quadratic pencil

$$L(\lambda) = E - (K_0 + B_0) - \lambda(K_1 + B_1)C - \lambda^2 C^2 \tag{1}$$

in separable Hilbert space H.

Here E is a unit operator in H, λ is a spectral parameter, the operator coefficients of quadratic pencil (1) satisfy the following conditions:

1) C is a normal completely continuous operator whose spectrum is contained in the sector $S_\lambda = \{\lambda : |\arg\lambda| \leq \varepsilon\}\,,\ 0 \leq \varepsilon \leq \frac{\pi}{2}$;

2) $K_j,\ j = 0,1$ are completely continuous operators in H, i.e. $K_j \in \sigma_\infty, l = 0,1$;

3) $B_j,\ j = 0,1$ are bounded operators in H, i.e. $B_j \in L(H), j = 0,1$.

Notice that origin of spectral theory of operatory pencils was put in fundamental works of M.V. Keldysh [1,2]. Further, the results of M.V. Keldysh were developed in [3-7].

In particular, in [3-5,7], the quadratic operator pencils were studied at different situations.

Definition 1. *The number λ_0 is called a characteristical number of the operator pencil $L(\lambda)$ if these exists a non-zero vector $\varphi_0 \in L$ such that $L(\lambda_0)\varphi_0 = 0$, and φ_0 is called an eigen vector of the pencil responding to the characteristical number λ_0. If the vectors $\varphi_0, \varphi_l, ..., \varphi_m$ satisfy the equations*

$$\sum_{j=0}^{k} \frac{L^{(j)}(\lambda_0)}{j!}\varphi_{j-k} = 0, \quad j = 0, ..., m,$$

they are called a chain of eigen and adjoint vectors of the eigen vector φ_0.

If $\{\varphi_0, \varphi_l, ..., \varphi_m\}$ is the chain of eigen and adjoint vectors of pencil (1) responding to the characteristical number λ_0, then assuming $\widetilde{\varphi}_h \in H^2 = H \oplus H$, where $h = \overline{0,m}$, $\widetilde{\varphi}_h = (\varphi_h^0, \varphi_h^{(l)})$, $\varphi_h^0 = \varphi_h$, $\varphi_h^{(l)} = \lambda_0\varphi_n + \varphi_{n-1}$.

Definition 2. *If the system $\{\widetilde{\varphi}_n\}$ constructed for all characteristical numbers and all possible chains of eigen and adjoint vectors is complete in the space H^2, then we say that the system of eigen and adjoint vectors is doubly complete in H.*

Notice that from M.V. Keldysh's paper [2] it follows that for the system of eigen and adjoint vectors to be double complete in H, it is necessary and sufficient that from the holomorphic property of the vector-function $\left(L^*(\overline{\lambda})\right)^{-1}(f_0+\lambda f_1)$ on all the plane for any collection of two vectors $f_0, f_1 \in H$ there should follow $f_0 = f_1 = 0$.

Let C be a normal completely continuous operator, $\lambda_1, \lambda_2, ..., \lambda_n$ be its eigen values from the sector S_ε, $e_1, e_2, ..., e_n$ be appropriate orthonormal eigen vectors, and $\lambda_n = \mu_n e^{i\varphi_n}$, $|\arg \varphi_n| \le \varepsilon$, $\mu_1 \ge \mu_2 \ge \mu_3 \ge ... \ge \mu_n \ge ...$,. Then the operator C is representable in the form:

$$C = \sum_{n=1}^{\infty} \mu_n e^{ike_n} (\cdot, e_n) e_n.$$

If $K \in \sigma_\infty$, and s_n are the eigen values of the operator $(K^*K)^{1/2}$, and $\sum\limits_{n=1}^{\infty} s_n^\rho < < \infty$, we say that $K \in \sigma_p \;\; (0 < p < \infty)$.

Formulate the basic result of the paper.

Theorem. *Let conditions 1)-3) and one of the following conditions be fulfilled:*

a) $C \in \sigma_p \;\; (0 < \rho < \frac{\pi}{2\varepsilon})$, *and it holds the inequality*

$$\delta(\varepsilon, \rho) = \sum_{j=0}^{1} d_j(\varepsilon, \rho) \, \|B_j\| < 1, \tag{2}$$

where

$$d_0(\varepsilon,\rho) = \begin{cases} 1, & 0 < \rho \le \dfrac{2\pi}{\pi + 2\varepsilon}, \\ \dfrac{1}{\sqrt{2}\sin\left(\frac{\pi}{2\rho} - \varepsilon\right)}, & \dfrac{2\pi}{\pi+2\varepsilon} \le \rho < \dfrac{\pi}{2\varepsilon}. \end{cases}$$

$$d_j(\varepsilon,\rho) = \begin{cases} \dfrac{1}{2\cos}, & 0 < \rho \le 1, \\ \dfrac{1}{2\sin\left(\frac{\pi}{2\rho} - \varepsilon\right)}, & 1 \le \rho < \dfrac{\pi}{2\varepsilon}. \end{cases}$$

b) $A \in \sigma_\rho, \;\; B_j \in \sigma_\rho (j = 0, 1)$.

Then the system of eigen and adjoint vectors of pencil (1) is doubly complete in H the sense of M.V. Keldysh.

In order to prove the theorem we denote $L_0(\lambda) = E - \lambda^2 C^2$, $K(\lambda) = = K_0 + \lambda K_1 C$, $B(\lambda) = B_0 + \lambda\, BC_1$. Let condition a) be fulfilled, and $0 \le \le \varepsilon < \frac{\pi}{2}$ and $\frac{1}{2} \le \rho < \frac{\pi}{2\varepsilon}$. In this case $\frac{\pi}{2\rho} > \varepsilon$. Show that in this case in the sectors $\Lambda_{\pm\rho} = \left\{\lambda : \left|\arg\lambda \pm \frac{\pi}{2}\right| \le \rho\right\}$ the operator pencil $L(\lambda)$ is invertible at rather large $|\lambda|$. Since the spectrum of the operator C is in the sector S_ε, then for any $\theta \in \left[0, \frac{\pi}{2} - \varepsilon\right]$ the operator pencil $L_0(\lambda)$ is invertible in the sectors $\Lambda_{\pm\rho} = \left\{\lambda : \left|\arg\lambda \pm \frac{\pi}{2}\right| \le \theta\right\}$.

Therefore from the condition $\frac{\pi}{2\rho} > \varepsilon$ it follows that the operator pencil $L_0(\lambda)$ is invertible in the sectors $\Lambda_{\pm\rho}$. Since in these sectors

$$L(\lambda) = (E - K(\lambda)L_0^{-1}(\lambda) - B(\lambda)L_0^{-1}(\lambda)L_0(\lambda), \tag{3}$$

then we must prove the invertibility of the pencil

$$L_1(\lambda) = ((E - K(\lambda)) - B(\lambda))L_0^{-1}(\lambda) \tag{4}$$

in the sectors $\Lambda_{\pm\ \frac{\pi}{1}-\frac{\pi}{2\rho}}$. Since $K_0, K_1 \in \sigma_\infty$, then by the M.V. Keldysh lemma [2]

$$\left\|K(\lambda)L_0^{-1}(\lambda)\right\| \le \left\|K_0(E-\lambda^2C^2)^{-1}\right\| + \left\|K_1\lambda C(E-\lambda^2C^2)^{-1}\right\| \to 0$$

as $|\lambda| \to 0$, $\lambda \in \Lambda_{\pm\ \frac{\pi}{1}-\frac{\pi}{2\rho}}$. Now by choosing the number $R > 0$ rather large, for $|\lambda| > R$ we provide that the inequality

$$\left\|K(\lambda)L_0^{-1}(\lambda)\right\| \le \frac{1-\delta(\varepsilon,\rho)}{2} \tag{5}$$

for $\lambda \in \Lambda_{\pm\ \frac{\pi}{1}-\frac{\pi}{2\rho}}$, where $\delta(\varepsilon,\rho)$ is determined from equality (2), is fulfilled. Further show that in these sectors

$$\left\|B(\lambda)L_0^{-1}(\lambda)\right\| \le \|B_0\| \left\|L_0^{-1}(\lambda)\right\| + \|B_1\| \left\|\lambda C L_0^{-1}(\lambda)\right\| \to 0 \tag{6}$$

Estimate each addend of (6) separately. Since for $\lambda \in \Lambda_{\pm\ \frac{\pi}{1}-\frac{\pi}{2\rho}}$ $\lambda = \mu e^{i\psi}$, $\mu > 0$, $\left|\frac{\pi}{2} \pm \psi\right| < \frac{\pi}{2} - \frac{\pi}{2\rho}$. Then from the spectral expansion of the operator A it follows that

$$\left\|L_0^{-1}(\lambda)\right\| \le \sup_n \left|\left(1-\lambda^2\lambda_n^2\right)^{-1}\right| = \sup_n \left(1+\mu^4\mu_n^4 - 2\mu\mu_n\cos 2(\psi+\alpha_n)\right)^{1/2} \le$$

$$\le \sup_n \left(1+\mu^4\mu_n^4 - 2\mu^2\mu_n^2\cos 2\left(\frac{\pi}{2\rho}-\varepsilon\right)\right)^{1/2} \tag{7}$$

Obviously, for $\frac{1}{2} \le \rho < \frac{2\pi}{2\pi+\varepsilon}$ the number $\cos 2\left(\frac{\pi}{2\rho}-\varepsilon\right) \ge 0$, therefore from inequality (7) we get

$$\left\|L_0^{-1}(\lambda)\right\| \le \left(1+\mu^4\mu_n^4\right)^{-\frac{1}{2}} \le d_0(\varepsilon,\rho) \tag{8}$$

In the case $\frac{2\pi}{2\pi+\varepsilon} \le \rho < \frac{\pi}{2\rho}$ $\quad\cos 2\left(\frac{\pi}{2\rho}-\varepsilon\right) \le 0$, therefore from inequality (7), applying the Cauchy inequality, we get:

$$\left\|L_0^{-1}(\lambda)\right\| \le \sup_n \left(1+\mu^4\mu_n^4 - (1+\mu^4\mu_n^4)\cos 2\left(\frac{\pi}{2\rho}-\varepsilon\right)\right)^{-1/2} =$$

$$= \left(1+\mu^4\mu_n^4\right)^{-\frac{1}{2}}\left(1-\cos 2\left(\frac{\pi}{2\rho}-\varepsilon\right)\right) \le \frac{1}{\sqrt{2}\sin\left(\frac{\pi}{2\rho}-\varepsilon\right)}. \tag{9}$$

Thus,

$$\|B_0\| \left\|L_0^{-1}(\lambda)\right\| \le \|B_0\|\, d_0(\varepsilon,\rho), \quad \left(\frac{1}{2} \le \rho < \frac{\pi}{2\varepsilon}\right).$$

Now, for $\frac{1}{2} \le \rho < \frac{\pi}{2\varepsilon}$ we estimate the second addend in (6).

Obviously, in this case again we apply the Cauchy inequality and get

$$\left\|\lambda C(E-\lambda^2C^2)^{-1}\right\| \le \sup_n \left\|\lambda\lambda_n(1-\lambda^2\lambda_n)^{-1}\right\| \le$$

$$\le \sup_n(\mu^2\mu_n^2\left(1+\mu^4\mu_n^4-2\mu^2\mu_n^2\cos 2\left(\frac{\pi}{2\rho}-\varepsilon\right)\right)^{-1/2} \le$$

$$\le \sup_n(\mu^2\mu_n^2\left(2\mu^2\mu_n^2-2\mu^2\mu_n^2\cos 2\left(\frac{\pi}{2\rho}-\varepsilon\right)\right)^{-1/2} =$$

$$= \frac{1}{\sqrt{2}\sin\left(\frac{\pi}{2\rho}-\varepsilon\right)} = d_1(\varepsilon,\rho)\,. \tag{10}$$

Consequently, taking into account inequalities (8)-(10) in (6), we get

$$\left\|B(\lambda)\,L_0^{-1}(\lambda)\right\| \le \delta(\varepsilon,\rho)\,,$$

where $\delta(\varepsilon,\rho)$ is determined from inequality (2).

Thus, for $\frac{1}{2} \le \rho < \frac{\pi}{2\varepsilon}$, $\lambda \in \Lambda_{\pm\ \frac{\pi}{1}-\frac{\pi}{2\rho}}$ and $|\lambda| \ge R$ we get

$$\|E-L_1(\lambda)\| \le \left\|K(\lambda)L_0^{-1}(\lambda)\right\| + \left\|B(\lambda)\,L_0^{-1}(\lambda)\right\| \le$$

$$\le \frac{1-\delta(\varepsilon,\rho)}{2} + \delta(\varepsilon,\rho) = \frac{1}{2}\left(1+\delta(\varepsilon,\rho)\right) < 1.$$

Consequently, the operator pencil $L(\lambda)$ is invertible for $\lambda \in \Lambda_{\pm\ \frac{\pi}{1}-\frac{\pi}{2\rho}}$, $|\lambda| \ge R$ and for $\frac{1}{2} \le \rho < \frac{\pi}{2\varepsilon}$. Then from the M.V. Keldysh theorem it follows that the operator pencil $L(\lambda)$ has a discrete spectrum with a unique limit point at infinity, since $\|B_0\| < \delta(\varepsilon,\rho) < 1$, therefore $E+B_0$ is invertible and

$$L(\lambda) = (E+B_0)(E+Q(\lambda)),$$

where

$$Q(\lambda) = (E+B_0)^{-1}K_0 + \lambda(E+B_0)^{-1}K_1C + \lambda^2(E+B_0)^{-1}C^2.$$

Since $Q(\lambda) \in \sigma_\infty$ for all λ from the complex plane $Q(0)=0$, therefore, $L(\lambda)$ is invertible everywhere except some points that are eigen values of the operator A. Here we notice that since the operators $(E+B_0)^{-1}K_0 \in \sigma_\infty$, $(E+B_0)^{-1}(K_1+B_1)G \in \sigma_\rho$ and $(E+B_0)^{-1} \in \sigma_{\frac{\rho}{2}}$, from the M.V. Keldysh lemma [2] it follows that $(E+Q(\lambda))^{-1}$ is represented in the form of ratios of two entire functions of finite order ρ and of minimal type for order ρ.

Now prove the theorem. Let $\frac{1}{2} \le \rho < \frac{\pi}{2\varepsilon}$ and there exist the vectors $f_0, f_1 \in H$ such that $\|f_0\| + \|f_1\| \ne 0$. Then the vector-function $R(\lambda) = (L^*(\overline{\lambda}))^{-1}(f_0+\lambda f_1)$ is an entire vector-function, and consequently, $R(\lambda)$ is an entire function of order ρ and of minimal type for order ρ. Since on the rays the angle between of which is at most $\dfrac{\pi}{2\rho}$ it holds the estimation $\left\|L^{-1}(\lambda)\right\| \leq const$, then by the Fragmen-Lindeloff theorem $R(\lambda) = (a_0 + \lambda a_1)$.Therefore $L^*(\overline{\lambda})(a_0 + \lambda a_1) = f_0 + \lambda f_1$.

Since $L^*(\overline{\lambda}) = E-(K_0^*+B_0^*)-\lambda(K_1^*+B_1^*)C-\lambda^2 C$, then comparing the coefficients for λ^3 we get $C^2 a_1 = 0$, i.e. $a_1 = 0$. In same way, the coefficient for λ^2 also equals zero, i.e. $C^2 a_0 = 0$.

Consequently, $a_1 = a_0 = 0$ i.e. $f_0 = f_1 = 0$.

Let $0 < \rho < \dfrac{1}{2}$. Then for $\rho = \dfrac{1}{2}$ it holds the estimation $\left\|L^{-1}(\lambda)\right\| \leq const$ on the imaginary axis for large $|\lambda|$. Therefore for $0 < \rho < \dfrac{1}{2}$ the vector-function $R(\lambda)$ is an entire function of order ρ and of minimal type for order ρ, and on the whole plane $|R(\lambda)| \leq C\,|\lambda|$. Thus, in this case the statement of the theorem is also true. In case b) the Theorem is proved by the M.V. Keldysh method. The theorem is proved.

Corollary. *Let the conditions of the theorem be fulfilled for $\varepsilon = 0$, i.e. A be a positive-definite self-adjoint operator and it hold the inequality*

$$\delta\left(\rho\right) = \sum_{j=0}^{1} d_j(\rho) < 1,$$

where

$$d_0\left(\rho\right) = \begin{cases} 1, & 0 < \rho \leq 2; \\ \dfrac{1}{\sqrt{2}} \sin \pi/2\rho, & 2 \leq \rho < \infty, \end{cases}$$

$$d_1(\rho) = \begin{cases} \dfrac{1}{2}, & 0 < \rho \leq 1; \\ \dfrac{1}{2\sin \pi/2\rho}, & 1 \leq \rho < \infty. \end{cases}$$

Then the system of eigen and adjoint vectors of the pencil $L(\lambda)$ is doubly complete in H.

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