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ON COMPLETENESS OF SOME SUBSYSTEMS OF DERIVATIVE CHAINS OF A SECOND ORDER OPERATOR BUNDLE

Abstract

In the paper we consider on a semi-axis a second order operator-differential equation with an integral boundary condition, and indicate sufficient conditions on operator coefficients of the boundary value problem at which it is regular solvable. Furthermore, we find the completeness conditions of the chain of eigen and adjoint vectors generated by the considered boundary value problem, and establish the completeness of descending elementary solutions of the operator-differential equation under consideration.

Consider in separable Hilbert space H a polynomial operator bundle

$$P(\lambda) = -\lambda^2 E + \lambda A_1 + A_2 + A^2 \tag{1}$$

and the appropriate boundary value problem

$$P(d/dt)u = -\frac{d^2u}{dt^2} + A^2u + A_1\frac{du}{dt} + A_2u(t) = 0, \quad t \in R_+ = (0, \infty) \tag{2}$$

$$Lu = u(0) - \int\limits_0^\infty K(s)u(s)ds - \int\limits_0^\infty K_1(s)u'(s)ds = \varphi, \tag{3}$$

where $u(t)$ is a vector-function with the values in H, $\varphi \in H$, A, A_1, A_2 are linear, generally speaking, unbounded operators in H, the operators $K(s)$ and $K_1(s)$ for any $s \in R$ are linear operators in H, the derivatives are understood in the sense of distributions theory [1].

Let A be a positive-definite self-adjoint operator in H. Denote by $H_\gamma = D(A^\gamma)$, $\gamma \geq 0$ the space of Hilbert scales generated by the operator A, i.e. in space H the scalar derivative is determined as follows $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$.

Denote by $L_2(R_+;H)$ a Hilbert space of vector-functions $f(t)$ with the values in H, for which the norm is finite [1]

$$\|f\|_{L_2(R_+;H)} = \left(\int\limits_0^\infty \|f(t)\|^2 dt\right)^{1/2}.$$

Then the linear space [1]

$$W_2^l(R_+;H) = \left\{u : u^{(l)}, A^{(l)}u \in L_2(R_+;H)\right\}, \quad l = 1,2.$$

is a complete Hilbert space with respect to the norm

$$\|u\|_{W_2^l(R_+;H)} = \left(\left\|u^{(l)}\right\|^2_{L_2(R_+;H)} + \left\|A^{(l)}u\right\|^2_{L_2(R_+;H)}\right)^{1/2}, \quad l = 1,2.$$

In the sequel, we assume that the following conditions are fulfilled:
1) A is a positive-definite self-adjoint operator;
2) The operators $B_j = A_j A^{-j} (j = 1, 2)$ are bounded in H:
3) K and K_1 generated by the integral operators

$$Ku = \int_0^\infty K(s)u(s)ds, \qquad L_1\vartheta = \int_0^\infty K_1(s)\vartheta(s)ds,$$

are bounded operators from $W_2^2(R_+; H)$ to $H_{3/2}$ and from $W_2^1(R_+; H)$ to $H_{3/2}$, respectively

Definition 1. *It the vector-function $u(t) \in W_2^2(R_+; H)$satisfies the equation (1) almost everywhere in R_+, we'll call it a regular solution of the equation (2).*

Definition 2. *If for any $\varphi \in H_{3/2}$ there exists the regular solution $u(t)$ of the equation (2) that satisfies the boundary condition (3) in the sense*

$$\lim_{t\to 0}\left\| u(t) - \int_0^\infty K(s)u(s)ds - \int_0^\infty K_1(s)u'(s)ds - \varphi \right\|_{3/2} = 0,$$

and the following estimation

$$\|u\|_{W_2^2(R_+;H)} \le const\, \|\varphi\|_{3/2}\,,$$

the problem (2),(3) is said to be regularly solvable.

Definition 3. *If the non-zero element φ_0 is the solution of the equation*

$$P(\lambda_0)\varphi_0 = 0,$$

then λ_0 is said to be an eigen value of the operator bundle $P(\lambda)$, and φ_0 an eigen vector responding to the number λ_0. The system of elements $\varphi_1, \varphi_2, ..., \varphi_m$ is called the chain of vectors adjoint to φ_0 if these elements satisfy the equations

$$P(\lambda_0)\varphi_1 + \frac{1}{1!}P'(\lambda_0)\varphi_0 = 0$$

$$P(\lambda_0)\varphi_2 + \frac{1}{1!}P'(\lambda_0)\varphi_1 + \frac{1}{2!}P''(\lambda_0)\varphi_0 = 0$$

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$$P(\lambda_0)\varphi_k + \frac{1}{1!}P'(\lambda_0)\varphi_{k-1} + \frac{1}{2!}P''(\lambda_0)\varphi_{k-2} = 0, \quad k = \overline{1,m},$$

where

$$P'(\lambda_0) = -2\lambda_0 E + A_1,$$

$$P''(\lambda_0) = -2E.$$

Definition 4. *Let $\{\varphi_0, \varphi_1, ..., \varphi_m\}$ be a chain of eigen and adjoint vectors responding to the eigen value $\lambda_0 \in \prod_- = \{\lambda : \operatorname{Re}\lambda < 0\}$.. Then the vector-functions*

$$\varphi_h(t) = e^{\lambda_0 t}\left(\varphi_n + \frac{t}{1!}\varphi_{n-1} + ... + \frac{t^n}{n!}\varphi_0\right) \in W_2^2(R_+; H),$$

satisfy the equation $P(d/dt)u(t) = 0$. *These vector-functions are called the descending elementary solutions of the equation* $P(d/dt)u(t) = 0$, *responding to the eigen value* λ_0.

If $\lambda_i(i = 1, 2, 3, ...0$ are the eigen values of the bundle $P(\lambda)$, and $\left\{\varphi_{i,0}^{(l)}, \varphi_{i,1}^{(l)}, ..., \varphi_{i,m_i l}^{(l)}\right\}$ a chain of eigen and adjoint vectors responding to the eigen value $\lambda_i(\operatorname{Re}\lambda_i < 0)$, then the descending elementary solutions

$$\varphi_{i,h}^{(l)}(t) = e^{\lambda_i t}\left(\varphi_{i,h}^{(l)} + \frac{t}{1!}\varphi_{i,h-1}^{(l)} + ... + \frac{t^n}{n!}\varphi_{i,0}^{(l)}\right), \quad h = \overline{0, m_{i,l}}, \quad l = 1, 2, 3, ...,$$

in zero satisfy the following condition

$$\varphi_{i,h}^{(l)}(0) - \int_0^\infty K(s)\varphi_{i,h}^{(l)}(s)ds - \int_0^\infty K_1(s)\frac{d}{ds}\varphi_{i,h}^{(l)}(s)ds \equiv \psi_{i,h}^{(l)}, \quad h = \overline{0, m_{i,l}}, \quad l = 1, 2, ...,$$

Obviously,

$$\varphi_{i,h}^{(l)}(0) = \varphi_{i,h}^{(l)}.$$

Definition 5. *It the system* $\left\{\psi_{i,h}^{(l)}\right\}_{h=\overline{0,m_{i,l}}, l=1,2,...}$ *is complete in the space* $H_{3/2}$, *we say that the system* $K\left(\prod_-\right)$ *is the system of derivative chains of eigen and adjoint vectors responding to eigen values from the left half plane corresponding to the problem (1), (2), is complete in the space of traces. In the present paper, we find the conditions providing the completeness of the system* $K\left(\prod_-\right)$ *in the space* $H_{3/2}$ *and completeness of descending elementary solutions in the space of regular solutions of problem (2), (3). Notice that for* $K(s) = K_1(s) = 0$ *the appropriate results where obtained in [2,3].*

In order to obtain main results, we'll use the following theorem from the papers [4,5].

Theorem 1. *Let conditions 1)-3) be fulfilled, and the norms*

$$\chi_0 = \|K\|_{W_2^2(R_+;H)\to H_{3/2}}, \quad \chi_1 = \|K_1\|_{W_2^1(R_+;H)\to H_{3/2}},$$

satisfy the condition: $\chi = \chi_0 + \chi_1 < 1/2$, *and*

$$q(x) = \frac{1}{2}\left(\frac{1 + 2\chi}{1 - 2\chi}\right)^{1/2}\|B_1\| + \left(\frac{1}{1 - 2\chi}\right)^{1/2}\|B_2\| < 1.$$

Then for any $f(t) \in L_2(R_+; H)$ *there exists the vector-function* $u(t) \in W_2^2(R_+; H)$ *that satisfies the equation,*

$$P(d/dt)u(t) = f(t), \quad t \in R_+, \tag{4}$$

and and the boundary condition

$$L_0 u = u(0) - \int_0^\infty K(s)u(s)ds - \int_0^\infty K_1(s)u'(s)ds = 0, \tag{5}$$

in the sense of convergence of $H_{3/2}$*, and*

$$\|u\|_{W_2^2(R_+;H)} \leq const\,\|f\|_{L_2(R_+;H)}\,.$$

Show that subject to the conditions of theorem 1, problem (2), (3) is regularly solvable.

Theorem 2. *Let the conditions of theorem 1 be fulfilled. Then the problem (2), (3) is regularly solvable.*

Proof. For $A_1 = A_2 = 0$ the problem

$$P_0(d/dt)u = -\frac{d^2u}{dt^2} + A^2u = 0, \quad t \in R_+, \tag{6}$$

$$Lu = u(0) - \int\limits_0^\infty K(s)u(s)ds - \int\limits_0^\infty K_1(s)u'(s)ds = \varphi, \tag{7}$$

for $\varphi \in H_{3/2}$ is regularly solvable. Indeed, the general solution of the equation (6) from the space $W_2^2(R_+;H)$ is of the form

$$u_0(t) = e^{-tA}\theta, \quad \theta \in H_{3/2}.$$

Then from (7), to determine the vector θ we get:

$$\theta - \int\limits_0^\infty K(s)e^{-sA}\theta ds + \int\limits_0^\infty K_1(s)Ae^{-sA}\theta ds = \varphi$$

or $(E-Q)\theta = \varphi$, where $Q = \int\limits_0^\infty K(s)e^{-sA}ds - \int\limits_0^\infty K_1(s)Ae^{-sA}ds.$

Since

$$\left\| \left(\int\limits_0^\infty K(s)e^{-sA}ds - \int\limits_0^\infty K_1(s)Ae^{-sA}ds \right) \theta \right\|_{3/2} \leq$$

$$\leq \left\|K(e^{-sA}\theta)_{3/2}\right\| + \left\|K_1(Ae^{-sA}\theta)\right\|_{3/2} \leq$$

$$\leq \chi_0 \left\|e^{-sA}\theta\right\|_{W_2^2(R_+;H)} + \chi_1 \left\|Ae^{-sA}\theta\right\|_{W_2^1(R_+;H)} \leq$$

$$\leq \chi_0 \left(2\left\|A^2e^{-sA}\theta\right\|^2_{L_2(R_+;H)}\right)^{1/2} + \chi_1 \left(2\left\|A^2e^{-sA}\theta\right\|^2_{L_2(R_+;H)}\right)^{1/2} =$$

$$= \sqrt{2}(\chi_0+\chi_1)\left\|A^2e^{-sA}\theta\right\|_{L_2(R_+;H)} \leq$$

$$\leq \sqrt{2}\,(\chi_0+\chi_1)\cdot\frac{1}{\sqrt{2}}\,\|\theta\|_{3/2} = (\chi_0+\chi_1)\,\|\theta\|_{3/2}$$

i.e. the operator $E-Q$ is invertible in $H_{3/2}$. Then $\theta = (E-Q)^{-1}\varphi$ and

$$\|\theta\|_{3/2} \leq \left\|(E-Q)^{-1}\right\|_{H_{3/2}\to H_{3/2}} \cdot \|\varphi\|_{3/2} = const\,\|\varphi\|_{3/2}$$

i.e.

$$\|u_0(t)\|_{W_2^2(R_+;H)} = \left\|e^{-tA}\theta\right\|_{W_2^2(R_+;H)} = \sqrt{2}\left\|A^2 e^{-tA}\theta\right\|_{L_2(R_+;H)} \leq$$

$$\leq \sqrt{2}\cdot\frac{1}{\sqrt{2}}\|\theta\|_{3/2} = \|\theta\|_{3/2} \leq const\,\|\varphi\|_{3/2}.$$

Notice that here we used the estimation [2,6]

$$\left\|A^2 e^{-tA}\theta\right\|_{L_2(R_+;H)} \leq \frac{1}{\sqrt{2}}\|\theta\|_{3/2}.$$

Now suppose that $A_1, A_2 \neq 0$. Then the regular solution of problem (2), (3) will be found in the form

$$u(t) = u_0(t) + u_1(t),$$

where

$$u_0(t) = e^{-tA}\theta.$$

Here $\theta \in H_{3/2}$ is an unknown vector, $u_1(t)$ is to be determined. Then from problem (2), (3) we get

$$P(u_0 + u_1) = 0,$$

$$\theta + u_1(t) - \int_0^\infty K(s)e^{-sA}\theta ds - \int_0^\infty K_0(s)e^{-sA}u_1(s)ds -$$

$$-\int_0^\infty K_1(s)Ae^{-sA}\theta ds - \int_0^\infty K_1(s)u_1'(s)ds = \varphi,$$

or

$$P_1 u_0 + P u_1 = 0, \tag{8}$$

$$u_0(0) - \int_0^\infty K(s)u_1(s)ds - \int_0^\infty K_1(s)u_1(s)ds = (E-Q)\theta + \varphi, \tag{9}$$

where

$$P_1(d/dt)u = P_1 u = A_1\frac{du}{dt} + A_2 u.$$

Hence we get

$$Pu_1(t) = -P_1 u_0(t) \equiv f(t), \tag{10}$$

$$u_1(0) - \int_0^\infty K(s)u_1(s)ds - \int_0^\infty K_1(s)u_1(s)ds = (E-Q)\theta + \varphi. \tag{11}$$

Choose θ so that $\varphi = -(E-Q)\theta$, i.e. $\theta = -(E-Q)^{-1}\varphi \in H_{3/2}$.

Since

$$\|f\|_{L_2(R_+;H)} = \|P_1 u_0(t)\|_{L_2(R_+;H)} = \left\|A_1 u_0'(t)\right\|_{L_2(R_+;H)} + \|A_2 u_0(t)\|_{L_2(R_+;H)} \leq$$

$$\leq \|B_1\|\left\|Au_0'\right\|_{L_2(R_+;H)} + \|B_2\|\left\|A^2 u_0\right\|_{L_2(R_+;H)} \leq$$

$$\leq (\|B_1\| + \|B_2\|)\left\|A^2 e^{-tA}\right\|_{L_2(R_+;H)} \leq const\cdot\|\theta\|_{3/2} \leq const\cdot\|\varphi\|_{3/2},$$

then $f \in L_2(R_+; h)$.

Thus, by theorem 1, problem (10), (11) has the solution $u_1 \in W_2^2(R_+; H)$, and

$$\|u_1(t)\|_{W_2^2(R_+;H)} \leq const \, \|P_1 u_0(t)\| \leq const \, \|\varphi\|_{3/2} \, .$$

Thus, $u(t) = u_0(t) + u_1(t) \in W_2^2(R_+; H)$ is the regular solution of problem (2), (3), and

$$\|u\|_{W_2^2(R_+;H)} \leq \|u_0\|_{W_2^2(R_+;H)} + \|u_1\|_{W_2^2(R_+;H)} \leq const \, \|\varphi\|_{3/2} \, .$$

The problem is proved.

Now we'll use the following theorem from [2,3].

Theorem 3. *Let A be a positive-definite self-adjoint operator. The operators $B_j = A_j A^{-j} (j = 1, 2)$ be bounded in H, moreover*

$$\frac{1}{2} \|B_1\| + \|B\| < 1.$$

If one of the following conditions are fulfilled:

a) $A^{-1} \in \sigma_p \quad (0 < p \leq 1)$;

b) $A^{-1} \in \sigma_p \quad (0 < p < \infty), \quad B_j \in \sigma_\infty \quad (j = 1, 2)$,

then the system $\left\{\varphi_{i,h}^{(l)}(0)\right\} = \left\{\varphi_{i,h}^{(l)}\right\}$ is complete in $H_{3/2}$, and the system of elementary solution of the equation $P(d/dt)u(t) = 0$ is complete in the space of regular solutions of the problem

$$P(d/dt)u(t) = 0, \tag{12}$$

$$u(0) = \varphi. \tag{13}$$

It holds

Theorem 4. *Let the conditions of theorem 1 or one of the conditions, either a) or b) from theorem3 be fulfilled. Then the system $K\left(\prod_-\right)$ is complete in $H_{3/2}$.*

Proof. It is obvious that subject to the conditions of theorem 1, the conditions of theorem 3 are also fulfilled.

$$\frac{1}{2} \|B_1\| + \|B_2\| \leq \frac{1}{2} \left(1 + \frac{4\chi}{1 - 2\chi}\right)^{1/2} \|B_1\| + \left(1 + \frac{2\chi}{1 - 2\chi}\right)^{1/2} \|B_2\| =$$

$$= \frac{1}{2} \left(\frac{1 + 2\chi}{1 - 2\chi}\right)^{1/2} \|B_1\| + \left(\frac{1}{1 - 2\chi}\right)^{1/2} \|B_2\| < 1$$

Thus, the system $\left\{\varphi_{i,h}^{(l)}(0)\right\} = \left\{\varphi_{i,h}^{(l)}\right\}$ is complete in $H_{3/2}$.

Now let's construct a bounded invertible operator $\mathbf{S}$ acting in $H_{3/2}$ and mapping $\left\{\varphi_{i,h}^{(l)}(0)\right\}$ to the system $\left\{\psi_{i,h}^{(l)}\right\}$, i.e. $\mathbf{S}\varphi_{i,h}^{(l)}(0) = \psi_{i,h}^{(l)}$.

This operator is constructed as follows:

Let $\varphi \in H_{3/2}$. Then the problems (2), (3) and (12), (13) have the solutions $u(t) \in P_0$ and $\widetilde{u}(t) \in P_k$, respectively, moreover

$$c_1 \|\varphi\|_{3/2} \leq \|u(t)\| \leq c \, \|\varphi\|_{3/2} \tag{14}$$

and

$$\widetilde{c}_1 \|\varphi\|_{3/2} \leq \|\widetilde{u}(t)\| \leq \widetilde{c} \, \|\varphi\|_{3/2} \, , \tag{15}$$

where P_k is the space of regular solutions of problem (2), (3), P_0 is the space of regular solutions of problem (12), (13). Now determine the operators Γ and $\widetilde{\Gamma}$ as follows $\left(\Gamma : H_{3/2} \to P_0, \widetilde{\Gamma} : H_{3/2} \to P_k\right)$:

$$\Gamma\varphi = u(t), \quad \widetilde{\Gamma}\varphi = \widetilde{u}(t).$$

From (14),) (15) it follows that the operators Γ and $\widetilde{\Gamma}$ are bounded and invertible operators. Then it is obvious that

$$\psi_{i,h}^{(l)} = \Gamma^{-1} u_{i,h}^{(l)}(t), \quad \varphi_{i,h}^{(l)}(0) = \widetilde{\Gamma}^{-1} u_{i,h}^{(l)}(t),$$

i.e.

$$\psi_{i,h}^{(l)} = \Gamma^{-1}\widetilde{\Gamma}\varphi_{i,h}^{(l)}(0).$$

Since $\Gamma^{-1}\widetilde{\Gamma}$ is an invertible operator, then denoting by $\mathbf{S} = \Gamma^{-1}\widetilde{\Gamma}$ we construct the operator $\mathbf{S}$ that we need. Thus, $\mathbf{S}$ is a continuous and invertible operator in $H_{3/2}$, and

$$\mathbf{S}\varphi_{i,h}^{(l)}(0) = \psi_{i,h}^{(l)}.$$

The completeness of the system $\left\{\psi_{i,h}^{(l)}(0)\right\}$ in $H_{3/2}$ yields the completeness of the system $\left\{\psi_{i,h}^{(l)}\right\}$.

The theorem is proved.

Now prove a theorem on completeness of elementary descending solutions in the space P_k.

Theorem 5. *Let the conditions of theorem 4 be fulfilled. Then the elementary descending solutions are complete in the space of all regular solutions of problem (2), (3).*

The proof follows from the inequality (15). Indeed, from the completeness of the system $\left\{\psi_{i,h}^{(l)}\right\}$, it follows that for any $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ and the numbers $c_{i,1,h}(\varepsilon), ..., c_{i,N,h}(\varepsilon)$ such that

$$\left\| \varphi - \sum_{i=1}^{N(\varepsilon)} \sum_{l=0}^{u_{i,j}} c_{i,j,l}(\varepsilon)\psi_{i,j,l}^{(l)} \right\| < \varepsilon.$$

Since $u(0) = \varphi$, $u_{i,j}^{(l)}(0) = \psi_{i,j}^{(l)}$ we get

$$\left\| u(t) - \sum_{i=1}^{N(\varepsilon)} \sum_{l=0}^{u_{i,j}} c_{i,j,l}(\varepsilon) u_{i,j}^{(l)}(t) \right\|_{W_2^2(R_+;H)} \leq \varepsilon_1,$$

where $\varepsilon_1 > 0$ is any number. The theorem is proved.

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