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BEST APPROXIMATION OF LIPSCHITZ CLASS FUNCTIONS

Abstract

In the paper we establish the best approximation order of the Lipschitz class functions of many groups of variables by the sums of a fewer number variables functions in the parallelepiped $\Pi(a,h)$ by means of the least upper bound of the modulus of finite mixed differences $\Delta_{\tau_1\ldots\tau_m}f$ with appropriate constants A_m and B_m.

Following [1] we introduce the following class of monotone functions of many groups of variables.

Consider an n-dimensional parallelepid

$$\Pi = \Pi(a,h) = \left\{ x \in R^n \middle| \, a_i \le x_i \le a_i + h_i, \ i = \overline{1,n} \right\}.$$

Having chosen the numbers $0 = k_0 < k_1 < ... < k_m$ denote $\mathcal{K} = (k_0, ..., k_m)$, $|\mathcal{K}| = m$.

Assume

$$t = (t_1, ..., t_m), \quad t_j = \left(x_{k_{j-1}+1}, ..., x_{k_j} \right), \quad j = \overline{1,m}.$$

Further, let

$$\mathcal{D}^m = \left\{ \varepsilon = (\varepsilon_1, ..., \varepsilon_m), \ \varepsilon_j = 0,1; \ j = \overline{1,m} \right\}$$

be a set of vertices of an m-dimensional unit cube; denote

$$\delta(\varepsilon) = \sum_{j=1}^{m} (1 - \varepsilon_j).$$

Consider the mapping $\delta_{(\xi,\tau)} : \mathcal{D}^m \to \Pi(\xi,\tau)$ of the set $\mathcal{D}^m$ into the set of vertices of n-dimensional parallelepiped $\Pi(\xi,\tau)$

$$g_{(\xi,\tau)}(\varepsilon) = \left(\xi_1 + \varepsilon_1\tau_1, ..., \xi_{k_1} + \varepsilon_1\tau_{k_1}, ..., \xi_{k_{m-1}+1} + \varepsilon_m\tau_{k_{m-1}+1}, ..., \xi_{k_m} + \varepsilon_m\tau_{k_m} \right).$$

Denote by $M_{\mathcal{K}} = M_{\mathcal{K}}(\Pi(a,h))$ the class of functions $f = f(x) : R^n \to R$, $x \in \Pi(a,h)$ for an arbitrary parallelepiped $\Pi(\xi,\tau) \subset \Pi(a,h)$ satisfying the condition

$$\mathcal{L}_{\mathcal{K}}(f, \Pi(\xi,\tau)) \overset{df}{=} 2^{-|\mathcal{K}|} \sum_{\varepsilon \in \mathcal{D}^{|\mathcal{K}|}} (-1)^{\delta(\varepsilon)} f\left(g_{(\xi,\tau)}(\varepsilon)\right) \ge 0.$$

We need the following result.

Theorem 1 [1]. *The precise estimations are valid for an arbitrary bounded real function f :*

$$|\mathcal{L}_{\mathcal{K}}(f, \Pi(a,h))| \le E_f \le 2S_f \prod_{i=1}^{m} h_i - |\mathcal{L}_{\mathcal{K}}(f, \Pi(a,h))|, \tag{1}$$

where

$$S_f = \sup_{\Pi(x,\Delta x)\subset\Pi(a,h)} \frac{\left|\mathcal{L}_K\left(f,\Pi\left(x,\Delta x\right)\right)\right|}{\prod\limits_{i=1}^{m}\sum\limits_{j\in\overline{k_j}}\Delta x_j}.$$

Define the class

$$Lip_k 1 = \left\{ f = f\left(x,y\right) \middle| \left|\Delta_{h\tau} f\right| \le K\left|\Delta_{h\tau} xy\right| \right\},$$

where $\Delta_{h\tau} f = \left|f\left(x+h,\ y+\tau\right) - f\left(x+h,y\right) - f\left(x,\ y+\tau\right) + f\left(x,y\right)\right|.$

(Sometimes, when it is clear from the context instead of $Lip_k 1$ we'll write $Lip1$).

Consider the best approximation

$$E_f = \inf_{\varphi(x)+\psi(y)} \sup_{(x,y)\in T} \left|f\left(x,y\right) - \varphi\left(x\right) - \psi\left(y\right)\right|, \quad T = [0,1;0,1].$$

Lemma 1. $f \in Lip\ 1 \Longrightarrow E_f \le C \sup\limits_{\substack{0\le h,\tau\le 1 \\ 0\le x+h,\ y+\tau\le 1}} \left|\Delta_{h\tau} f\right|.$

Proof. The right inequality of relation (1) in the case $m = n = 2,\ \Pi\left(a,h\right) = T$ allows to write

$$E_f \le \frac{1}{2} S_f - \left|\Delta_{11} f\right|. \tag{2}$$

Further we have $f \in Lip\ 1 \Longrightarrow f$ is continuous on $T \Longrightarrow f$ bounded on $T \Longrightarrow \sup\limits_{h,\tau}\left|\Delta_{h\tau} f\right| \stackrel{df}{=} K_1 < \infty.$

$$S_f = \sup_{0\le h,\tau\le 1} \frac{\left|\Delta_{h\tau} f\right|}{h\tau}$$

$$f \in Lip\ 1 \Longrightarrow S_f = \sup_{h,\tau} \frac{\left|\Delta_{h\tau} f\right|}{h\tau} \stackrel{df}{=} K_2 \le K.$$

Now, using (2) we get

$$E_f \le \frac{1}{2} S_f - \left|\Delta_{11} f\right| = \underbrace{K\left(\frac{S_f}{2K} - \frac{\left|\Delta_{11} f\right|}{K_1}\right)}_{C} \stackrel{df}{=} C_1 \sup_{h,\tau}\left|\Delta_{h\tau} f\right|.$$

Consider the general case. Define the class of functions $f = f\left(x_1, ..., x_n\right) = f\left(x\right)$ determined on the parallelepiped

$$\Pi\left(a,h\right) = \left\{ x \in R^n \middle| a_i \le x_i \le a_i + h_i,\ i = \overline{1,n} \right\}$$

$$Lip_k\ 1 = \left\{ f \middle| \left|\Delta_{\tau_1\ldots\tau_n} f\right| \le K \left|\Delta_{\tau_1\ldots\tau_n} \prod_{i=1}^{n} x_i\right| \right\}.$$

Consider the best approximation

$$E_f = \inf_{\sum\limits_{\nu=1}^{m}\varphi_\nu\left(x\backslash x_\nu\right)} \sup_{x\in\Pi(a,h)} \left| f\left(x\right) - \sum_{\nu=1}^{m} \varphi_\nu\left(x\backslash x_\nu\right) \right|.$$

Lemma 2. $f \in Lip\ 1 \Longrightarrow E_f \leq C \sup\limits_{\substack{a_i \leq \tau_i \leq a_i + h_i \\ a_i \leq x_i + \tau_i \leq a_i + h_i \\ i = \overline{1,n}}} \left| \Delta_{\tau_1 \ldots \tau_n} f \right|.$

Proof: The right inequality in (1) in the case $m = n$ has the form

$$E_f \leq 2^{1-n} S_f \prod_{i=1}^{n} h_i - \frac{1}{2^n} \left| \Delta_{h_1 \ldots h_n} f \right|. \tag{3}$$

Further we have $f \in Lip1 \Longrightarrow 1)$ f is continuous on $\Pi(a,h) \Longrightarrow f$ is bounded on $\Pi(a,h) \Longrightarrow \sup\limits_{\tau_1,\ldots,\tau_n} \left| \Delta_{\tau_1,\ldots,\tau_n} f \right| \overset{df}{=} K_1 < \infty.$

2) $S_f = \sup\limits_{\tau_1,\ldots,\tau_n} \left| \dfrac{\Delta_{\tau_1 \ldots \tau_n} f}{\tau_1, \ldots, \tau_n} \right| = K_2 \leq K.$

Using the scheme of the proof of lemma 1, we get

$$E_f \leq 2^{1-n} \prod_{i=1}^{n} h_i S_f \frac{K_1}{K_1} - \frac{1}{2^n} \left| \Delta_{h_1 \ldots h_n} f \right| \cdot \frac{K_1}{K_1} =$$

$$= K_1 2^{1-n} \left(\frac{S_f \prod\limits_{i=1}^{n} h_i}{K_1} - \frac{1}{2} \left| \frac{\Delta_{h_1 \ldots h_n} f}{K_1} \right| \right) = C \sup_{\tau_i} \left| \Delta_{\tau_1,\ldots,\tau_n} f \right|.$$

Lemma 3. $f \in Lip\ 1 \Longrightarrow E_f \leq C \sup\limits_{\substack{a_i \leq x_i \leq a_i + h_i \\ a_i \leq x_i + \theta_i \leq a_i + h_i}} \left| \Delta_{\tau_1 \ldots \tau_n} f \right|.$

Denote $\tau_i = \left(\theta_{k_{j-1}+1}, \ldots, \theta_{k_j} \right), \quad j = \overline{1,m}$ and consider the difference

$$\Delta_{\tau_j} f = f \left(t \backslash t_j, t_j + \tau_j \right) - f(t)$$

$$\Delta_{\tau_i \tau_j} f = \Delta_{\tau_j} \left(\Delta_{\tau_j} f \right).$$

Introduce the Lipschitz class on the groups of variables $\overline{k};\ \tau_1, \ldots, \tau_m$

$$Lip\ 1 = \left\{ f \middle| \exists s < \infty;\ \left| \Delta_{\tau_1,\ldots,\tau_m} f \right| \leq s \left| \Delta_{\tau_1,\ldots,\tau_m} \prod_{i=1}^{n} x_i \right| \right\}.$$

Theorem 2.

$$f \in Lip\ 1 \Longrightarrow A_m \sup_{a_i \leq x_i \leq x_i + a_i \leq a_i + h_i} \left| \Delta_{\tau_1 \ldots \tau_m} f \right| \leq$$

$$\leq E_f \leq B_m \sup_{a_i \leq x_i \leq x_i + a_i \leq a_i + h_i} \left| \Delta_{\tau_1 \ldots \tau_m} f \right|. \tag{4}$$

Proof. Earlier in [1] it was shown that $\Delta_{\tau_1 \ldots \tau_m}$ is an annihilator of the fanctions of the form $\sum\limits_{\nu=1}^{m} \varphi_\nu \left(t \backslash t_\nu \right)$, i.e. for the function f to have the form $\sum\limits_{\nu=1}^{m} \varphi_\nu \left(t \backslash t_\nu \right)$ it is necessary and sufficient $\Delta_{\tau_1 \ldots \tau_m} f = 0.$

Then, taking into account the linearity of $\Delta_{\tau_1\dots\tau_m} f$ we have $\forall 0 \le x_i \le x_i + \theta_i \le a_i + h_i$

$$|\Delta_{\tau_1\dots\tau_m} f| = \left|\Delta_{\tau_1\dots\tau_m}\left(f - \sum \varphi_\nu\right)\right| \le 2^m \left\|f - \sum \varphi_\nu\right\|_{C(\Pi(a,h))} \Longrightarrow$$

$$\Longrightarrow 2^{-m} \sup_{a_i \le x_i \le x_i + a_i \le a_i + h_i} |\Delta_{\tau_1\dots\tau_n} f| \le E_f. \tag{5}$$

Further, $f \in Lip\ 1 \Longrightarrow f$ is continuous on $\Pi(a,h) \Longrightarrow f$ is bounded on $\Pi(a,h) \Longrightarrow \sup\limits_{a_i \le x_i \le x_i + a_i \le a_i + h_i} |\Delta_{\tau_1\dots\tau_m} f| \le +\infty$;

Use estimation (1). It is easy to note that we can write the right relation in (1) in the form

$$E_f \le 2^{1-m} S_f \prod_{j=1}^{m} \sum_{i \in k_j} h_i - |\mathcal{L}_k(f, \Pi(a,h))|, \tag{6}$$

where

$$S_f = \sup \left| \frac{\Delta_{\tau_1,\dots,\tau_n} f}{\Delta_{\tau_1,\dots,\tau_n} \prod\limits_{i=1}^{n} x_i} \right| \quad \text{and} \quad \mathcal{L}_k(f, \Pi(x,\theta)) \overset{df}{=} 2^{-m} \Delta_{\tau_1,\dots,\tau_m} f.$$

We have

$$\Delta_{\tau_1,\dots,\tau_m} \prod_{i=1}^{n} x_i = \prod_{j=1}^{m} \sum_{i \in k_j} \theta_i, \quad \text{where } \overline{k}_j = \{k_{j-1}+1, \dots, k_j\}.$$

Therefore

$$f \in Lip\ 1 \Longrightarrow S_f = \sup \left| \frac{\Delta_{\tau_1,\dots,\tau_m} f}{\Delta_{\tau_1,\dots,\tau_n} \prod\limits_{i=1}^{n} x_i} \right| \le S < +\infty$$

Taking into account what has been said, from relation (6) we get

$$E_f \le 2^{1-m} S_f \prod_{j=1}^{m} \sum_{i \in \overline{k}_j} h_i - |\mathcal{L}_k(f, \Pi(a,h))| = B_m \sup_{a_i \le x_i \le x_i + a_i \le a_i + h_i} |\Delta_{\tau_1\dots\tau_m} f|.$$

The last relation with the functions from $Lip 1$ completes the proof of the cited inequality of the theorem.

Using the sheme of the proof of the left inequality (1) in [1] we can establish also the left inequality in (4) that completes the proof of the theorem.

References

[1]. Babayev M-B.A. *Estimations of the best uniform approximation by the sums of the function of a few number of variavles. Special Issues of theory of functions.* Baku, 1989, pp. 38-46 (Russian).

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Received: May 21, 2012; Revised: October 08, 2012.