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CHARACTERIZATION OF n-POSITIVE HOMOGENEOUS FUNCTIONS

Abstract

In the paper, an n-positive homogeneous function is compared with a sublinear function which is determined on tensor product spaces. A series of properties of sublinear functions determined on tensor product spaces is studied. A link between an n-positive homogeneous function and a sublinear function determined on tensor product spaces is also studied.

1. Introduction

The paper consists of five paragraphs. In the second paragraph, the formulation of the basic results of the paper is given. In the third paragraph, some properties of the representation of an element of tensor product spaces and the properties of convergence on tensor product spaces are studied. In case of tensor product of two spaces, such kind of questions are studied in [1].

In the third paragraph, the convex hull of a function is also considered. The convex hull of a function in a finite-dimensional space is considered in the works [3, 4]. As usual, the convex hull of a function in a finite dimensional space is investigated with the use of Caratheodory theorem. In an infinite dimensional space, such a question is considered in the works [1],[2]. In the third paragraph, a similar question is considered in an infinite-dimensional space.

In the fourth and fifth paragraphs, an n-positive homogeneous function is compared with a sublinear function, determined on a tensor product spaces. A series of properties of sublinear functions determined on tensor product spaces is studied. A link between an n-positive homogeneous function and a sublinear function determined on tensor product spaces is also studied. Properties of n-positive homogeneous function from $X \times ... \times X$ in R are studied in the fourth paragraph and those of n-positive homogeneous function from $K \times ... \times K$ in R, where $K \subset X$ is a convex closed salient cone are studied in the fifth paragraph. Note that corollary 4.5 when $n = 2$ and X is a Banach space such that each point x with $\|x\| = 1$ is a strongly exposed point of the unit ball is an analogue of theorem 6.4 [5], which is proved by another method for even positive-homogeneous functions of second order in [5]. Such kind of questions are also studied, particularly, for n-sublinear functions in the works [6], [7], for a bipositively- homogeneous function in [1] and for an n-positively homogeneous function in [2].

Note that the problem of investigation of n-positive homogeneous functions arises when one obtains necessary optimality conditions of high order of the solutions to no n-smooth extreme problems (see corollary 3.5.5 [1]), but such problem is also of independent interest.

2. Formulation of basic results

Let X be a real Banach space, $R = (-\infty, +\infty)$, $\bar{R} = (-\infty, +\infty]$ and $q : X \times ... \times X \to \bar{R}$. The function q is called n-positive homogeneous if the functions $x_i \to$

$q(x_1,...,x_i,...,x_n)$ are positive homogeneous and $q(x_1,...,x_{i-1},0,x_{i+1},...,x_n) = 0$ for $i=\overline{1,n}$. The function q is called n-sublinear if q is an n-positive-homogeneous function and the functions $x_i \to q(x_1,...,x_i,...,x_n)$ are convex.

The set of all continuous n-linear functions from $X \times ... \times X$ in R is denoted by $B(X^n,R)$. An n-linear function from $X \times ... \times X$ in R is called symmetric if it takes on the same value at all permutations of its variables. If there exists $b \in B(X^n,R)$ such that $Q(x) = b(x,...,x)$, then Q is called n-polynomial. Note that for each function $b \in B(X^n,R)$ there exists a symmetric function $b_1 \in B(X^n,R)$ such that $b(x,...,x) = b_1(x,...,x)$ for $x \in X$. The set of all n-polynomial functions from X in R is denoted by $B_0(X^n)$. Tensor product of n number spaces X is denoted by $X \otimes .... \otimes X$ (see[1], [2], [8]). In the paper, we generally use the definition of tensor product introduced in [8] .

As usual, the set of all linear continuous functions from X in R is denoted by X^*. Note that $X^* \otimes ... \otimes X^*$ is identified with some subspaces of n-linear continuous functions on $X \times ... \times X$ by means of the identity $(x_1^* \otimes ... \otimes x_n^*)(x_1,...,x_n) = x_1^*(x_1)...x_n^*(x_n)$. It is easily verified that for each element $\nu \in X \otimes ... \otimes X$ the representation $\nu = \sum\limits_{i_1=1}^{k_1} ... \sum\limits_{i_n=1}^{k_n} \alpha_{i_1...i_n} x_1^{i_1} \otimes ... \otimes x_n^{i_n}$ holds true, where $\{x_s^{i_s}\}_{i_s=1}^{k_s}$ are such that subsets of these elements differ from each other, i.e. they are linearly independent for $s=\overline{1,n}$ acting as sets. Therefore, similar to [8] (see p. 120) it is verified that for each nonzero element $\nu \in X \otimes ... \otimes X$ there exists the function $x_1^* \otimes ... \otimes x_n^* \in X^* \otimes ... \otimes X^*$ such that $(x_1^* \otimes ... \otimes x_n^*)(\nu) \neq 0$, i.e. $X^* \otimes ... \otimes X^*$ separates the elements of space $X \otimes ... \otimes X$.

We assume that the space $X \otimes ... \otimes X$ is supplied with a topology generated with respect to the norm

$$\|\nu\| = \inf\left\{ \sum_{i=1}^{m} \|x_1^i\| \cdots \|x_n^i\| : \quad \nu = \sum_{i=1}^{m} x_1^i \otimes ... \otimes x_n^i,\ x_j^i \in X,\ m \in N \right\},$$

i.e. $X \otimes ... \otimes X$ is supplied with a projective topology (see [8]). Next, we shall identify (see [8,9]) $(X \otimes \cdots \otimes X)^*$ and $B(X^n,R)$.

Let $\{E_\alpha : \alpha \in A\}$ be a family of finite-dimensional subspaces of the space X directed along the growth and satisfying the condition $\bigcup\limits_{\alpha \in A} E_\alpha = X$, where $E_\alpha \neq E_\beta$ at $\alpha \neq \beta$; A is a set of indices directed (reflective, transitive, antisymmetric) by the relation $\leq$. Thus, A is directed along the growth $\alpha \leq \beta$ if $E_\alpha \subset E_\beta$. As any linear system has algebraic basis, then the existence of the given family of finite-dimensional subspaces E_α, $\alpha \in A$, in X follows from Zorn lemma. It is clear that E_α, $\alpha \in A$, is a Banach space with respect to the induced topology from the Banach space X. Assuming

$$E_\alpha \otimes ... \otimes E_\alpha = Lin\{x^1 \otimes ... \otimes x^n \in X \otimes ... \otimes X : x^1,...,x^n \in E_\alpha\},$$

we have $E_\alpha \otimes ... \otimes E_\alpha \subset E_\beta \otimes ... \otimes E_\beta$ for $\alpha \leq \beta$. It is clear that $E_\alpha \otimes ... \otimes E_\alpha$ is a Banach space with respect to the norm

$$\|\nu\| = \inf\left\{\sum_{i=1}^{m} \|x_1^i\| \dots \|x_n^i\| : \right.$$
$$\left. \nu = \sum_{i=1}^{m} x_1^i \otimes \dots \otimes x_n^i,\ \ x_1^i \otimes \dots \otimes x_n^i \in E_\alpha \otimes \dots \otimes E_\alpha,\ \ m \in N \right\}$$

and a subspace in $X \otimes \dots \otimes X$.

Note that all norms are equivalent in a finite-dimensional space. Therefore, it is possible to take s-norms (see §3) that are independent of the choice of the elements representations.

Let g_α denote a canonical imbedding $E_\alpha \otimes \dots \otimes E_\alpha$ in $X \otimes \dots \otimes X$. As is known (see [8]) an inductive topology in $X \otimes \dots \otimes X$ with respect to the family $(E_\alpha \otimes \dots \otimes E_\alpha,\ g_\alpha,\ \alpha \in A)$ is a local convex space. We denote by $(X \otimes \dots \otimes X)_s$ a space $X \otimes \dots \otimes X$ supplied with the topology introduced. Similarly, from 2.6.4 and 2.6.5 [8], we have that $\{\nu_k\} \subset X \otimes \dots \otimes X$ converges to ν with respect to the topology in $(X \otimes \dots \otimes X)_s$ if and only if $\{\nu_k\}$ converges to $\{\nu\}$ in $E_\alpha \otimes \dots \otimes E_\alpha$ for some $\alpha \in A$. Therefore, the topology in $(X \otimes \dots \otimes X)_s$ is stronger than the one in $X \otimes \dots \otimes X$. Then we have that $(X \otimes \dots \otimes X)_s$ is a Hausdorff space.

From the definition of the topology in $(X \otimes \dots \otimes X)_s$ it follows that $B(X^n, R) \subset \subset (X \otimes \dots \otimes X)_s^*$. We denote by $clB(X^n, R)$ a closure $B(X^n, R)$ in the topology $\sigma((X \otimes \dots \otimes X)_s^*, X \otimes \dots \otimes X)$. As $clB(X^n, R)$ is closed in $(X \otimes \dots \otimes X)_s^*$ with respect to the topology $\sigma((X \otimes \dots \otimes X)_s^*, X \otimes \dots \otimes X)$ and $B(X^n, R)$ separates the points of the set $X \otimes \dots \otimes X$, then, using separation theorems (see theorems 3.4 [10]), we have that $clB(X^n, R) = (X \otimes \dots \otimes X)_s^*$. And vice versa, if $x^* \in (X \otimes \dots \otimes X)_s^*$, then, it is easily verified that $b(x_1, \dots, x_n) = x^*(x_1 \otimes \dots \otimes x_n)$ is an n-linear function. Besides, from 2.6.1 [8] it follows that $x^* \in (X \otimes \dots \otimes X)_s^*$ if and only if $x^*|_{E_\alpha \otimes \dots \otimes E_\alpha} \in (E_\alpha \otimes \dots \otimes E_\alpha)^*$. Therefore $b|_{E_\alpha^n} \in B(E_\alpha^n, R)$, where $E_\alpha^n = E_\alpha \times \dots \times E_\alpha$.

Besides if X is a seperabel space, having chosen the denumerable system of elements $x_1, x_2, \dots, x_k, \dots$ generating all X and having put $E_k = Lin\{x_1, x_2, \dots, x_k\}$, $E_k \otimes \dots \otimes E_k = Lin\{x^1 \otimes \cdots \otimes x^n : x^1, \dots, x^n \in E_k\}$, we have $\bigcup\limits_{k \in N} E_k \otimes \dots \otimes E_k = = X \otimes \dots \otimes X$ and $A = N$. It is known that (see 2.6.6 [8]) the inductive topology in $X \otimes \dots \otimes X$ with respect to family $(E_k \otimes \dots \otimes E_k,\ g_k,\ k \in N)$ is a complete local convex space. As E_k is reflexive, from 4.5.8 [8] it follows that $(X \otimes \dots \otimes X)_s$ is reflexive.

If $f : X \times \dots \times X \to \bar{R}$ and $f(-x, \dots, -x) = f(x, \dots, x)$, then f is called even.

Assume that $B_0(X^n) = \{Q :\ Q(x) = x^*(x, \dots, x), x^* \in B(X^n, R)\}$, $B_1(X^n) = = \{Q :\ Q(x) = x^*(x, \dots, x), x^* \in clB(X^n, R) = (X \otimes \dots \otimes X)_s^*\}$, $\bar{\partial}_n \varphi = \{Q \in \in B_1(X^n) :\ \varphi(x) \geq Q(x)$ for $x \in X\}$, where $\varphi(x) = f(x, \cdots, x)$ (see the details in paragraphs 4 and 5). Next, we assume that the number X in $X \times \dots \times X$ equals n and $n \geq 2$.

Theorem 1. *Let n be even, $f : X \times \dots \times X \to \bar{R}$ an n-positive-homogeneous even and lower semicontinuous function in each finite-dimensional subspace of space $X \times \dots \times X$, there exist $\alpha > 0$ such that $\alpha \|x\|^n \leq f(x, \dots, x)$ for $x \in X$. Then $f(x, \dots, x) = \sup\{Q(x) : Q \in \bar{\partial}_n \varphi\}$ for $x \in X$.*

Theorem 2. *Let X be a real Banach space, n even, $f : X \times \dots \times X \to \bar{R}$ a lower semicontinuous n-positive homogeneous even function, there exist $b \in (X \otimes \dots \otimes X)_s^*$*

and $\alpha > 0$ *such that* $\|x\|^n \le b(x,...,x)$ *and* $-\alpha b(x,...,x) \le f(x,...,x)$ *for* $x \in X$. *Then* $f(x,...,x) = \sup\left\{Q(x)\ :\ Q \in \bar{\partial}_n\varphi\right\}$ *for* $x \in X$.

If X *is a Hilbert space, then in theorem 2, we can assume* $b(x,...,x) = \|x\|^n$.

Corollary 1. *Let* X *be a real Banach space,* n *be even,* $f : X \times ... \times X \to R$ *a lower semicontinuous* n*-positive homogeneous even function, there exist* $b \in (X \otimes ... \otimes X)^*_s$ *such that* $\|x\|^n \le b(x,...,x)$. *Then* $f(x,...,x) = \sup\left\{Q(x)\ :\ Q \in \bar{\partial}_n\varphi\right\}$ *for* $x \in X$.

If $q : X \to \bar{R}$, *we assume* $d_nq = \{Q \in B_0(X^n) : q(x) \ge Q(x)\ \ for\ \ x \in X\}$.

Theorem 3. *Let* $X = R^k$, n *be even,* $f : R^k \times ... \times R^k \to \bar{R}$ *an* n*-positive homogeneous lower semicontinuous even function. Then* $f(x,...,x) = \sup\{Q(x)\ :\ Q \in d_n\varphi\}$ *for* $x \in R^k$.

The set $K \subset X$ *is called a cone if* $\lambda x \in K$ *for* $x \in K$ *and* $\lambda \ge 0$. *The cone* K *is called a salient (or pointed) cone if* $K \bigcap \{-K\} = \{0\}$.

If there exists such a convex closed salient cone K_1 *and number* $d > 0$ *is such that* $\{x \in X\ :\ \|x - x_0\| \le d\,\|x_0\|\} \subset K_1$ *for each point* $x_0 \in K$, *then it is said that the convex closed salient cone* K *allows plastering (see [11], p.40), where the number* d *is independent of* $x_0 \in K$.

Let $K \subset X$ be a convex closed cone, $f : K \times ... \times K \to R$ an n-positive homogeneous function. Assume $\varphi(x) = \begin{cases} f(x,...,x) : & x \in K, \\ +\infty : & x \notin K. \end{cases}$

Theorem 4. *Let* X *be a real Banach space,* $K \subset X$ *a convex closed salient cone (cone* K *allows plastering cone if* n *is odd),* $f : K \times ... \times K \to R$ *a lower semicontinuous* n*-positive homogeneous function, there exist* $b \in (X \otimes ... \otimes X)^*_s$ *and* $\alpha > 0$ *such that* $\|x\|^n \le b(x,...,x)$ *and* $-\alpha b(x,...,x) \le f(x,...,x)$ *for* $x \in K$. *Then* $f(x,...,x) = \sup\left\{Q(x)\ :\ Q \in \bar{\partial}_n\varphi\right\}$ *for* $x \in K$.

A linear function $x^* \in X^*$ is called uniformly positive if there exists such $a > 0$ that $x^*(x) \ge a\,\|x\|$ for $x \in K$ (see [11], p. 40). A convex closed salient cone K allows plastering cone if and only if there exists a uniform positive function $x^* \in X^*$ (see [11], p. 40). Therefore $\|x\|^n \le b(x,...,x)$ for $x \in K$, where $\frac{1}{a^n}(x^*(x))^n = b(x,...,x)$, $b \in B(X^n, R)$, i.e. if K allows plastering cone, then there exist $b \in B(X^n, R)$ such that $\|x\|^n \le b(x,...,x)$ for $x \in K$.

Corollary 2. *Let* X *be a real Banach space, cone* K *allow plastering,* $f : K \times ... \times K \to R$ *a lower semicontinuous* n*-positive homogeneous function. Then* $f(x,...,x) = \sup\left\{Q(x)\ :\ Q \in \bar{\partial}_n\varphi\right\}$ *for* $x \in K$.

Theorem 5. *Let* $X = R^k$, $K \subset R^k$ *be a convex closed salient cone,* $f : K \times ... \times K \to R$ *an* n*-positive homogeneous lower semicontinuous function. Then* $f(x,...,x) = \sup\{Q(x)\ :\ Q \in d_n\varphi\}$ *for* $x \in K$.

3. Some properties of tensor product

In the third paragraph, some properties of the representation of an element and convergence of a sequence in tensor product spaces are studied.

Let X be a real Banach space and $R_+ = [0, -\infty)$.

Lemma 1 [2]. *If* $x \otimes x \otimes ... \otimes x = \sum\limits_{i=1}^{m} y^i \otimes y^i \otimes ... \otimes y^i$ *and* n *is even, then there* *exists $\lambda_i \in R$, $i = \overline{1,m}$, such that $y^i = \lambda_i x$ and $\sum\limits_{i=1}^{m} \lambda_i^n = 1$.*

Corollary 1. *If X is a Banach space, n is even, $x \otimes x \otimes ... \otimes x = \sum\limits_{i=1}^{m} x^i \otimes x^i \otimes ... \otimes x^i$ and $\{x^i \otimes .... \otimes x^i\}_{i=1}^m$ are linearly independent, where $x, x^i \in X$ for $i = \overline{1,m}$, then $m = 1$ and $x^1 = \pm x$.*

Lemma 2. *If X is a separable Banach space, $K \subset X$ is a convex closed salient cone, $x \otimes x \otimes ... \otimes x = \sum\limits_{i=1}^{m} x^i \otimes x^i \otimes ... \otimes x^i$ and x, $x^i \in \mathrm{K}$ at $i = \overline{1,m}$, then there exist $\lambda_i \in R_+$ for $i = \overline{1,m}$, such that $x^i = \lambda_i x$ and $\sum\limits_{i=1}^{m} \lambda_i^n = 1$.*

Proof. If n is even, the validity of lemma 2 follows from lemma 1. Let n be odd. According to theorem 5.9 (see [11], p. 42) there exists $x^* \in X$ such that $x^*(z) > 0$ for $z \in \mathrm{K}$, $z \neq 0$.

Let $x \otimes x \otimes ... \otimes x = \sum\limits_{i=1}^{m} x^i \otimes x^i \otimes ... \otimes x^i$ and x, $x^i \in \mathrm{K}$ for $i = \overline{1,m}$.

The case $x = 0$ is trivial. Let $x, x^i \in \mathrm{K}$, $x \neq 0$, $x^i \neq 0$.

If $b_1 \in B(X^{n-1}, R)$, then $b = b_1 \otimes x^* \in B(X^n, R)$ and

$$b_1(x, ..., x)x^*(x) = \sum_{i=1}^{m} b_1(x^i, ..., x^i)x^*(x^i).$$

Hence it follows that

$$x^*(x)x \otimes x \otimes ... \otimes x = \sum_{i=1}^{m} x^*(x^i)x^i \otimes x^i \otimes ... \otimes x^i \tag{1}$$

and $x^*(x) > 0$. As $x^i \in K$, then $x^*(x^i) > 0$. Then from (1) we obtain

$$x \otimes x \otimes ... \otimes x = \sum_{i=1}^{m} \frac{x^*(x^i)}{x^*(x)} x^i \otimes x^i \otimes ... \otimes x^i \tag{2}$$

As the number x in equality (2) is even, assuming that $\bar{x}^i = \sqrt[n-1]{\frac{x^*(x^i)}{x^*(x)}}\, x^i$ according to lemma 1 we have $\bar{x}^i = \beta_i x$, i.e. $x^i = \beta_i \sqrt[n-1]{\frac{x^*(x)}{x^*(x^i)}}\, x = \lambda_i x$ at $i = \overline{1,m}$. Also it is clear that $x \otimes x \otimes ... \otimes x = \sum\limits_{i=1}^{m} \lambda_i^n x \otimes x \otimes ... \otimes x$. Hence it follows that $\sum\limits_{i=1}^{m} \lambda_i^n = 1$. As x, $x^i \in K$, we have $\lambda_i > 0$ for $i = \overline{1,m}$. The lemma is proved.

Corollary 2. *If X is a separable Banach space, $K \subset X$ a convex closed salient cone, $x \otimes x \otimes ... \otimes x = \sum\limits_{i=1}^{m} x^i \otimes x^i \otimes ... \otimes x^i$ and $\{x^i \otimes .... \otimes x^i\}_{i=1}^m$ are linearly independent, where $x, x^i \in K$ for $i = \overline{1,m}$, then $m = 1$ and $x^1 = x$.*

Let K be a convex closed salient cone. A linear function $x^* \in X^*$ is called uniformly positive if there exists $a > 0$ such that $x^*(x) \geq a \|x\|$ for $x \in K$. A convex closed salient cone K allows plastering if and only if there exists a uniformly positive function $x^* \in X^*$ (see [11], p. 40).

Remark 1. If X is a Banach space, $K \subset X$ is a convex closed salient cone and cone K allows plastering, $x \otimes x \otimes ... \otimes x = \sum\limits_{i=1}^{m} x^i \otimes x^i \otimes ... \otimes x^i$ and $x,\ x^i \in$ K for $i = \overline{1,m}$, then there exist $\lambda_i \in R_+$ for $i = \overline{1,m}$, such that $x^i = \lambda_i x$ and $\sum\limits_{i=1}^{m} \lambda_i^n = 1$.

Lemma 3 [2]. *If $x_m \in X$ weakly converges to x and $x_m \otimes .... \otimes x_m$ weakly converges to $\nu \in X \otimes ... \otimes X$, then $\nu = x \otimes .... \otimes x$.*

Lemma 4 [2]. *If x_m^i strongly converges to x^i, then $\nu_m = \sum\limits_{i=1}^{k} x_m^i \otimes ... \otimes x_m^i$ strongly converges to $\nu = \sum\limits_{i=1}^{k} x^i \otimes ... \otimes x^i$.*

Let Z be a vector space. If $S \subset Z$ is a nonempty set, then assume

$$coneS = \left\{ \sum_{i=1}^{m} \alpha_i z^i \ : \ z^i \in S,\ \ \alpha_i \geq 0,\ \ m \in N \right\}.$$

Introduce the notation $M = \{x \otimes ... \otimes x\ : x \in X\}$.

Assuming $\lambda \cdot x \otimes \cdots \otimes x = \sqrt[n]{\lambda} x \otimes \cdots \sqrt[n]{\lambda} x$ for $\lambda \geq 0$, we have

$$co\,M = \left\{ \sum_{i=1}^{m} \alpha_i x^i \otimes \cdots \otimes x^i \ : \sum_{i=1}^{m} \alpha_i = 1,\ \ x^i \in X,\ \alpha_i \geq 0,\ m \in N \right\} =$$
$$= \left\{ \sum_{i=1}^{m} y^i \otimes ... \otimes y^i : \ \ y^i \in X,\ \ m \in N \right\},$$

$$cone\,M = \left\{ \sum_{i=1}^{m} \alpha_i x^i \otimes \cdots \otimes x^i \ : \ x^i \in X,\ \alpha_i \geq 0, m \in N \right\} =$$
$$= \left\{ \sum_{i=1}^{m} y^i \otimes \cdots \otimes y^i : \ \ y^i \in X, m \in N \right\}.$$

If n is even, we have that

$$Lin\,M = \left\{ \sum_{i=1}^{m} \lambda_i x^i \otimes \cdots \otimes x^i \ : \ \ x^i \in X,\ \lambda_i \in R,\ m \in N \right\} = co\,M - co\,M.$$

If n is odd, we also have

$$Lin\,M = \left\{ \sum_{i=1}^{m} \lambda_i x^i \otimes \cdots \otimes x^i \ : \ \ x^i \in X,\ \lambda_i \in R,\ m \in N \right\} =$$
$$= \left\{ \sum_{i=1}^{m} y^i \otimes \cdots \otimes y^i : \ \ y^i \in X, m \in N \right\} = co\,M.$$

If $X = R^k$, from lemma 1.20 [10], it follows that $Lin\,M$ is closed in space $R^k \otimes \cdots \otimes R^k$.

Let X be a real Banach space and $B_* = \{x^* \in X^* : \|x^*\| \leq 1\}$ a unit ball in X^*. As is known (see [8], p.153; [9], p.40)

$$s(\nu) = \sup \left\{ \sum_{i=1}^{m} x_1^*(x_1^i)...x_n^*(x_n^i) : \quad \nu = \sum_{i=1}^{m} x_1^i \otimes ... \otimes x_n^i,\ x_j^i \in X,\ x_1^*, ..., x_n^* \in B_* \right\}.$$

is a cross norm (s-norm) on $X\otimes....\otimes X$, where $\sum\limits_{i=1}^{m} x_1^*(x_1^i)...x_n^*(x_n^i)$ does not depend on the choice of the representation of the element $\nu=\sum\limits_{i=1}^{m} x_1^i\otimes...\otimes x_n^i$.

If X is a Hılbert space, then using the other definition of a tensor product (see [12]), we have that Euclidian norm does not depend on the representation of the element $\nu\in X\otimes...\otimes X$.

In lemma 5, the convex hull of a function is considered in an infinite-dimensional space.

Lemma 5 [2]. *If X_i, $i=\overline{1,n}$, are normalized spaces, $P:X_1\times...\times X_n\to\bar{R}$ an n-positive homogeneous function and $(conv\ P)(\nu)>-\infty$, then*

$$(conv\ P)(\nu)=\bar{P}(\nu)=$$

$$=\inf\left\{\sum_i P(x_1^i,...,x_n^i):\nu=\sum_i x_1^i\otimes...\otimes x_n^i,\ (x_1^i,...,x_n^i)\in X_1\times...\times X_n\right\}=$$

$$=\inf\left\{\sum_{i=1}^{r} P(x_1^i,...,x_n^i):\right.$$

$$\left.\nu=\sum_{i=1}^{r}(x_1^i\otimes...\otimes x_n^i),\ \ x_1^i\otimes...\otimes x_n^i, i=\overline{1,r},\ \ lin.\,indepen.,\ \ r\in N\right\}.$$

Remark 2. If X is normalized space, n is even, $f:X\times...\times X\to\bar{R}$ is an n-positive homogeneous function, then the solution to problem

$$\sum_{i=1}^{m}\alpha_i f(x^i,...,x^i)\to\inf,\qquad \alpha_i\geq 0,\quad \nu=\sum_{i=1}^{m}\alpha_i x^i\otimes...\otimes x^i \tag{3}$$

exists (The set $(\alpha_1,...,\alpha_m)\in R^m$ satifying the conditions $\alpha_i\geq 0$, $\nu=\sum\limits_{i=1}^{m}\alpha_i x^i\otimes...\otimes x^i$ is compact). Therefore

$$f_S(\nu)=\inf\left\{\sum_{i=1}^{m} f(x^i,...,x^i)\ :\nu=\sum_{i=1}^{m} x^i\otimes...\otimes x^i\ ,x^i\in\right.$$
$$\left.\in X,\ m\in N\right\}=\inf\left\{\sum_{i=1}^{m} f(x^i,...,x^i)\ :\nu=\right.$$
$$\left.=\sum_{i=1}^{m} x^i\otimes...\otimes x^i,\ \left\{x^i\otimes...\otimes x^i\right\}_{i=1}^{m}-lin.indepen.,\ x^i\in X,\ \ m\in N\right\},$$

where as usually, we assume that $\inf\emptyset=+\infty$.

Remark 3. Note that if X is a real separable Banach space, $K\subset X$ is a convex closed salient cone, $f:K\times...\times K\to\bar{R}$ is an n-positive homogeneous function, then

$$f_S(\nu)=\inf\left\{\sum_{i=1}^{m} f(x^i,...,x^i)\ :\nu=\sum_{i=1}^{m} x^i\otimes...\otimes x^i\ ,x^i\in\right.$$
$$\left.\in K,\ m\in N\right\}=\inf\left\{\sum_{i=1}^{m} f(x^i,...,x^i)\ :\nu=\right.$$
$$\left.=\sum_{i=1}^{m} x^i\otimes...\otimes x^i,\ \left\{x^i\otimes...\otimes x^i\right\}_{i=1}^{m}-lin.indepen.,\ x^i\in K,\ \ m\in N\right\}.$$

Remark 4. Note that if X is a real Banach space, $K\subset X$ is a convex closed salient cone and cone K allows plastering if n is odd, $f:K\times ...\times K\to \bar{R}$ is an n-positive homogeneous function, then

$$f_S(\nu)=\inf\left\{\sum_{i=1}^{m} f(x^i,...,x^i)\ :\nu=\sum_{i=1}^{m} x^i\otimes ...\otimes x^i\ ,x^i\in \right.$$
$$\in K,\ m\in N\} = \inf\left\{\sum_{i=1}^{m} f(x^i,...,x^i)\ :\nu=\right.$$
$$\left.=\sum_{i=1}^{m} x^i\otimes ...\otimes x^i,\ \left\{x^i\otimes ...\otimes x^i\right\}_{i=1}^{m} - lin.indepen.,\ x^i\in K,\ m\in N\right\}.$$

4. Properties of n-positive homogeneous functions

Let X be a real Banach space, $M=\{x\otimes ...\otimes x\ :\ x\in X\}$ and $f:X\times ...\times X\to \bar{R}$ be an n-positive homogeneous function. Assume that

$$f_S(\nu)=\inf\left\{\sum_{i=1}^{m} f(x^i,...,x^i)\ :\ \ \nu=\sum_{i=1}^{m} x^i\otimes ...\otimes x^i,\ \ m\in N,\ x^i\in X\right\},$$

where as usually, we assume that $\inf\emptyset=+\infty$.

A function $f:X\times ...\times X\to \bar{R}$ is called even if $f(-x,...,-x)=f(x,...,x)$ for $x\in X$.

Lemma 1 [2]. *If n is even, $f:X\times ...\times X\to \bar{R}$ an n-positive homogeneous even function, then $f_s(x\otimes ...\otimes x)=f(x,...,x)$ for $x\in X$.*

Let $f:X\times\cdots\times X\to \bar{R},\ \ \nu\in X\otimes ...\otimes X$. Assume

$$\bar{f}(\nu)=\inf\left\{\sum_{i=1}^{k} f(x_1^i,...,x_n^i):\ \ \ \nu=\sum_{i=1}^{k} x_1^i\otimes ...\otimes x_n^i,\ x_j^i\in X,\ k\in N\right\}.$$

Lemma 2[2]. *If n is even, $f_1:X\times ...\times X\to \bar{R}$ is an n-positive homogeneous even function and*

$$f(x_1,...,x_n)=\begin{cases} f_1(x,...,x)\ : & for\ \ x_i=x,\ \ i=\overline{1,n},\ \ x\in X,\\ +\infty\ \ \ \ : & otherwise,\end{cases}$$

then $\bar{f}(x\otimes ...\otimes x)=f(x,...,x)$ for $x\in X$.

If $\nu=\sum\limits_{i=1}^{k} x_1^i\otimes ....\otimes x_n^i$, we assume $r=rid\,\nu=\max\{\dim\{x_1^i\}_{i=1}^{k},...,\dim\{x_n^i\}_{i=1}^{k}\}$.

Remark 1. Let $\nu=\sum\limits_{j=1}^{k} x^j\otimes ...\otimes x^j$, $rid\ \nu=r$ and $\{x^1,....,x^r\}$ be linearly independent. Then there exist $\alpha_{ij}\in R$ such that $x^i=\alpha_{1i}x^1+...+\alpha_{ri}x^r$ for $i=\overline{r+1,k}$. Therefore,

$$\nu=\sum_{j=1}^{r} x^j\otimes ...\otimes x^j+\sum_{i=r+1}^{k}(\alpha_{1i}x^1+...+\alpha_{ri}x^r)\otimes x^i\otimes ...\otimes x^i=$$
$$=\sum_{j=1}^{r} x^j\otimes ...\otimes x^j+\sum_{i=r+1}^{k}(\alpha_{1i}x^1+...+\alpha_{ri}x^r)\otimes ...\otimes(\alpha_{1i}x^1+...+\alpha_{ri}x^r).$$

Lemma 3. *Let n be even, $f : X\times...\times X \to \bar{R}$ an n-positive homogeneous lower semicontinuous even function, there exist $\alpha > 0$ such that $\alpha\,\|x\|^n \leq \leq f(x,...,x)$ for $x \in X$. Then there exist $\{x^i\}_{i=1}^{r^n}$, where $r = rid\ \nu$ and $\nu = \sum\limits_{i=1}^{r^n} x^i\otimes...\otimes x^i$ such that $f_S(\nu) = \sum\limits_{i=1}^{r^n} f(x^i,...,x^i)$ for $\nu \in coM$.*

Proof. It is clear that if $\nu = \sum\limits_{i=1}^{k} x^i\otimes...\otimes x^i$, where $\{x^i\otimes...\otimes x^i\}_{i=1}^{k}$ are linearly independent, then $k \leq r^n$ (see [16], p.20). Therefore, from lemma 3.5 it follows, that $f_S(\nu) = \inf\left\{\sum\limits_{i=1}^{r^n} f(x^i,...,x^i):\ \ \nu = \sum\limits_{i=1}^{r^n} x^i\otimes...\otimes x^i,\ \ x^i \in X\right\}$.

The case $f_S(\nu) = +\infty$ is trivial.

Let $f_S(\nu) < +\infty$. By definition of $f_S(\nu)$, there exist $x_m^i \in X$, $1 \leq i \leq r^n$, where $\nu = \sum\limits_{i=1}^{r^n} x_m^i \otimes ... \otimes x_m^i$ such that $\sum\limits_{i=1}^{r^n} f(x_m^i,...,x_m^i) \leq f_S(\nu) + \frac{1}{m}$. By the data we have that $\sum\limits_{i=1}^{r^n} \alpha\left\|x_m^i\right\|^n \leq \sum\limits_{i=1}^{r^n} f(x_m^i,\,...,x_m^i) \leq f_S(\nu)+1$. Therefore, $\alpha\sum\limits_{i=1}^{r^n}\left\|x_m^i\right\|^n \leq f_S(\nu)+1$. Hence $\left\|x_m^i\right\| \leq \sqrt[n]{\frac{f_S(\nu)+1}{\alpha}}$.

According to remark 1, the element $\nu = \sum\limits_{j=1}^{k} x^j\otimes...\otimes x^j$ has the representation

$\nu = \sum\limits_{j=1}^{r} x^j\otimes...\otimes x^j + \sum\limits_{i=r+1}^{k}(\alpha_{1i}x^1+...+\alpha_{ri}x^r)\otimes x^i...\otimes x^i$, where $\{x^i\}_{i=1}^{r}$ are linearly independent.

From the equality $\sum\limits_{i=1}^{r^n} x_m^i\otimes...\otimes x_m^i = \sum\limits_{j=1}^{r} x^j\otimes...\otimes x^j + \sum\limits_{i=r+1}^{k}(\alpha_{1i}x^1+...+\alpha_{ri}x^r)\otimes x^i...\otimes x^i$ it follows that $x_m^i \in Lin\{x^1,...,x^r\}$. Suppose the contrary. Let there exist $i \in \overline{1,r^n}$ such that the set $\left\{x^1,x^2,...,x^r,x_m^i\right\}$ are linearly independent. Then, according to the lemma of biorthogonal basis (see [4],p.25) there exist $x^* \in X^*$ such that $x^*(x_m^i) = 1$ and $x^*(x^j) = 0$ for $j = \overline{1,r}$. Assuming $(x^*\otimes...\otimes x^*)(x_1,...,x_n) = x^*(x_1)...x^*(x_n)$ for $(x_1,...,x_n) \in X\times...\times X$ we have that $\sum\limits_{i=1}^{r^n} x^*(x_m^i)^n = 0$. As $\sum\limits_{i=1}^{r^n} x^*(x_m^i)^n \geq 1$, then we have a contradiction, i.e. the set $\left\{x^1,x^2,...,x^k,x_m^i\right\}$ is linearly dependent. Therefore, choosing the convergent subsequence $\left\{x_{m_k}^1\right\}$ from $\left\{x_m^1\right\}$, $\left\{x_{m_{k_s}}^1\otimes...\otimes x_{m_{k_s}}^1\right\}$ from $\left\{x_{m_k}^1\otimes...\otimes x_{m_k}^1\right\}$, $\left\{x_{m_{k_{s_j}}}^2\right\}$ from $\left\{x_{m_{k_s}}^2\right\}$, $\left\{x_{m_{k_{s_{jl}}}}^2\otimes...\otimes x_{m_{k_{s_{jl}}}}^2\right\}$ from $\left\{x_{m_{k_{s_j}}}^2\otimes...\otimes x_{m_{k_{s_j}}}^2\right\}$ and continuing the process, it is possible to assume that x_m^i converges to $\bar{x}^i \in X$ and $x_m^i\otimes...\otimes x_m^i$ converges to $\bar{x}^i\otimes...\otimes\bar{x}^i$ (according to lemma 3.4 it is enough to choose the convergent subsequences $\left\{x_{m_k}^1\right\}$ from $\left\{x_m^1\right\}$, $\left\{x_{m_{k_s}}^2\right\}$ from $\left\{x_{m_k}^2\right\}$, $\left\{x_{m_{k_{s_j}}}^3\right\}$ from $\left\{x_{m_{k_s}}^3\right\}$

and etc.). It is clear that $\nu = \sum\limits_{i=1}^{r^n} \bar{x}^i \otimes ... \otimes \bar{x}^i$ and

$$f_S(\nu) \geq \varliminf_{m\to\infty} \sum_{i=1}^{r^n} f(x_m^i, ..., x_m^i) \geq \sum_{i=1}^{r^n} \varliminf_{m\to\infty} f(x_m^i, ..., x_m^i) \geq \sum_{i=1}^{r^n} f(\bar{x}^i, ..., \bar{x}^i)\,.$$

Hence it follows that $f_S(\nu) = \sum\limits_{i=1}^{r^n} f(\bar{x}^i, ..., \bar{x}^i)$. The lemma is proved.

It is known (see [10], p.23) that $LinM \bigcap (E_\alpha \otimes ... \otimes E_\alpha)$ is closed for $\alpha \in A$. Then we have that $LinM$ is closed in $(X \otimes ... \otimes X)_s$. Therefore, if n is odd, then $coM = LinM$ and it is closed in space $(X \otimes ... \otimes X)_s$.

Lemma 4. *If n is even, then coM is closed in space $(X \otimes ... \otimes X)_s$.*

Proof. It is clear that, E_α, $\alpha \in A$ is a Banach space with respect to induced topology from Banach space X(see §2).

It is known (see [13], p.9) that this topology is induced by any Euclidian metric determined by scalar product introduced on $E_\alpha \times E_\alpha$ with the help of positive-definite symmetric bilinear function $b(x,y)$. Similarly, from 2.6.4 and 2.6.5 [8] we have that $\{\nu_k\} \subset X \otimes ... \otimes X$ converges to ν with respect to topology in $(X \otimes ... \otimes X)_s$ iff for some $\alpha \in A$ subsequence $\{\nu_k\}$ converges to ν in space $E_\alpha \otimes ... \otimes E_\alpha$. We assume $|x|_e = \sqrt{b(x,x)}$ and

$$\|\nu\|_e = \inf\left\{ \sum_{i=1}^{m} |x_1^i|_e ... |x_n^i|_e :\ \nu = \sum_{i=1}^{m} x_1^i \otimes ... \otimes x_n^i, \quad x_1^i, ..., x_n^i \in E_\alpha,\ m \in N \right\}.$$

It is known that if $\dim E_\alpha = k$ and $\nu = \sum\limits_{i=1}^{s} x^i \otimes ... \otimes x^i \in E_\alpha \otimes ... \otimes E_\alpha$, where $\{x^i \otimes ... \otimes x^i\}_{i=1}^s$ are linearly independent, then $s \leq k^n$.

If n is even, $\nu = \sum\limits_{i=1}^{l} x_1^i \otimes ... \otimes x_n^i$ and $\tilde{b}(x_1, ..., x_n) = b(x_1, x_2)...b(x_{n-1}, x_n)$, then from Cauchy-Schwartz inequality (see [8]), it follows that

$$\tilde{b}(\nu) = \sum_{i=1}^{l} b(x_1^i, x_2^i)...b(x_{n-1}^i, x_n^i) \leq \sum_{i=1}^{l} |x_1^i|_e ... |x_{n-1}^i|_e \cdot |x_n^i|_e\,.$$

Therefore, $\tilde{b}(\nu) = \sum\limits_{i=1}^{l} b(x_1^i, x_2^i)...b(x_{n-1}^i, x_n^i) \leq \|\nu\|_e$. Hence it follows that if $\nu = \sum\limits_{i=1}^{l} x^i \otimes ... \otimes x^i$, then $\|\nu\|_e = \sum\limits_{i=1}^{k^n} |x^i|_e^n$.

Let $\nu_m \in coM$ and $\nu_m \to \nu$. Then there exists $\alpha > 0$ such that $\nu_m \in E_\alpha \otimes ... \otimes E_\alpha$ and $\nu_m \to \nu$. Let $\dim E_\alpha = k$. If n is even and $x_m^i \in E_\alpha, i = \overline{1,k}$, such that $\nu_m = \sum\limits_{i=1}^{k^n} x_m^i \otimes ... \otimes x_m^i$, we have $\|\nu_m\|_e = \sum\limits_{i=1}^{k^n} |x_m^i|_e^n$. As ν_m converges to ν, there exists $\lambda \in R_+$ such that $\|\nu_m\|_e \leq \lambda$, i.e $\sum\limits_{i=1}^{k^n} |x_m^i|_e^n \leq \lambda$. Hence it follows that $|x_m^i|_e \leq \sqrt[n]{\lambda}$. Therefore, choosing the convergent subsequences $\{x_{m_k}^1\}$ from $\{x_m^1\}$, $\left\{x_{m_{k_s}}^2\right\}$ from $\{x_{m_k}^2\}$, $\left\{x_{m_{k_{s_j}}}^3\right\}$ from $\left\{x_{m_{k_s}}^3\right\}$ and continuing the process, we can assume that x_m^i converges to $\bar{x}^i \in E_\alpha$ and $x_m^i \otimes ... \otimes x_m^i$ converges to $\bar{x}^i \otimes ... \otimes \bar{x}^i$ for $i = \overline{1, k^n}$. It is clear that $\nu = \sum\limits_{i=1}^{r^n} \bar{x}^i \otimes ... \otimes \bar{x}^i$, i.e $\nu \in coM$.

As the norms $\|\cdot\|$, $s(\cdot)$ and $\|\cdot\|_e$ are equivalent in $E_\alpha \otimes ... \otimes E_\alpha$ we obtain that $coM \bigcap E_\alpha \otimes ... \otimes E_\alpha$ is closed in space $E_\alpha \otimes ... \otimes E_\alpha$. Therefore, coM is closed in space $(X \otimes ... \otimes X)_s$. The lemma is proved.

Lemma 5. *Let $f : X \times ... \times X \to \bar{R}$ be an n-positive-homogeneous continuous function.Then there exists $\alpha > 0$ such that $|f(x_1, ..., x_n)| \leq \alpha \|x_1\| ... \|x_n\|$ for $(x_1, ..., x_n) \in X \times ... \times X$.*

Proof. As $f : X \times ... \times X \to R$ is an n-positive-homogeneous continuous function, then for $\varepsilon > 0$ there exists $\nu > 0$ such that $|f(x_1, ..., x_n)| \leq \varepsilon$ for $\|(x_1, ..., x_n)\| = \sum\limits_{i=1}^{n} \|x_i\| \leq \nu$. Then

$$|f(\frac{\nu}{n\|x_1\|}x_1, ..., \frac{\nu}{n\|x_n\|}x_n)| = \frac{\nu^n}{n^n\|x_1\| ... \|x_n\|}|f(x_1, ..., x_n)| \leq \varepsilon$$

for $(x_1, ..., x_n) \in X \times ... \times X$, $x_i \neq 0$. Therefore, $|f(x_1, ..., x_n)| \leq \varepsilon \frac{n^n}{\nu^n} \|x_1\| ... \|x_n\|$ for $(x_1, ..., x_n) \in X \times ... \times X$. The lemma is proved.

Lemma 6. *Let $f : X \times ... \times X \to \bar{R}$ be an n-positive-homogeneous lower semicontinuous function. Then there exists α such that $\alpha \|x_1\| ... \|x_n\| \leq f(x_1, ..., x_n)$ for $(x_1, ..., x_n) \in X \times ... \times X$.*

Proof. As $f : X \times ... \times X \to \bar{R}$ is an n-positive-homogeneous lower semicontinuous function, then for $\varepsilon < 0$ there exists $\nu > 0$ such that $\varepsilon \leq f(x_1, ..., x_n)$ for $\|(x_1, ..., x_n)\| \leq \nu$. Then

$\varepsilon \leq f(\frac{\nu}{n\|x_1\|}x_1, ..., \frac{\nu}{n\|x_n\|}x_n) = \frac{\nu^n}{n^n\|x_1\| ... \|x_n\|}f(x_1, ..., x_n)$

for $(x_1, ..., x_n) \in X \times ... \times X$, $x_i \neq 0$. Therefore, $\varepsilon \frac{n^n}{\nu^n} \|x_1\| ... \|x_n\| \leq f(x_1, ..., x_n)$ for $(x_1, ..., x_n) \in X \times ... \times X$. The lemma is proved.

Proposition 1. *Let n be even, $f : X \times ... \times X \to \bar{R}$ an n-positive-homogeneous lower semicontinuous even function, there exists $\alpha > 0$ such that $\alpha \|x\|^n \leq f(x, ..., x)$ for $x \in X$. Then $f_S(\nu)$ is a lower semicontinuous function in co M with respect to the topology of space $(X \otimes ... \otimes X)_s$.*

Proof. We are to show that $S(f_S, \lambda) = \{\nu \in co\,M : f_S(\nu) \leq \lambda\}$ is a closed set for $\lambda \in R_+$. Let $\nu_k \in S(f_S, \lambda)$ and $\nu_k \in E_\alpha \otimes ... \otimes E_\alpha$, where $\alpha \in A$ is a fixed element, $\dim E_\alpha = r$ and ν_k converges to ν in space $(X \otimes ... \otimes X)_s$. As coM is closed in space $(X \otimes ... \otimes X)_s$ we have $\nu \in coM$. According to lemma 3, there exist $\{x_k^i \otimes ... \otimes x_k^i\}$, where $\nu_k = \sum\limits_{i=1}^{r^n} x_k^i \otimes ... \otimes x_k^i$, such that $f_S(\nu_k) = \sum\limits_{i=1}^{r^n} f(x_k^i, ..., x_k^i)$. Similarly to the proof of lemma 3, we conclude that $x_k^i \in E_\alpha$ for $i = \overline{1, r^n}$. By the data, we have that $\sum\limits_{i=1}^{r^n} \alpha \|x_k^i\|^n \leq \sum\limits_{i=1}^{r^n} f(x_k^i, ..., x_k^i) \leq \lambda$. Therefore, $\alpha \sum\limits_{i=1}^{r^n} \|x_k^i\|^n \leq \lambda$. Hence we have $\|x_k^i\| \leq \sqrt[n]{\frac{\lambda}{\alpha}}$. Without loss of generality, we assume that(see the proof of lemma 3) that x_m^i converges to $\bar{x}^i \in X$ and $x_m^i \otimes ... \otimes x_m^i$ converges to

$\bar{x}^i \otimes ... \otimes \bar{x}^i$. Then $\nu_k = \sum\limits_{i=1}^{r^n} x_k^i \otimes ... \otimes x_k^i$ converges to $\nu = \sum\limits_{i=1}^{r^n} \bar{x}^i \otimes ... \otimes \bar{x}^i$. Therefore,

$$\varliminf_{k\to\infty} f_S(\nu_k) = \varliminf_{k\to\infty} \sum_{i=1}^{r^n} f(x_k^i, ..., x_k^i) \geq \sum_{i=1}^{r^n} \varliminf_{k\to\infty} f(x_k^i, ..., x_k^i) \geq$$

$$\geq \sum_{i=1}^{r^n} f(\bar{x}^i, ..., \bar{x}^i) \geq f_S(\nu).$$

Hence it follows that $f_S(\nu) \leq \lambda$, i.e $\nu \in S(f_S, \lambda)$. We obtain that the set $S(f_S, \lambda)$ is closed. The proposition 1 is proved.

From the proof of proposition 1, the validity of the following proposition 2 follows.

Proposition 2. Let n be even, $f : X \times ... \times X \to \bar{R}$ an n-positive-homogeneous even and lower semicontinuous function in each finite-dimensional subspace of space $X \times ... \times X$, there exist $\alpha > 0$ such that $\alpha \|x\|^n \leq f(x, ..., x)$ for $x \in X$. Then $f_S(\nu)$ is a lower semicontinuous function in $co\,M$ with respect to the topology of space $(X \otimes ... \otimes X)_s$.

Remark 2. Let n be even, $f : X \times ... \times X \to \bar{R}$ an n-positive-homogeneous even and lower semicontinuous function in each finite-dimensional subspace of space $X \times ... \times X$, there exists $c \geq 0$ and function $h : R_+ \to R_+$, where $h(t) \to +\infty$ for $t \to +\infty$, such that $h(\|x\|) - c \leq f(x, ..., x)$ for $x \in X$. Then from the proof of lemma 3 and proposition 1, it follows that $f_S(\nu)$ is a lower semicontinuous function in $co\,M$ with respect to the topology of space $(X \otimes ... \otimes X)_s$.

We assume that $\partial g = (x^* \in (X \otimes ... \otimes X)_s^* : g(\nu) \geq x^*(\nu)$ for $\nu \in X \otimes ... \otimes X\}$.

The proof of theorem 1. Under the condition of proposition 2, $f_S(\nu)$ is a lower semicontinuous sublinear function in coM. Therefore, under the condition of proposition 2, $g(\nu) = \begin{cases} f_S(\nu) \ : \ \nu \in co\,M \\ +\infty \ : \quad \nu \notin co\,M \end{cases}$ is a lower semicontinuous sublinear function in $(X \otimes ... \otimes X)_s$ (see [14]). Then, according to Hormander theorem (see [15]) it follows that $g(\nu) = \sup\{b(\nu) : \ b \in \partial g\}$. As $g(x \otimes ... \otimes x) = f(x, ..., x)$ for $x \in X$ we have $f(x, ..., x) = \sup\left\{Q(x) \ : \ Q \in \bar{\partial}_n \varphi\right\}$ for $x \in X$, where $\varphi(x) = f(x, ..., x)$, $\bar{\partial}_n \varphi = \{Q \in B_1(X^n) : \ \varphi(x) \geq Q(x) \, for \, x \in X\}$, $B_1(X^n) = \{Q : \ Q(x) = x^*(x, ..., x), \, x^* \in \ clB(X^n, R) = (X \otimes ... \otimes X)_s^*\}$. Theorem 1 is proved.

If $f : X \times ... \times X \to \bar{R}$ is an n-positive-homogeneous lower semicontinuous function, then from lemma 6 it follows that there exists $\alpha > 0$ such that $-\alpha \|x_1\| ... \|x_n\| \leq f(x_1, ..., x_n)$ for $(x_1, ..., x_n) \in X \times ... \times X$. Therefore, if n is even, $f : X \times ... \times X \to \bar{R}$ is an n-positive-homogeneous lower semicontinuous even function, then $f(x, ..., x) + (\alpha + \varepsilon) \|x\|^n = \sup\left\{Q(x) \ : \ Q \in \bar{\partial}_n \varphi_1\right\}$ for $x \in X$, where $\varphi_1(x) = f(x, ..., x) + (\alpha + \varepsilon) \|x\|^n$, $\varepsilon > 0$. It is clear that, $f(x, ..., x) = \sup\left\{Q(x) - (\alpha + \varepsilon) \|x\|^n \ : \ Q \in \bar{\partial}_n \varphi_1\right\}$ for $x \in X$.

The proof of theorem 2. Assuming $f^\alpha(x_1, ..., x_n) = f(x_1, ..., x_n) + (\alpha + 1) b(x_1, ..., x_n)$, from proposition 1 we have that $f_S^\alpha(\nu)$ is a lower semicontinuous function in $co\,M$ with respect to the topology of space $(X \otimes ... \otimes X)_s$. As

$f_S^\alpha(\nu) = f_S(\nu) + (\alpha+1)b_S(\nu)$, we have that f_S is also a lower semicontinuous function in $co\,M$. Assuming $g(\nu) = \begin{cases} f_S(\nu) & : \ \nu \in co\,M, \\ +\infty & : \quad \nu \notin co\,M \end{cases}$, we have that $g(\nu)$ is a lower semicontinuous function in space $(X \otimes ... \otimes X)_s$. Then, according to Hormander theorem it follows that $g(\nu) = \sup\{b(\nu) : \ b \in \partial g\}$. As $g(x \otimes ... \otimes x) = f(x, ..., x)$ for $x \in X$ we have $f(x, ..., x) = \sup\left\{Q(x) \ : \ Q \in \bar{\partial}_n \varphi\right\}$ for $x \in X$. Theorem 2 is proved.

Note that if X is a locally convex space, theorem 2 also holds true. Furthermore, the function b can be taken from $B(X^n, R)$.

From theorem 2 and lemma 6, the validity of the following corollary 1 follows.

Corollary 1. *Let X be a real Banach space, n even, $f : X \times ... \times X \to \bar{R}$ a lower semicontinuous n-positive homogeneous even function, there exist $b \in (X \otimes ... \otimes X)_s^*$ such that $\|x\|^n \leq b(x, ..., x)$ for $x \in X$. Then $f(x, ..., x) = = \sup\left\{Q(x) \ : \ Q \in \bar{\partial}_n \varphi\right\}$ for $x \in X$.*

Note that in spaces L_1, C and C^1 the inequality $\|x\|^2 \leq b(x, x)$, where $b \in B(X^2, R)$ is symmetric, is not satisfied.

Corollary 2. *Let X be a real Hilbert space, n even, $f : X \times ... \times X \to \bar{R}$ a lower semicontinuous n-positive homogeneous even function. Then $f(x, ..., x) = = \sup\left\{Q(x) \ : \ Q \in \bar{\partial}_n \varphi\right\}$ for $x \in X$.*

Let $X \otimes \cdots \otimes X$ be supplied with a projective topology. Further, we are to identify $(X \otimes \cdots \otimes X)^*$ and $B(X^n, R)$(see [8,9]). Denote by $B_0(X^n)$ a set of all n-polynomial functions from X in R. If $q : X \to \bar{R}$, we take

$$d_n q = \{Q \in B_0(X^n) : q(x) \geq Q(x) \ for \ \ x \in X\}$$

The proof of theorem 3. If $f : R^k \times ... \times R^k \to \bar{R}$ is an n-positive homogeneous lower semicontinuous function, then there exist α such that $\alpha \|x\|^n \leq \leq f(x, ..., x)$ for $x \in R^k$.

It is clear that $(R^k \otimes ... \otimes R^k)_s^* = (R^k \otimes ... \otimes R^k)^* = B((R^k)^n, R)$. Therefore, if $X = R^k$, then $d_n\varphi = \bar{\partial}_n\varphi$. Then the validity of theorem 3 follows from theorem 2. Theorem 3 is proved.

Note that if X is a Hilbert space and $n = 2$, according to Hellinger and Teoplits theorem (see [10], p.132) it follows that $B_0(X^2) = B_1(X^2)$. Therefore $d_2\varphi = \bar{\partial}_2\varphi$.

From lemma 1 and theorem 2 the validity of the following corollary follows.

Corollary 3. If n is even, $\varphi : X \to \bar{R}$ a positive-homogeneous even function of order n and there exists $b \in B(X^n, R)$ such that $b(x, ..., x) \leq \varphi(x)$ for $x \in X$ and $f(x_1, ..., x_n) = \sqrt[n]{(\varphi(x_1) - b(x_1, ..., x_1))...(\varphi(x_n) - b(x_n, ..., x_n))}$, then $f_S(x \otimes ... \otimes x) = \varphi(x) - b(x, ..., x)$ for $x \in X$.

Corollary 4. *If n is even, $\varphi : X \to \bar{R}$ a lower semicontinuous even positive-homogeneous function of order n and there exists $b \in B(X^n, R)$ such that $\|x\|^n \leq b(x, ..., x)$ for $x \in X$, then $\varphi(x) = \sup\left\{Q(x) \ : \ Q \in \bar{\partial}_n\varphi\right\}$ for $x \in X$.*

Proof. As $\varphi : X \to \bar{R}$ is a lower semicontinuous function, then for $\varepsilon > 0$ there exists $\nu > 0$ such that $-\varepsilon \leq \varphi(x)$ for $x \in X$, $\|x\| \leq \nu$. Then $-\varepsilon \leq \frac{\nu^n}{\|x\|^n}\varphi(x)$ for $x \in X$, $x \neq 0$. Therefore, $-\frac{\varepsilon}{\nu^n}\|x\|^n \leq \varphi(x)$ for $x \in X$. Then $-\frac{\varepsilon}{\nu^n} b(x, ..., x) \leq \varphi(x)$.

Assuming $f(x_1,...,x_n) = \sqrt[n]{(\varphi(x_1)+\frac{\varepsilon}{\nu^n}b(x_1,...,x_1))...(\varphi(x_n)+\frac{\varepsilon}{\nu^n}b(x_n,...,x_n))}$ according to theorem 2 we have $\varphi(x)=\sup\left\{Q(x)\ :\ Q\in\bar{\partial}_n\varphi\right\}$ for $x\in X$.The corollary is proved.

Corollary 5. *If n is even, $\varphi: X\to\bar{R}$ is a lower semicontinuous even positive-homogeneous function of order n, then*

$$\varphi(x)=\sup\left\{Q(x)+\alpha\left\|x\right\|^n\ :\varphi(x)\geq Q(x)+\alpha\left\|x\right\|^n,\alpha\leq 0,Q\in B_1(X^n)\right\}$$

for $x\in X$.

Proof. As $\varphi: X\to\bar{R}$ is a lower semicontinuous positive-homogeneous function of degree n, then $\varepsilon>0$ there exists $\nu>0$ such that $-\frac{\varepsilon}{\nu^n}\left\|x\right\|^n\leq\varphi(x)$ for $x\in X$ (see corollary 4). Assuming

$$f(x_1,...,x_n)=\sqrt[n]{(\varphi(x_1)+(1+\frac{\varepsilon}{\nu^n})\left\|x_1\right\|^n)...(\varphi(x_n)+(1+\frac{\varepsilon}{\nu^n})\left\|x_n\right\|^n))}$$

from proposition 1 and the theorem of Hormander we have $\varphi(x)+(1+\frac{\varepsilon}{\nu^n})\left\|x\right\|^n=\sup\left\{Q(x)\ :\ Q\in\bar{\partial}_n(\varphi+(1+\frac{\varepsilon}{\nu^n})\left\|x\right\|^n)\right\}$ for $x\in X$. Therefore,

$$\varphi(x)=\sup\left\{Q(x)+\alpha\left\|x\right\|^n\ :\varphi(x)\geq Q(x)+\alpha\left\|x\right\|^n,\alpha\leq 0,Q\in B_1(X^n)\right\}$$

for $x\in X$.The corollary is proved.

Corollary 6. *If n is even, $\varphi: X\to\bar{R}$ is a lower semicontinuous even positive-homogeneous function of order n and X is a finite-dimensional space, then $\varphi(x)=\sup\left\{Q(x)\ :\ Q\in d_n\varphi\right\}$.*

Corollary 7. *If X is a Hilbert space, $\varphi: X\to\bar{R}$ is a lower semicontinuous even positive- homogeneous function of second order, then*

$$\varphi(x)=\sup\left\{Q(x)\ :\ Q\in d_2\varphi\right\}.$$

5. Properties of n-positive homogeneous functions in a cone

Let X be a real Banach space, $X\otimes\cdots\otimes X$ supplied with a projective topology, $K\subset X$ a convex closed cone, $M_+=\left\{x\otimes...\otimes x\ :x\in K\right\}$ and $f: K\times...\times K\to R$ an n-positive homogeneous function.

It is clear that

$$co\,M_+=\left\{\sum_{i=1}^m\alpha_i x^i\otimes\cdots\otimes x^i\ :\sum_{i=1}^m\alpha_i=1,\ \ x^i\in K,\ \alpha_i\geq 0,\ m\in N\right\}=$$
$$=\left\{\sum_{i=1}^m y^i\otimes...\otimes y^i:\ \ y^i\in K,\ \ m\in N\right\},$$

$$cone\,M=\left\{\sum_{i=1}^m\alpha_i x^i\otimes\cdots\otimes x^i\ :\ \ x^i\in K,\ \alpha_i\geq 0,\ m\in N\right\}=$$
$$=\left\{\sum_{i=1}^m y^i\otimes\cdots\otimes y^i:\ \ y^i\in K,\ m\in N\right\},$$

$$Lin\,M_+=\left\{\sum_{i=1}^m\lambda_i x^i\otimes\cdots\otimes x^i\ :\ \ x^i\in K,\ \lambda_i\in R,\ m\in N\right\}=co\,M_+-co\,M_+.$$

Assume

$$f_S(\nu)=\inf\left\{\sum_{i=1}^{m} f(x^i,...,x^i)\ :\ \ \nu=\sum_{i=1}^{m} x^i\otimes...\otimes x^i,\ \ m\in N,\ \ x^i\in K\right\},$$

where as usual we assume that $\inf\emptyset=+\infty$.

Next, we assume that the number K in $K\times...\times K$ equals n.

Lemma 1. *If n is even, $K\subset X$ a convex closed salient cone, $f:K\times...\times K\to R$ an n-positive homogeneous function, then $f_s(x\otimes...\otimes x)=f(x,...,x)$ for $x\in K$.*

Proof. Using corollary 3.1 and lemma 3.5 we obtain

$$f_S(x\otimes...\otimes x)=$$
$$=\inf\left\{\sum_{i=1}^{m} f(x^i,...,x^i)\ :\ x\otimes...\otimes x=\sum_{i=1}^{m} x^i\otimes...\otimes x^i,\ \ x^i\in K,\, m\in N\right\}=$$
$$=\inf\Big\{\sum_{i=1}^{m} f_1(x^i,...,x^i)\ :\ x\otimes...\otimes x=\sum_{i=1}^{m} x^i\otimes...\otimes x^i,\ \ \left\{x^i\otimes...\otimes x^i\right\}_{i=1}^{m}-$$
$$-lin.\,indepen.,\ x^i\in K, m\in N\Big\}=$$
$$=\inf\left\{f(x^1,..,x^1):x\otimes...\otimes x=x^1\otimes...\otimes x^1,\ \ x^1\in K\right\}=f(x,...,x)$$

for $x\in K$. The lemma is proved.

Lemma 2. *If X is a real separable Banach space, n odd, $K\subset X$ a convex closed salient cone, $f:K\times...\times K\to R$ an n-positive homogeneous function, then $f_s(x\otimes...\otimes x)=f(x,...,x)$ for $x\in K$.*

Proof. Using corollary 3.2 and lemma 3.5 we obtain

$$f_S(x\otimes...\otimes x)=$$
$$=\inf\left\{\sum_{i=1}^{m} f(x^i,...,x^i)\ :\ x\otimes...\otimes x=\sum_{i=1}^{m} x^i\otimes...\otimes x^i,\ \ x^i\in K,\, m\in N\right\}=$$
$$=\inf\Big\{\sum_{i=1}^{m} f_1(x^i,...,x^i)\ :\ x\otimes...\otimes x=\sum_{i=1}^{m} x^i\otimes...\otimes x^i,\ \ \left\{x^i\otimes...\otimes x^i\right\}_{i=1}^{m}-$$
$$-lin.\,indepen.,\ x^i\in K, m\in N\Big\}=$$
$$=\inf\left\{f(x^1,..,x^1):x\otimes...\otimes x=x^1\otimes...\otimes x^1,\ \ x^1\in K\right\}=f(x,...,x)$$

for $x\in K$. The lemma is proved.

Remark 1. If X is a real Banach space, n odd, $K\subset X$ a convex closed salient cone and the cone K allows plastering, $f:K\times...\times K\to R$ an n-positive-homogeneous function, then $f_s(x\otimes...\otimes x)=f(x,...,x)$ for $x\in K$.

Assume $M_+=\{x\otimes...\otimes x\ :x\in K\}$.

Lemma 3. *If n is even, then coM_+ is closed in space $(X\otimes...\otimes X)_s$.*

Proof. It is clear that, E_α, $\alpha\in A$, is a Banach space with respect to the induced topology from Banach space X (see §2).

It is known (see [13], p. 9) that this topology is induced by any Euclidian metrics determined by the scalar product, introduced in $E_\alpha\times E_\alpha$ with the help of positive-definite symmetric bilinear function $b(x,y)$. Similarly, from 2.6.4 and 2.6.5 [8] we conclude that $\{\nu_k\} \subset X \otimes ... \otimes X$ converges to ν with respect to the topology in $(X \otimes ... \otimes X)_s$ if for some $\alpha \in A$ $\{\nu_k\}$ converges to ν in $E_\alpha \otimes ... \otimes E_\alpha$. Assume $|x|_e = \sqrt{b(x,x)}$ and

$$\|\nu\|_e = \inf \left\{ \sum_{i=1}^{m} |x_1^i|_e ... |x_n^i|_e : \ \nu = \sum_{i=1}^{m} x_1^i \otimes ... \otimes x_n^i, \quad x_1^i, ..., x_n^i \in E_\alpha, \ m \in N \right\}.$$

It is known that if $\dim E_\alpha = k$ and $\nu = \sum\limits_{i=1}^{s} x^i \otimes ... \otimes x^i \in E_\alpha \otimes ... \otimes E_\alpha$, where $\{x^i \otimes ... \otimes x^i\}_{i=1}^{s}$ are linearly independent, then $s \le k^n$. Except for that, from the proof of lemma 3.5 it follows that such representation exists.

If n is even, $\nu = \sum\limits_{i=1}^{l} x_1^i \otimes ... \otimes x_n^i$ and $\tilde{b}(x_1, ..., x_n) = b(x_1, x_2)...b(x_{n-1}, x_n)$ from the Cauchy-Schwartzs inequality, it follows that

$$\tilde{b}(\nu) = \sum_{i=1}^{l} b(x_1^i, x_2^i)...b(x_{n-1}^i, x_n^i) \le \sum_{i=1}^{l} |x_1^i|_e ... |x_{n-1}^i|_e \cdot |x_n^i|_e .$$

Therefore $\tilde{b}(\nu) = \sum\limits_{i=1}^{l} b(x_1^i, x_2^i)...b(x_{n-1}^i, x_n^i) \le \|\nu\|_e$. Hence it follows that if $\nu = \sum\limits_{i=1}^{l} x^i \otimes ... \otimes x^i$, then $\|\nu\|_e = \sum\limits_{i=1}^{l} |x^i|_e^n$.

Let $\nu_m \in coM_+$ and $\nu_m \to \nu$. Then there exists $\alpha > 0$ such that $\nu_m \in \in E_\alpha \otimes ... \otimes E_\alpha$ and $\nu_m \to \nu$. Let $\dim E_\alpha = k$. If $x_m^i \in E_\alpha \bigcap K$ at $i = \overline{1, k^n}$, are such that $\nu_m = \sum\limits_{i=1}^{k^n} x_m^i \otimes ... \otimes x_m^i$, then we have $\|\nu_m\|_e = \sum\limits_{i=1}^{k^n} |x_m^i|_e^n$. As ν_m converges to ν, there exists $\lambda \in R_+$ such that $\|\nu_m\|_e \le \lambda$, i.e $\sum\limits_{i=1}^{k^n} |x_m^i|_e^n \le \lambda$. Hence it follows that $|x_m^i|_e \le \sqrt[n]{\lambda}$. Therefore, choosing convergent subsequences $\{x_{m_k}^1\}$ from $\{x_m^1\}$, $\left\{x_{m_{k_s}}^2\right\}$ from $\{x_{m_k}^2\}$, $\left\{x_{m_{k_{s_j}}}^3\right\}$ from $\left\{x_{m_{k_s}}^3\right\}$ and continuing the process, we can assume that x_m^i converges to $\bar{x}^i \in E_\alpha \bigcap K$ and $x_m^i \otimes ... \otimes x_m^i$ converges to $\bar{x}^i \otimes ... \otimes \bar{x}^i$ at $i = \overline{1, k^n}$. It is clear that $\nu = \sum\limits_{i=1}^{r^n} \bar{x}^i \otimes ... \otimes \bar{x}^i$, i.e $\nu \in coM_+$.

As the norms $\|\cdot\|$, $s(\cdot)$ and $\|\cdot\|_e$ are equivalent in $E_\alpha \otimes ... \otimes E_\alpha$, then we conclude that $coM_+ \bigcap E_\alpha \otimes ... \otimes E_\alpha$ is closed in $E_\alpha \otimes ... \otimes E_\alpha$. Therefore coM_+ is closed in space $(X \otimes ... \otimes X)_s$. The lemma is proved.

Lemma 4. *If $K \subset X$ is a convex closed salient cone, the cone K allows plastering, n is odd, then coM_+ is closed in space $(X \otimes ... \otimes X)_s$.*

Proof. It is clear that E_α, $\alpha \in A$, is a Banach spaces with respect to the induced topology from Banach space X (see §2).

It is known (see [13], p. 9) that this topology is induced by any Euclidian metrics determined by the scalar product introduced in $E_\alpha \times E_\alpha$ with the help of positive definite symmetric bilinear function $b(x, y)$. Assume $|x|_e = \sqrt{b(x,x)}$ and

$$\|\nu\|_e =$$

$$= \inf\left\{\sum_{i=1}^{m} \left|x_1^i\right|_e ... \left|x_{n-1}^i\right|_e \left\|x_n^i\right\| : \nu = \sum_{i=1}^{m} x_1^i \otimes ... \otimes x_n^i,\ x_1^i, ..., x_n^i \in E_\alpha,\ m \in N\right\}.$$

From lemma 3.5 it follows that if $\dim E_\alpha = k$ and $\nu = \sum_{i=1}^{s} x^i \otimes ... \otimes x^i \in \in E_\alpha \otimes ... \otimes E_\alpha$, where $\{x^i \otimes ... \otimes x^i\}_{i=1}^{k}$ are linearly independent and $x^i \in E_\alpha \bigcap K$, then $s \leq k^n$. Furthermore, from the proof of lemma 3.5 it follows that such representation exists.

Let $\nu_m \in co\ M_+$ and $\nu_m \to \nu$. From the definition of $\|\nu_m\|_e$ it follows that for any $\delta > 0$ there exist $\tilde{x}_j^i \in E_a$, $i = \overline{1,l}$, $j = \overline{1,n}$, such that $\nu_m = \sum_{i=1}^{l} \tilde{x}_1^i \otimes ... \otimes \tilde{x}_n^i$ and $\sum_{i=1}^{l} \left|\tilde{x}_1^i\right|_e ... \left|\tilde{x}_{n-1}^i\right|_e \left\|\tilde{x}_n^i\right\| < \|\nu_m\|_e + \delta$. As $\nu_m \to \nu$, there exists $\alpha > 0$ such that $\nu_m \in E_\alpha \otimes ... \otimes E_\alpha$ and $\nu_m \to \nu$ in $E_\alpha \otimes ... \otimes E_\alpha$. Introduce the notations $\dim E_\alpha = k$. Let $x_m^i \in K$,$i = \overline{1, k^n}$, be such that $\nu_m = \sum_{i=1}^{k^n} x_m^i \otimes ... \otimes x_m^i$. According to theorem 5.6 (see [11], p. 40) there exists $x* \in X^*$ and $a > 0$ such that $x^*(x) \geq a\|x\|$ for $x \in K$. Then assuming $z^*(x_1, ..., x_n) = b(x_1, x_2)...b(x_{n-2}, x_{n-1})x^*(x_n)$ we conclude that $z^*(\nu_m) = z^*(\sum_{i=1}^{l} x_m^i \otimes ... \otimes x_m^i) < \|x^*\| (\|\nu_m\|_e + \delta)$. Introduce the notations $\varepsilon = \inf\{x^*(x)\ : x \in K, \|x\| = 1\}$ and $I_m = \{i \in \overline{1, k^n}\ : x^*(x_m^i) \geq \varepsilon\}$, we have $\|x^*\| (\|\nu_m\|_e + \delta) \geq \varepsilon \sum_{i \in I_m} \left|x_m^i\right|_e^{n-1}$. By the data ν_m converges to ν. Then there exists $\lambda \in R_+$ such that $\|\nu_m\|_e + \delta \leq \lambda$. We conclude that $\left|x_m^i\right|_e^{n-1} \leq \frac{\lambda}{\varepsilon}\|x^*\|$ for $i \in I_m$. It is clear that $\left\|x_m^i\right\| \leq 1$ for $i \in J_m$, where $J_m = \{i \in \in \overline{1, k^n}\ : x^*(x_m^i) \leq \varepsilon\}$. (If $z^*(\nu_m) = z^*(\sum_{i=1}^{l} x_m^i \otimes ... \otimes x_m^i) < \|x^*\| (\|\nu_m\|_e + \delta)$, then it is easily verified that $a\sum_{i=1}^{l} \left|x_m^i\right|_e^{n-1} \left\|x_m^i\right\| < \|x^*\| (\|\nu_m\|_e + \delta)$. Therefore, $a\left|x_m^i\right|_e^{n-1} \left\|x_m^i\right\| < \|x^*\| (\|\nu_m\|_e + \delta) \leq \lambda$, i.e. the sequence $\left\{x_m^i\right\}$ is also bounded). Therefore, without loss of generality, we can assume that x_m^i converges to $\bar{x}^i \in K$ and $x_m^i \otimes ... \otimes x_m^i$ converges to $\bar{x}^i \otimes ... \otimes \bar{x}^i$. Therefore, there exist $\bar{x}^i \in K$, $i = \overline{1, k^n}$, such that $\nu = \sum_{i=1}^{k^n} \bar{x}^i \otimes ... \otimes \bar{x}^i$, i.e $\nu \in co\, M_+$. As the norms $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent in $E_\alpha \otimes ... \otimes E_\alpha$, we conclude that $co\, M_+$ is closed in $(X \otimes ... \otimes X)_s$. The lemma is proved.

Remark 2. Let X be a Banach space, $K \subset X$ a convex closed salient cone. If $0 \notin c\bar{o}\{x \in K\ : \|x\| = 1\}$, then according to the separation theorem (see [10], p. 71) there exists $x* \in X$ such that $\varepsilon = \inf\{x^*(x)\ : x \in K, \|x\| = 1\} > 0$. Therefore, if n is odd, then similarly to lemma 4 we have that the set coM_+ is closed in space $(X \otimes ... \otimes X)_s$.

Remark 3. Let $K \subset R^n$ be a convex closed salient cone. According to theorem 5.9 (see [11], p. 42) there exists $x* \in X$ such that $x^*(z) > 0$ for $z \in$ K, $z \neq 0$. Then $\varepsilon = \min\{x^*(x)\ : x \in K, \|x\| = 1\} > 0$. If n is odd, then similarly to lemma 4 we have that the set coM_+ is closed in space $(R^n \otimes ... \otimes R^n)$.

Lemma 5. *Let X be a separable Banach space, $K \subset X$ a convex closed salient* *cone, n odd, $f: K\times ...\times K\to R$ an n-positive homogeneous lower semicontinuous function, there exists $\alpha>0$ such that $\alpha\left\|x\right\|^n\le f(x,...,x)$ for $x\in X$ and $\nu\in coM_+$. Then there exists $\{x^i\}_{i=1}^{r^n}\subset K$, where $r=rid\ \nu$ and $\nu=\sum\limits_{i=1}^{r^n}x^i\otimes...\otimes x^i$ such that $f_S(\nu)=\sum\limits_{i=1}^{r^n}f(x^i,...,x^i)$.*

Proof. If $\nu=\sum\limits_{i=1}^{k}x^i\otimes...\otimes x^i$, where $\{x^i\otimes...\otimes x^i\}_{i=1}^k$ are linearly independent, we conclude that $k\le r^n$. Therefore, according to lemma 3.5 we obtain

$$f_S(\nu)=\inf\left\{\sum_{i=1}^{r^n}f(x^i,...,x^i):\ \ \nu=\sum_{i=1}^{r^n}x^i\otimes...\otimes x^i,\ \ x^i\in K\right\}.$$

By the definition of $f_S(\nu)$, there exist $x_m^i\in K$, $1\le i\le r^n$, where $\nu=\sum\limits_{i=1}^{r^n}x_m^i\otimes...\otimes x_m^i$ such that $\sum\limits_{i=1}^{r^n}f(x_m^i,...,x_m^i)\le f_S(\nu)+\frac{1}{m}$. By the data, we have that $\sum\limits_{i=1}^{r^n}\alpha\left\|x_m^i\right\|^n\le\sum\limits_{i=1}^{r^n}f(x_m^i,...,x_m^i)\le f_S(\nu)+1$. Therefore, $\alpha\sum\limits_{i=1}^{r^n}\left\|x_m^i\right\|^n\le f_S(\nu)+1$. Hence we conclude that $\left\|x_m^i\right\|\le\sqrt[n]{\frac{f_S(\nu)+1}{\alpha}}$.

Let $\nu=\sum\limits_{j=1}^{k}x^j\otimes...\otimes x^j$, $rid\ \nu=r$ and $\{x^1,....,x^r\}$ be linearly independent. Then there exist $\alpha_{ij}\in R$ such that $x^i=\alpha_{1i}x^1+...+\alpha_{ri}x^r$ for $i=\overline{r+1,k}$. According to theorem 5.9 (see [11], p.42) there exists $x*\in X$ such that $x^*(z)>0$ for $z\in$ K, $z\ne0$.

From equality $\sum\limits_{i=1}^{r^n}x_m^i\otimes...\otimes x_m^i=\sum\limits_{j=1}^{k}x^j\otimes...\otimes x^j$ it follows that

$$\sum_{i=1}^{r^n}x^*(x_m^i)x_m^i\otimes...\otimes x_m^i=\sum_{j=1}^{k}x^*(x^j)x^j\otimes...\otimes x^j,$$

where the number x^j in the equality is even. Then

$$\sum_{i=1}^{r^n}x^*(x_m^i)x_m^i\otimes...\otimes x_m^i=\sum_{j=1}^{r}x^*(x^j)x^j\otimes...\otimes x^j+$$

$$+\sum_{i=r+1}^{k}x^*(x^i)(\alpha_{1i}x^1+...+\alpha_{ri}x^r)\otimes x^i...\otimes x^i,$$

where $\{x^i\}_{i=1}^r$ are linearly independent. Hence it follows that $x_m^i\in Lin\{x^1,...,x^r\}$. Therefore, choosing a convergent subsequence $\left\{x_{m_k}^1\right\}$ from $\left\{x_m^1\right\}$, $\left\{x_{m_{k_s}}^1\otimes...\otimes x_{m_{k_s}}^1\right\}$ from $\left\{x_{m_k}^1\otimes...\otimes x_{m_k}^1\right\}$, $\left\{x_{m_{k_{s_j}}}^2\right\}$ from $\left\{x_{m_{k_s}}^2\right\}$, $\left\{x_{m_{k_{s_{j_l}}}}^2\otimes...\otimes x_{m_{k_{s_{j_l}}}}^2\right\}$ from $\left\{x_{m_{k_{s_j}}}^2\otimes...\otimes x_{m_{k_{s_j}}}^2\right\}$ and continuing the process, we can assume that x_m^i converges to $\bar{x}^i \in X$ and $x^i_m \otimes ... \otimes x^i_m$ converges to $\bar{x}^i \otimes ... \otimes \bar{x}^i$ (According to lemma 3.4, it is enough to choose convergent subsequences $\left\{x^1_{m_k}\right\}$ from $\left\{x^1_m\right\}$, $\left\{x^2_{m_{k_s}}\right\}$ from $\left\{x^2_{m_k}\right\}$, $\left\{x^3_{m_{k_{s_j}}}\right\}$ from $\left\{x^3_{m_{k_s}}\right\}$ and so on). It is clear that $\nu = \sum\limits_{i=1}^{r^n} \bar{x}^i \otimes ... \otimes \bar{x}^i$ and

$$f_S(\nu) \geq \varliminf_{m\to\infty} \sum_{i=1}^{r^n} f(x^i_m, ..., x^i_m) \geq \sum_{i=1}^{r^n} \varliminf_{m\to\infty} f(x^i_m, ..., x^i_m) \geq \sum_{i=1}^{r^n} f(\bar{x}^i, ..., \bar{x}^i) .$$

Hence it follows that $f_S(\nu) = \sum\limits_{i=1}^{r^n} f(\bar{x}^i, ..., \bar{x}^i)$. The lemma is proved.

The following lemma 6 and lemma 7 is proved similarly to the proof of lemma 5.

Lemma 6. *Let a convex closed salient cone $K \subset X$ allow plastering, n be odd, $f : K \times ... \times K \to R$ an n-positive homogeneous lower semicontinuous function, there exist $\alpha > 0$ such that $\alpha \|x\|^n \leq f(x, ..., x)$ for $x \in X$ and $\nu \in coM_+$. Then there exist $\{x^i\}_{i=1}^{r^n} \subset K$, where $r = rid\ \nu$ and $\nu = \sum\limits_{i=1}^{r^n} x^i \otimes ... \otimes x^i$, such that $f_S(\nu) = \sum\limits_{i=1}^{r^n} f(x^i, ..., x^i)$.*

Lemma 7. *Let $K \subset X$ be a convex closed salient cone, n even, $f : K \times ... \times K \to R$ an n-positive homogeneous lower semicontinuous function, there exists $\alpha > 0$ such that $\alpha \|x\|^n \leq f(x, ..., x)$ for $x \in X$ and $\nu \in coM_+$. Then, there exists $\{x^i\}_{i=1}^{r^n} \subset K$, where $r = rid\ \nu$ and $\nu = \sum\limits_{i=1}^{r^n} x^i \otimes ... \otimes x^i$, such that $f_S(\nu) = \sum\limits_{i=1}^{r^n} f(x^i, ..., x^i)$.*

Proposition 1. *Let $K \subset X$ be a convex closed salient cone (cone K allows plastering if n is odd), $f : K \times ... \times K \to R$ an n-positive homogeneous lower semicontinuous function, there exists $\alpha > 0$ such that $\alpha \|x\|^n \leq f(x, ..., x)$ for $x \in K$. Then $f_S(\nu)$ is lower semicontinuous function in $co\,M_+$ with respect to the topology of space $(X \otimes ... \otimes X)_s$.*

Proof. We are to show that $S(f_S, \lambda) = \{\nu \in co\,M_+ : f_S(\nu) \leq \lambda\}$ is a closed set for $\lambda \in R_+$. Let $\nu_k \in S(f_S, \lambda)$ and ν_k converges to ν in space $(X \otimes ... \otimes X)_s$. Then there exists $\alpha \in A$ such that $\nu_k \in E_\alpha \otimes ... \otimes E_\alpha$ and ν_k converges to ν in space $E_\alpha \otimes ... \otimes E_\alpha$, where $\dim E_\alpha = r < +\infty$.According to lemma 7 (or lemma 6), there exist $\left\{x^i_k \otimes ... \otimes x^i_k\right\}$, where $x^i_k \in K$ and $\nu_k = \sum\limits_{i=1}^{r^n} x^i_k \otimes ... \otimes x^i_k$, such that $f_S(\nu_k) = \sum\limits_{i=1}^{r^n} f(x^i_k, ..., x^i_k)$. Similarly to the proof of lemma 7 (or lemma 5), we obtain that $x^i_k \in E_\alpha$ at $j = \overline{1, r^n}$. By the data, we have $\sum\limits_{i=1}^{r^n} \alpha \left\|x^i_k\right\|^n \leq \sum\limits_{i=1}^{r^n} f(x^i_k, ..., x^i_k) \leq \lambda$. Therefore $\alpha \sum\limits_{i=1}^{r^n} \left\|x^i_k\right\|^n \leq \lambda$. Hence we have $\left\|x^i_k\right\| \leq \sqrt[n]{\frac{\lambda}{\alpha}}$. Without loss of generality, we assume (see the proof of lemma 3 and 4) that x^i_m converges to $\bar{x}^i \in K$ and $x^i_m \otimes ... \otimes x^i_m$ converges to $\bar{x}^i \otimes ... \otimes \bar{x}^i$. Then $\nu_k = \sum\limits_{i=1}^{r^n} x^i_k \otimes ... \otimes x^i_k$ converges to

$\nu=\sum\limits_{i=1}^{r^n}\bar{x}^i\otimes...\otimes\bar{x}^i$. Therefore

$$\varliminf_{k\to\infty} f_S(\nu_k)=\varliminf_{k\to\infty}\sum_{i=1}^{r^n} f(x_k^i,...,x_k^i)\geq$$

$$\geq\sum_{i=1}^{r^n}\varliminf_{k\to\infty} f(x_k^i,...,x_k^i)\geq\sum_{i=1}^{r^n} f(\bar{x}^i,...,\bar{x}^i)\ \geq f_S(\nu).$$

Hence it follows that $f_S(\nu)\leq\lambda$, i.e $\nu\in S(f_S,\lambda)$. We conclude that the set $S(f_S,\lambda)$ is closed. The proposition is proved.

According to the condition of proposition 1, $f_S(\nu)$ is a lower semicontinuous function in $co\,M_+$. Assuming $g(\nu)=\begin{cases} f_S(\nu)\ :\ \nu\in co\,M_+,\\ +\infty\ :\quad \nu\notin co\,M_+\end{cases}$, according to the condition of proposition 1, we have that $g(\nu)$ is a lower semicontinuous sublinear function in $(X\otimes...\otimes X)_s$. Then according to the Hormander theorem it follows that $f(x,...,x)=\sup\left\{Q(x)\ :\ Q\in\bar{\partial}_n\varphi\right\}$, where $\varphi(x)=f(x,...,x)$ for $x\in$ K and $\varphi(x)=+\infty$ for $x\notin K$, $\bar{\partial}_n\varphi=\{Q\in B_1(X^n):\ \varphi(x)\geq Q(x)\ \text{ for }\ x\in X\}$, $B_1(X^n)=\{Q:\ Q(x)=x^*(x,...,x),\ x^*\in\ clB(X^n,R)=(X\otimes...\otimes X)_s^*\}$.

Lemma 8. *If $f:K\times...\times K\to R$ is an n-positive-homogeneous continuous function, then there exists $\alpha>0$ such that $|f(x_1,...,x_n)|\leq\alpha\left\|x_1\right\|...\left\|x_n\right\|$ for $(x_1,...,x_n)\in K\times...\times K$.*

Lemma 9. *If $f:K\times...\times K\to R$ is an n-positive-homogeneous lower semicontinuous function, then there exists α such that $\alpha\left\|x_1\right\|...\left\|x_n\right\|\leq\ f(x_1,...,x_n)$ for $(x_1,...,x_n)\in K\times...\times K$.*

The proof of theorem 4. Assuming $f^\alpha(x_1,...,x_n)=f(x_1,...,x_n)+(\alpha+$ $+1)b(x_1,...,x_n)$, according to proposition 1, we have that $f_S^\alpha(\nu)$ is a lower semicontinuous function in $co\,M_+$ with respect to the topology in $(X\otimes...\otimes X)_s$. As $f_S^\alpha(\nu)=f_S(\nu)+(\alpha+1)b_S(\nu)$ for $\nu\in co\,M_+$, we have that f_S is a lower semicontinuous function in $co\,M_+$. Assuming $g(\nu)=\begin{cases} f_S(\nu)\ :\ \nu\in co\,M_+,\\ +\infty\ :\quad \nu\notin co\,M_+\end{cases}$ we have that $g(\nu)$ is a lower semicontinuous function in $(X\otimes...\otimes X)_s$. Then according to Hormander theorem, it follows that $g(\nu)=\sup\{b(\nu):\ b\in\partial g\}$. As $g(x\otimes...\otimes x)=f(x,...,x)$ for $x\in K$, we have that $f(x,....,x)=\sup\left\{Q(x)\ :\ Q\in\bar{\partial}_n\varphi_\varepsilon\right\}$ for $x\in K$. Theorem 4 is proved.

Note that in theorem 3 the function b can be taken from $B(X^n,R)$.

From theorem 3 and lemma 9, the validity of the following corollary 1 follows.

Corollary 1. *Let X be a real Banach space, cone K allow plastering, f : $K\times...\times K\to R$ a lower semicontinuous n-positive homogeneous function. Then $f(x,...,x)=\sup\left\{Q(x)\ :\ Q\in\bar{\partial}_n\varphi\right\}$ for $x\in K$.*

In particular, if $X=R^k$ and $K\subset X$ is a convex closed salient cone, then introducing the notation

$$f_S(\nu)=\inf\left\{\sum_{i=1}^m f(x^i,...,x^i)\ :\ \ \nu=\sum_{i=1}^m x^i\otimes...\otimes x^i,\ \ m\in N,\ x^i\in K\right\}$$

and using lemmas 3.5 and a lemma 3.2, similarly to lemma 1, we conclude that $f_S(x \otimes ... \otimes x) = f(x,...,x)$ for $x \in K$. If $\varphi(x) = \begin{cases} f(x,...,x) & : x \in K, \\ +\infty : & x \notin K. \end{cases}$, we assume $d_n\varphi = \{Q \in B_0(X^n) : \varphi(x) \geq Q(x) \;\; for \;\; x \in X\}$.

The proof of theorem 5. As $K \subset R^k$ is a convex closed salient cone, then according to theorem 5.9 ([11], p.42), there exists $x^* \in X^*$ such that $x^*(z) > 0$ for $z \in$ K, $z \neq 0$. Then there exists a such that, $a = \min\{x^*(x) \; : x \in K, \|x\| = 1\} > 0$. It is clear that $x^*(x) \geq a\|x\|$ for $x \in K$, i.e. cone K allows plastering. Therefore $\|x\|^n \leq b(x,...,x)$ for $x \in K$, where $\frac{1}{a^n}(x^*(x))^n = b(x,...,x)$, $b \in B(X^n, R)$. As $f : K \times ... \times K \to R$ is an n-positive homogeneous lower semicontinuity function, then there exists α such that $\alpha\|x\|^n \leq f(x,...,x)$ for $x \in K$. If $X = R^k$, then $\bar{\partial}_n\varphi = d_n\varphi$. Therefore, the validity of theorem 5 follows from theorem 4, i.e. from theorem 4 it follows that $f(x,...,x) = \sup\{Q(x) \; : Q \in d_n\varphi\}$ for $x \in K$. Theorem 5 is proved.

Remark 4. Note that using the other definition of tensor product (see[9], p.38) one can also prove the obtained results.

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