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ON UNIFORMLY INTEGRABILITY OF FAMILY OF SOME NORMALIZED BOUNDARY FUNCTIONALS ASSOCIATED WITH NONLINEAR BOUNDARIES CROSSING BY RANDOM WALK

Abstract

In the paper we study the issues of finiteness of overshoot moments and also identical integrability of a family of boundary functional associated with nonlinear boundaries crossing by a random walk.

1. Introduction

Let $\xi_n,\ n \geq 1$ be a sequence of independent identically distributed random variables determined on some probability space (Ω, F, P).

Assume

$$S_0 = 0, \quad S_n = \sum_{k=1}^{n} \xi_k, \quad n \geq 1.$$

Consider a family of the first passage time

$$\tau_a = \inf\left\{n \geq 1 : S_n > f_a(n)\right\}, \tag{1}$$

where $f_a(t),\ t \geq 0$ is a family of positive nonlinear (non-random) functions (boundaries) from the growing parameter $a > 0$. We ‘ll assume that $\inf\{\varnothing\} = = \infty$.

Study of the issue of uniformly integrability of a family of boundary functionals associated with the first passage time τ_a of the form (1) was always of great theoretical and practical interest. This direction was investigated in ([1]-[9]).

In the present paper we study the issues of finiteness of the mean value of the overshoot $R_a = S_{\tau_a} - f_a(\tau_a)$ and also uniformly integrability of a family of boundary functionals associated with the passage of the random walk S_n for a nonlinear boundary $f_a(t)$.

Note that such problems under different suppositions for the boundary $f_a(t)$ were studied in [1-9].

2. Conditions and statement of the main results

We'll assume that $0 < \mu = E\xi_1 < \infty$, and for nonlinear boundary $f_a(t)$ the following regularity conditions are fulfilled:

1) For each a the function $f_a(t)$ is convex downwards and continuously-differentiable, and $f_a(1) \uparrow \infty$ as $a \to \infty$ and $f'_a(t) \geq 0$ for all $t > 0$.

2) For all a the function $\dfrac{f_a(t)}{t}$ monotonically decreases to zero as $t \to \infty$.

Denote by $N_a = N_a(\mu)$ the solution of the equation $f_a(n) = n\mu$ with respect to n, that exists and is unique by the made assumptions.

Assume

$$\xi^+ = \max(0, \xi) \ \text{ and } \ \xi^- = \max(0, -\xi).$$

The following results are valid.

Theorem 1. *Let the above-enumerated conditions for the boundary $f_a(t)$ and distribution of random variables ξ_n, $n \geq 1$ be fulfilled.*

1) If $E(\xi_1^-) < \infty$, then there exists a number $a_0 > 0$ such that the family $\left\{ \left(\frac{\tau_a}{N_a} \right), a \geq a_0 \right\}$ is uniformly integrable.

2) If $E(\xi_1^+) < \infty$, then there exists a number $a_0' > 0$ such that the family

$$\left\{ \left(\frac{\xi_{\tau_a}}{N_a} \right), \quad a \geq a_0' \right\}$$

is uniformly integrable.

3) It $E|\xi_1| < \infty$, then there exists a number $a_0'' >$ such that the family

$$\left\{ \left(\frac{\xi_{\tau_a}}{N_a} \right), \quad a \geq a_0'' \right\}$$

is uniformly integrable.

Theorem 2. *Let the above-enumerated conditions for the boundary $f_a(t)$ and distribution of random variables ξ_n, $n \geq 1$ be fulfilled.*

Assume that $E\left(\xi_1^+\right) < \infty$ and $f_a'(N_a) \to \theta \in [0, \mu)$ as $a \to \infty$.

Then

$$1) \quad ER_a < \infty \quad \text{for all } a > 0;$$

3. Proof of the main results

For proving theorem 1 and 2, we'll need the following fact formulated in the form of lemmas.

Denote

$$v_a = \inf \{n \geq 1 : S_n > a\}$$

is the linear first passage time of the random walk S_n for the level $a > 0$.

Lemma 1. *Let $E\left(\xi_1^-\right) < \infty$. Then the family*

$$\left\{ \left(\frac{v_a}{a} \right), \ a \geq 1 \right\}$$

is uniformly integrable.

Lemma 2. *Let X_n, $n \geq 1$ and Y_n, $n \geq 1$ be sequences of positive random variables such that the families*

$$\{X_n, \ n \geq 1\} \text{ and } \{Y_n, \ n \geq 1\}$$

are uniformly integrable. Then the family

$$\{(X_n + Y_n), \ n \geq 1\}$$

is also uniformly integrable.

Lemma 3. *Let $X_n \xrightarrow{P} X$ and the family $\{|X_n|, \ n \geq 1\}$ be uniformly integrable. Then*

$$E|X_n| \to E|X| \quad as \quad n \to \infty$$

These lemmas were proved in [4].

Lemma 4. *Let* $\left(\xi_1^+\right) < \infty$. *Then*

$$\frac{\xi_{\tau_a}}{(\tau_a)} \xrightarrow{a.s.} 0 \quad as \quad n \to \infty.$$

The confirmation of this lemma follows from [6].

Proof of theorem 1. For proving statement 1) we consider the following first passage time

$$T_a^{(1)} = \inf\left\{n \geq 1 : S_n > f_a(N_a) + f'_a(N_a)(n - N_a)\right\}.$$

It is clear that taking into account $f_a(N) = \mu N_a$, we can write

$$T_a^{(1)} = \inf\left\{n \geq 1 : S_n^{(1)} > N_a\right\}, \tag{1}$$

where

$$S_n^{(1)} = \sum_{k=1}^{n} \xi_k^{(1)} \quad \text{and} \quad \xi_k^{(1)} = \frac{\xi_k - f'_a(N_a)}{\mu - f'_a(N_a)}.$$

It is clear that $E\xi_k^{(1)} = 1$, and the step $\xi_k^{(1)}$ of the random walk $S_n^{(1)}, \quad n \geq 1$ depends on the parameter a. The linear boundary $y(t) = f_a(N_a) + f'_a(N_a)(t- -N_a)$ as a function of t is tangent to the nonlinear: boundary $f_a(t)$ at the point $(N_a, f_a(N_a))$.

As the boundary $f_a(t)$ is convex downwards (concave), we have

$$\tau_a \leq T_a^{(1)}, \tag{2}$$

since $f_a(n) \geq f_a(N_a) + f'_a(N_a) + f'_a(N_a)(n - N_a)$.

Further, consider the random variables

$$\xi_k^{(2)} = \frac{\xi_k - (\varepsilon + \theta)}{\mu - (\theta - \varepsilon)}, \quad k = 1, 2, ..., \quad 0 < \varepsilon < \mu - \theta$$

and the random walk

$$S_n^{(2)} = \sum_{k=1}^{n} \xi_k^{(2)}.$$

It is clear that $\xi_k^{(2)}$ is independent of the parameter a.

Consider

$$T_a^{(2)} = \inf\left\{n \geq 1 : \quad S_n^{(2)} > N_a\right\}.$$

From the condition $f'_a(N_a) \to \theta \in [0, \mu)$ as $a \to \infty$ it follows that there exists a number $a_1 > 0$ such that for $a > a_1$ it is fulfilled

$$0 \leq f'_a(N_a) < \mu.$$

It is clear that fro any $0 < \varepsilon < \mu - \theta$ there exists a number $a_2 \geq a_1$ such that for $a \geq a_2$ it holds

$$\theta - \varepsilon < f'_a(N_a) < \theta + \varepsilon.$$

For $a \geq a_2$ we have
$$\xi_k^{(2)} \leq \xi_k^{(1)}.$$

Consequently,
$$T_k^{(1)} \leq T_a^{(2)} \quad \text{for} \quad a \geq a_2.$$

It follows from (2) that for $a \geq a_2$
$$\tau_a \leq T_a^{(2)}. \tag{3}$$

From lemma 1, the family
$$\left(\frac{T_a^{(2)}}{N_a} \right), \quad a \geq a_2$$
is uniformly integrable. Then from (3) the family
$$\left(\frac{\tau_a}{N_a} \right), \quad a \geq a_2$$
is also uniformly integrable. Therefore, statement 1) of theorem 1 is fulfilled with $a_0 = a_2$.

Prove statement 2 of theorem 1. By definition of the variable τ_a and from monotonicity of the boundary $f_a(t)$ we have
$$0 < S_{\tau_a} - f_a(\tau_a) \leq S_{\tau_a} - f_a(\tau_a - 1) \leq$$
$$\leq S_{\tau_a} - S_{\tau_a - 1} = \xi_{\tau_a} \tag{4}$$

From (4) it follows that
$$\xi_{\tau_a} = \xi_{\tau_a}^+.$$

It is easy to see that
$$\xi_{\tau_a} \leq \left(\xi_1^+\right) + ... + \left(\xi_1^+\right) \tag{5}$$

It is known that [5] $E\tau_a < \infty$ for each $a > 0$, since $E\left|\xi_1\right| < \infty$. From the Wald's identity and (5) we get
$$E\xi_{\tau_a} \leq E\tau_a \cdot E\left(\xi_1^+\right)$$
or
$$E\left(\frac{\xi_{\tau_a}}{N_a}\right) \leq E\left(\frac{\tau_a}{N_a}\right) E\left(\xi_1^+\right). \tag{6}$$

From statement 1) it follows that the family $\frac{\tau_a}{N_a}$, $a \geq a_0$ is uniformly integrable. Therefore, statement 2) of the proved theorem follows from (6).

Prove statement 3). From the assumptions made for the boundary $f_a(t)$ it follows that there exists the numbers $\delta \in (0, \mu)$ and c_0 such that
$$f_a(t) \leq c_0 + \delta t, \quad t > 0. \tag{7}$$

From (4) it follows that
$$f_a(\tau_a) < S_{\tau_a} \leq f_a(\tau_a - 1) + \xi_{\tau_a}.$$

Hence, from (7) we get

$$S_{\tau_a} \leq c_0 + \delta(\tau_a - 1) + \xi_{\tau_a} \leq c_0 + \delta\tau_a + \xi_{\tau_a}. \tag{8}$$

Then we have

$$0 \leq \frac{S_{\tau_a}}{N_a} \leq \frac{c_0}{N_a} + \delta\frac{\tau_a}{N_a} + \frac{\xi_{\tau_a}}{N_a} \tag{9}$$

From condition $E|\xi_1| < \infty$it follows that by statements 1) and 2) the families

$$\left(\frac{\tau_a}{N_a}\right), \quad a \geq a_0$$

and

$$\left(\frac{\xi_{\tau_a}}{(N_a)}\right), \quad a \geq a_0$$

are uniformly integrable.

Then statement 3) of theorem 1 follows from lemma 2 and estimation (9).

Proof of theorem 2. From (4) it follows that

$$0 \leq R_a = S_{\tau_a} - f_a(\tau_a) \leq \xi_{\tau_a}.$$

Then

$$ER_a \leq E\xi_{\tau_a}. \tag{10}$$

From (6) it follows that

$$E\xi_{\tau_a} < \infty$$

for all $a > 0$. Therefore the affirmation of theorem 2 follows from (10).

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