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A HARDY TYPE GENERAL INEQUALITY IN $L^{p(\cdot)}(0,1)$ WITH DECREASING EXPONENT

Abstract

We derive a Hardy type inequality

$$\left\| W(.)^{-1}\sigma(.)^{\frac{1}{p(.)}} \int_0^x f(t)dt \right\|_{L^{p(\cdot)}(0,1)} \le C \left\| \omega(.)^{\frac{1}{p(.)}} f(.) \right\|_{L^{p(\cdot)}(0,1)}, f \ge 0.$$

for the exponent $p : (0,1) \to (1,\infty)$ *is a decreasing function on some interval* $(0,\epsilon)$, $\epsilon > 0$ *and* $\sigma = \omega(.)^{-\frac{1}{p(.)-1}} \in L^1(0,1)$, $W(x) = \int_0^x \sigma(t)dt$.

We study a Hardy type inequality

$$\left\| W(.)^{-1}\sigma(.)^{\frac{1}{p(.)}} \int_0^x f(t)dt \right\|_{L^{p(\cdot)}(0,1)} \le C \left\| \omega(.)^{\frac{1}{p(.)}} f(.) \right\|_{L^{p(\cdot)}(0,1)}, f \ge 0. \tag{1}$$

in the norms of variable exponent Lebesgue space $L^{p(\cdot)}(0,1)$, whenever the exponent p is a decreasing function on some interval $(0,\epsilon)$, $0 < \epsilon < 1$ and the functions $\sigma = \omega(.)^{-\frac{1}{p(.)-1}} \in L^1(0,1)$, $W(x) = \int_0^x \sigma(t)dt$.

As to the basic properties of spaces $L^{p(\cdot)}$, we refer to the works [2], [5], [17]. In this paper, we assume that $p(x)$ is a measurable function on $(0,1)$ and its values are in the interval $[1,\infty)$. Also $p^+ = \sup\{p(x) : x \in (0,1)\} < \infty$ and $p^- = \inf\{p(x) : x \in (0,1)\} > 1$. The space $L^{p(\cdot)}(0,1)$ is a class of measurable functions $f(x)$ on $(0,1)$ such that the modular $I_{p(.)}(f) = \int_0^1 |f|^{p(x)} dx$ is finite. A norm in $L^{p(\cdot)}(0,1)$ is defined as $\|f\|_{L^{p(\cdot)}(0,1)} = \inf\left\{\lambda > 0 : I_{p(.)}\left(\frac{f}{\lambda}\right) \le 1\right\}$. For $1 < p^-$, $p^+ < \infty$ the space $L^{p(\cdot)}(0,1)$ is a reflexive Banach space. For the function $1 \le p(x) < \infty$ $p'(x)$ denotes the conjugate function of $p(x)$, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and $p'(x) = \infty$ if $p(x) = 1$. We denote by $C, C_1, C_2, ...$ various positive constants whose values may vary at each appearance.

For a last time the variable exponent Hardy type inequalities was studied by several authors (see, f.e. [1], [3], [4], [6], [7], [8], [10], [11], [12], [13], [14], [15], [16]). There are several sufficient conditions on the function $p : (0,1) \to (1,\infty)$ for the inequality

$$\left\| x^{-1} \int_0^x f(t)dt \right\|_{L^{p(\cdot)}(0,1)} \le C \|f(.)\|_{L^{p(\cdot)}(0,1)}, \quad f \ge 0 \tag{2}$$

to hold. They are expressed in terms of regularity conditions for p at the origin. The inequality (2) follows from (1) for the case $\omega \equiv 1$. It follows from the results of works [3], [8], [14] (see, also [1], [11], [13]) that the inequality (2) holds if $p^- = \inf p > 1$, $p^+ = \sup p(x) < \infty$ and the condition

$$A := \limsup_{x\to 0} |p(x) - p(0)| \log\frac{1}{x} < \infty. \tag{3}$$

is satisfied.

In [10] (see, also [6]), Mamedov had proved that the condition

$$B := \limsup_{x\to 0} \left[p(x) - p\left(\frac{x}{2}\right)\right] \log\frac{1}{x} < \infty \tag{4}$$

is necessary for the inequality (2) if the exponent function p is increasing. The condition (4) is strictly weaker than (3). This condition is satisfied e.g. by $p(x) = p(0) + \frac{C}{\left(\ln\frac{1}{x}\right)^{\alpha}}$ and $\alpha > 0$, $C > 0$. For the exponent, that is increasing near the origin, the condition (4) is also sufficient if the number B be $B < p(0)\,(p(0)-1)$ (see, [10]). Unfortunately, such good condition (4) is no longer sufficient for the inequality (1) to hold if the condition on B be ignored. In this case, a necessary and sufficient condition is still an open problem.

Setting $\omega = x^{\beta(.)p(.)}$ in (1) we attain the inequality (9) if the condition (8) be satisfied for the functions p, β (see, Corollary). The inequality (9) also was much studied for a last time. Note a necessary and sufficient condition for (7) to hold is the condition

$$\beta(0) < 1 - \frac{1}{p(0)} \tag{5}$$

if the β, p satisfy (3)(see, e.g. [1], [3], [7], [8], [14]). It is interesting that the condition $\sigma \in L^1(0,1)$ replaces (5) in the Corollaries , . Notice the inequality (1) contains not only power type weights $x^{\beta(.)-1}, x^{\beta(.)}$ from the left and right hand sides respectively (the type of inequality (9)). For example, we can take a function σ for the inequalities (7) and (6) not necessarily power type.

In Theorem 1 below, we prove that the regularity condition is not needed if the exponent p is decreasing at small neighborhood of the origin.

Theorem 1. *Let $\omega : (0,1) \to (0,\infty)$ be a measurable function $\sigma = \omega^{-\frac{1}{p(.)-1}} \in L^1(0,1)$ and $W(x) = \int_0^x \sigma(t)dt$. Suppose $p : (0,1) \to [1,\infty)$ be decreasing on some interval $(0,\epsilon)$, $\epsilon > 0$; then it holds the inequality (1) for a positive measurable function f.*

Corollary 1. *Let $\omega : (0,1) \to (0,\infty)$ be a measurable function $\sigma = \omega^{-\frac{1}{p(.)-1}} \in L^1(0,1)$. Suppose $p : (0,1) \to [1,\infty)$ be decreasing on some interval $(0,\epsilon)$, $\epsilon > 0$ and the condition*

$$\int_0^x \sigma(t)dt \le Cx\sigma(x), \quad 0 < x < \epsilon \tag{6}$$

is satisfied; then it holds the inequality

$$\left\|\sigma^{-\frac{1}{p'(.)}}(.)\, x^{-1} \int_0^x f(t)dt\right\|_{L^{p(.)}(0,1)} \le C \left\|\sigma^{-\frac{1}{p'(.)}}(.) f(.)\right\|_{L^{p(.)}(0,1)}, f \ge 0. \tag{7}$$

for a positive measurable function f.

Corollary 2. *Let $\beta : (0,1) \to \boldsymbol{R}$ be a measurable function. Suppose $p : (0,1) \to [1,\infty)$ be decreasing on some interval $(0,\epsilon)$, $\epsilon > 0$ such that $\beta(x)p'(x) \le 1$ for $0 < x < 1$ and the condition*

$$\int_0^x t^{-\beta(t)p'(t)}dt \le Cx^{1-\beta(x)p'(x)}, \quad 0 < x < \epsilon \tag{8}$$

is satisfied. Then it holds the inequality

$$\left\|x^{\beta(.)-1} \int_0^x f(t)dt\right\|_{L^{p(.)}(0,1)} \le C \left\|x^{\beta(.)} f(.)\right\|_{L^{p(.)}(0,1)}, f \ge 0. \tag{9}$$

for a positive measurable function f.

Since there is a relation between modular and variable exponent norm (see, f.e. [17]):

$$\|f\|^{p^+}_{L^{p(.)}(0,l)} \le I_p(f) \le \|f\|^{p^-}_{L^{p(.)}(0,l)}, \quad 1 \ge \|f\|_{L^{p(.)}(0,l)} \tag{10}$$

$$\|f\|^{p^-}_{L^{p(.)}(0,l)} \le I_p(f) \le \|f\|^{p^+}_{L^{p(.)}(0,l)}, \quad 1 \le \|f\|_{L^{p(.)}(0,l)}. \tag{11}$$

we can perform our estimates in terms of the modular.

Proof of Theorem 1. To prove the inequality (1), it suffices to consider the case when f is a positive measurable function such that $\left\|\omega^{\frac{1}{p(.)}} f\right\|_{p(.)} \le 1$ (see, [2]). It follows from (10) that $I_{p(.)}\left(\omega^{\frac{1}{p(.)}} f\right) \le 1$. In order to prove Theorem 1 we have to show

$$\left\|W^{-1}\sigma^{\frac{1}{p(.)}} Hf\right\|_{L^{p(.)}(0,1)} \le C_1. \tag{12}$$

To prove (12), we shall establish the estimate

$$I_{p(.)}\left(W^{-1}\sigma^{\frac{1}{p(.)}} Hf\right) \le C_2.$$

Using the triangle inequality of $p(.)$-norms for $\delta \in (0,1)$, we have

$$\left\|W^{-1}(.)\sigma(.)^{\frac{1}{p(.)}} Hf(.)\right\|_{p(.)} \le \left\|W^{-1}(.)\sigma(.)^{\frac{1}{p(.)}} Hf(.)\right\|_{p(.);(0,\delta)} +$$

$$+\left\|W^{-1}(.)\sigma(.)^{\frac{1}{p(.)}} Hf(.)\right\|_{p(.);(\delta,1)} := i_1 + i_2.q \tag{13}$$

Define by $w(W)$ an inverse function for $W(w) = \int\limits_0^w \sigma(u)du$. A change of variable $z = W(x)t$ in the interior integral below gives

$$Hf(x) = \int_0^{W(x)} \frac{f(w(z))}{\sigma(w(z))} dz = W(x) \int_0^1 \frac{f(w(tW(x)))}{\sigma(w(tW(x)))} dt.$$

Using this and Minkowskii's inequality for $L^{p(.)}$ norms (see, f.e. [17], [2], [5], [9]), we see that

$$i_1 = \left\|W^{-1}(.)\sigma(.)^{\frac{1}{p(.)}} Hf(.)\right\|_{p(.);\,(0,\delta)} = \left\|\int_0^1 \sigma(.)^{\frac{1}{p(.)}} \frac{f(w(tW(.)))}{\sigma(w(tW(.)))} dt\right\|_{p(.);\,(0,\delta)}$$

$$\le \int_0^1 \left\|\sigma(.)^{\frac{1}{p(.)}} \frac{f(w(tW(.)))}{\sigma(w(tW(.)))}\right\|_{p(.);\,(0,\delta)} dt. \tag{14}$$

Now estimate the norm $\left\|W^{-1}(.)\sigma(.)^{\frac{1}{p(.)}} Hf(.)\right\|_{p(.);(0,\delta)}$ for $0 < t < 1$. Since p is decreasing on $(0,\delta)$, we have $p\left(w\left(\frac{W(v)}{t}\right)\right) \le p(v)$ for $v \in (0,\delta)$. Therefore,

$$\int_0^\delta \left(\frac{\sigma(x)^{\frac{1}{p(x)}}}{t^{-\frac{1}{p^-_{(0,\delta)}}}} \frac{f(w(tW(x)))}{\sigma(w(tW(x)))}\right)^{p(x)} dx = \int_0^\delta \left(\frac{f(w(tW(x)))}{\sigma(w(tW(x)))} t^{\frac{1}{p^-_{(0,\delta)}}}\right)^{p(x)} \sigma(x)dx$$

$$\leq \int_0^\delta \left(\frac{f(w(tW(x))}{\sigma(w(tW(x))}\right)^{p(x)} t\sigma(x)dx = \left[\begin{array}{c} w(tW(x)) = v, \quad tW(x) = W(v) \\ t\sigma(x)dx = dW(v) = \sigma(v)dv \end{array}\right]$$

$$\leq \int_0^{w(tW(\delta))} \left(\frac{f(v)}{\sigma(v)}\right)^{p(w(\frac{W(v)}{t}))} \chi_{\{\frac{f(v)}{\sigma(v)}\geq 1\}}\sigma(v)dv + \int_0^{w(tW(\delta))} \chi_{\{\frac{f(v)}{\sigma(v)}<1\}}\sigma(v)dv$$

$$\leq \int_0^\delta \left(\frac{f(v)}{\sigma(v)}\right)^{p(v)} \chi_{\{\frac{f(v)}{\sigma(v)}\geq 1\}}\sigma(v)dv + \int_0^\delta \chi_{\{\frac{f(v)}{\sigma(v)}<1\}}\sigma(v)dv \leq 1 + W(\delta),$$

where $p^-_{(0,\delta)}$ is minimum of p over $(0,\delta)$.

This implies

$$\int_0^\delta \left(\frac{\sigma(x)^{\frac{1}{p(.)}}}{(1+W(\delta))^{\frac{1}{p^-_{(0,\epsilon)}}} t^{-\frac{1}{p^-_{(0,\delta)}}}} \frac{f(w(tW(x))}{\sigma(w(tW(x))}\right)^{p(x)} dx \leq 1, \ \ 0<t<1.$$

Therefore and using the definition of $p(.)$ -norms, we get

$$\left\| W(.)^{-1}\sigma(.)^{\frac{1}{p(.)}}(.)f(.\,t)\right\|_{p(.);\,(0,\delta)} \leq (1+W(\delta))^{\frac{1}{p^-_{(0,\epsilon)}}} t^{-\frac{1}{p^-_{(0,\delta)}}}, \ \ 0<t<1. \qquad (15)$$

Using (15) and (14) for the first summand in (13) we get the estimate

$$i_1 \leq (1+W(\delta))^{\frac{1}{p^-}} \int_0^1 t^{-\frac{1}{p^-_{(0,\delta)}}} dt \leq \left(p^-_{(0,\delta)}\right)' (1+W(\delta))^{\frac{1}{p^-_{(0,\delta)}}} \leq C. \qquad (16)$$

Let us estimate $\left\|W^{-1}(.)\sigma(.)^{\frac{1}{p(.)}}Hf(.)\right\|_{p(.);(\delta,1)}$. For $x\in(\delta,1)$ using Young's inequality, we get

$$\left(W^{-1}(x)\sigma(x)^{\frac{1}{p(x)}}Hf(x)\right)^{p(x)} = \sigma(x)W^{-p(x)}(x)\left(\int_0^x \frac{f(s)}{\sigma(s)}\sigma(s)ds\right)^{p(x)}$$

$$\leq \sigma(x)\,(1+W(1))^{\frac{p^+}{p^-}-1}\, W^{-p^+}(\delta)\int_0^x \frac{f(s)}{\sigma(s)}\sigma(s)ds$$

$$\leq C\sigma(x)\int_0^x \frac{1}{p(x)}\left(\frac{f}{\sigma}\right)^{p(x)}\sigma(s)ds + C\sigma(x)\int_0^x \frac{1}{p'(x)}\sigma(s)ds$$

$$\leq C_1\sigma(x),$$

since

$$\left(\int_0^x \left(\frac{f}{\sigma}\right)\sigma(s)ds\right) \leq 2\left\|\sigma^{-\frac{1}{p'(.)}}f\right\|_{p(.)}\left\|\sigma^{\frac{1}{p(.)}}\right\|_{p(.)} \leq 2\,(1+W(1))^{\frac{1}{p^-}}$$

and

$$W(x)\geq W(\delta) \qquad \text{for} \qquad x\in(\delta,1).$$

Therefore,

$$I_{p(.);(\delta,1)}\left(W^{-1}(.)\sigma(.)^{\frac{1}{p(.)}}Hf(.)\right) \leq C_2W(1).$$

Hence

$$\left\|W^{-1}(.)\sigma(.)^{\frac{1}{p(.)}}Hf(.)\right\|_{p(.);(\delta,1)} \leq C_3$$

Inserting this estimate and (16) in (13) we complete the proof of Theorem 1.

Proof of Corollary 2. This Corollary follows from the Corollary 2 by using of the inequality (6) in the the estimate (1).

Proof of Corollary 3. This Corollary follows from the Theorem 2 by using of the inequality (8) in the the estimate (7) for $\sigma(x) = x^{-\beta(x)p'(x)}$.

Acknowledgement. We express our gratitude to the anonym referee for useful comments and detail examination of the paper.

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Received: September 11, 2012; Revised: December 05, 2012.