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ON ONE PROPERTY OF OPERATOR BUNDLES OF MULTIPARAMETER SPECTRAL PROBLEM

Abstract

By separating the variables of partial differential equations, depending on the number of variables we get a multiparameter spectral problem, i.e. the system of differential equations that are connected with each other only by number parameters. Having applied the Green function for the solution of boundary value problems with partial differential equations, we get a system of weakly-connected integral equations of the form

$$\begin{cases} \sum\limits_{j=1}^{n} \lambda_j K_{ij}\varphi_i = \varphi_i, \qquad \varphi_i \in H_i \\ \qquad\qquad i = 1;...n; \end{cases}$$

Therefore, investigation of such problems is of great interest in spectral theory. In this paper we study one property of operator bundles that is obtained by studying such a multiparameter problem. We investigate the conditions under which the operators separating the spectral parameters are defined on all the space.

Let's consider the two-parameter problem

$$\begin{cases} \sum\limits_{j=1}^{2} \lambda_j K_{ij}\varphi_i = \varphi_i, \qquad \varphi_i \in H_i \\ \qquad\qquad i = 1;2; \end{cases}, \tag{1}$$

where K_{i1}, K_{i2} are compact self-adjoint operators in Hilbert space $H_i, i = 1;2$.

Introduce the denotation

$$\Delta_0 = \det\left(K_{i,j}\right)^{\otimes}_{\substack{i=1,2\\ j=1,2}}, \quad \Delta_1 = \det\begin{pmatrix} I_1 & K_{1,2} \\ I_2 & K_{2,2} \end{pmatrix}^{\otimes}, \quad \Delta_2 = \det\begin{pmatrix} K_{1,1} & I_1 \\ K_{2,1} & I_2 \end{pmatrix}^{\otimes},$$

Δ_0, Δ_1, Δ_2 are linear operators determined on Hilbert space $H = H_1 \otimes H_2$ (the sign $\otimes$ is a tensor product).

Let in problem (1) the definetess conditions be fulfilled in the form

$$\Delta_i > 0, \quad i = 0;1;2, \tag{2}$$

By studying spectral theory, especially by researching the variational principle for two-parameter spectral problem (1), the family of the operators $\Gamma_i = \Delta_0^{-1}\Delta_i$, $i = 1;2$ separating the spectral parameters λ_i, $i = 1;2$ (see e.i. [1], [2], [3]) plays an important role. It is known that under condition (2), the joint spectrum of the family of the operators $\Gamma_i = \Delta_0^{-1}\Delta_i$, $i = 1;2$ separating the spectral parameters λ_i, $i = 1;2$ coincides with the spectrum of problem (1). Δ_i, $i = 0;1;2$ are linear, bounded operators in space H. Therefore, the ranges of values $R(\Delta_i)$, $i=0;1;2$ of the operators Δ_i, $i=0;1;2$ are open everywhere dense in H line elements, respectively. But, generally speaking, this fact doesn't mean that the intersections $R(\Delta_0)\cap R(\Delta_1)$; $R(\Delta_0)\cap R(\Delta_2)$ are not empty sets, i.e. the domain of definition of the operators $\Gamma_i=\Delta_0^{-1}\Delta_i$, $i=1;2$ are not empty sets. Therefore, it is interesting to know under which conditions the operators $\Gamma_i=\Delta_0^{-1}\Delta_i$, $i=1;2$ may be defined in a rather wide set.

It is easy to see that in order the operators $\Gamma_i=\Delta_0^{-1}\Delta_i$, $i=1;2$ be defined on all the space H, it is necessary and sufficient that the relation $R(\Delta_0)\subset R(\Delta_1)$ $\cap R(\Delta_2)$ be fulfilled. Therefore, it is enough to find the conditions from which the condition $R(\Delta_0)\subset R(\Delta_1)\cap R(\Delta_2)$ is obtained.

Theorem 1. *If in problem (1) it holds condition (2) and the inequality*

$$\|\Delta_0\varphi\|\le\|\Delta_i\varphi\|\,,\quad i=1;2 \tag{3}$$

for $\forall\varphi\in H$, then the relation $R(\Delta_0)\subset R(\Delta_1)\cap R(\Delta_2)$ is fulfilled.

Proof. Let $\varphi^0\in H$ be an arbitrarily fixed element. Prove that there exists $\exists h\in H$ that $\Delta_0\varphi^0=\Delta_1 h$.

In Hilbert space H we introduce a new scalar product $[\varphi,\psi]=(\Delta_1\varphi,\psi)$, $\forall\varphi,\psi\in H$. This scalar product defines a new norm in H. Denote by $\|\varphi\|_{\Delta_1}$ the norm of the element $\varphi\in H$. The obtained normalized space is not complete. Denote by H_{Δ_1} the completion of this space. In space H_{Δ_1} the scalar product is determined in the form $[\varphi,\psi]_{\Delta_1}=\lim\limits_{n\to\infty}(\Delta_1\varphi_n,\psi_n)$, where $\varphi,\psi\in H_{\Delta_1}$, and $\{\varphi_n\}\subset H$, $\{\psi_n\}\subset H$ are fundamental sequences with respect to the norm $\|\varphi\|_{\Delta_1}$.

Let's consider the functional $l_{\varphi^0}(\psi)=\left(\Delta_0\varphi^0,\psi\right)$ determined in everywhere dense subset H of the space H_{Δ_1}. From the conditions $\Delta_i>0$, $i=1;2$ we can get $(-1)^{i+j}K_{i,j}\ge 0$. Then using the Cauchy-Bunyakovsky generalized inequality, we get that for $\forall\psi\in H$

$$\left|l_{\varphi^0}(\psi)\right|^2=\left|\left(\Delta_0\varphi^0,\psi\right)\right|^2\le\left|\left(K_{11}^tK_{22}^t\varphi^0,\psi\right)\right|^2+2\left|\left(K_{11}^tK_{22}^t\varphi^0,\psi\right)\right|\left|\left(K_{12}^tK_{21}^t\varphi^0,\psi\right)\right|+$$

$$+\left|\left(K_{12}^tK_{21}^t\varphi^0,\psi\right)\right|^2\le 2\left|\left(K_{11}^tK_{22}^t\varphi^0,\psi\right)\right|^2+\left|\left(K_{12}^tK_{21}^t\varphi^0,\psi\right)\right|^2\le$$

$$\le 2\left(K_{22}^tK_{11}^t\varphi^0,K_{11}^t\varphi^0\right)\left(K_{22}^t\psi,\psi\right)+2\left(-K_{12}^tK_{21}^t\varphi^0,K_{21}^t\varphi^0\right)\left(-K_{12}^t\psi,\psi\right)\le C_\varphi^2\|\psi\|_{\Delta_1}^2$$

where $K_{1j}^t=K_{1j}\otimes I_2$, $K_{i,2}^t=I_1\otimes K_{i,2}$, $i,j=1;2$.

$C_\varphi^2=const$ (with respect to ψ)$=2\max\left\{\|K_{22}\|\left\|K_{11}^2\right\|\left\|\varphi^0\right\|^2;\|K_{12}\|\left\|K_{21}^2\right\|\left\|\varphi^0\right\|^2\right\}$.

So, we obtain the boundedness of the linear functional $l_{\varphi^0}(\psi)$ in everywhere dense subset $H\subset H_{\Delta_1}$ (for a fixed element $\varphi^0\in H$). Continuously continufing, we determine a bounded linear functional $l_{\varphi^0}(\psi)$ everywhere defined in space H_{Δ_1}. Therefore, by the Riesz theorem in Hilbert space H_{Δ_1}, there exists an element h such that for $\forall\psi\in H_{\Delta_1}$ the equality $l_{\varphi^0}(\psi)=[h;\psi]_{\Delta_1}$ is fulfilled. Then for $\forall\psi\in H\subset H_{\Delta_1}$ write

$$\left(\Delta_0\varphi^0,\psi\right)=[h;\psi]_{\Delta_1}\,. \tag{4}$$

Now prove that $h \in H$. Let the sequence $\{h_n\} \subset H$ be fundamental in space H_{Δ_1} and converge to the element h. Then for $\forall x \in H \subset H_{\Delta_1}$ we can write $[h,x]_{\Delta_1} = \lim\limits_{n\to\infty} (\Delta_1 h_n, x)$. Let $\psi \in R(\Delta_1)$; $\|\psi\| = 1$; $\psi = \Delta_1 x$, then

$\lim\limits_{n\to\infty} |(h_n,\psi)| = \lim\limits_{n\to\infty} |(h_n, \Delta_1 x)| = \lim\limits_{n\to\infty} |(\Delta_1 h_n, x)| = \left|[h;x]_{\Delta_1}\right| =$ (by equality (4)$= \left|\left(\Delta_0\varphi^0, x\right)\right| = \left|\left(\varphi^0, \Delta_0 x\right)\right| \le \left\|\varphi^0\right\| \|\Delta_0 x\|$).

By inequality (3) $\|\Delta_0 x\| \le c\|\Delta_1 x\| \le c\|\psi\| = c$. Hence it follows that $\lim\limits_{n\to\infty} |(h_n,\psi)|$ exists and is uniformly bounded with respect to the variable ψ (where $\|\psi\| = 1$). Consequently, without loss of generality, if it is necessary, passing to the subsequences we can affirm that $\lim\limits_{n\to\infty} (h_n,\psi)$ exists and is uniformly bounded with respect to the variable ψ (where $\|\psi\| = 1$). And this means that the scalar product (h,ψ) exists for all $\psi \in R(\Delta_1)$ and is uniformly bounded with respect to the variable ψ. The set $R(\Delta_1)$ is everywhere dense in Hilbert space H. Therefore, continuously continuing, we get $(h,\psi) \le c$ for $\forall\psi \in H$; $\|\psi\| = 1$. Then h is an element of the conjugate space H^*. Taking into account equality $H = H^*$, we can write $h \in H$. Then for $\forall\psi \in H \subset H_{\Delta_1}$ equality (4) has the form

$$\left(\Delta_0\varphi^0, \psi\right) = (\Delta_1 h, \psi)$$

i.e. $\Delta_0\varphi^0 = \Delta_1 h$. Thus, we get $R(\Delta_0) \subset R(\Delta_1)$. Similarly we can prove that $R(\Delta_0) \subset R(\Delta_2)$. The theorem is proved.

The obtained result prompts that the system of separating operators $\Gamma_i = \Delta_0^{-1}\Delta_i$, $i = 1;2$ of problem (1) is determined everywhere in Hilbert space $H = H_1 \otimes H_2$.

The same property holds also for the multi-parameter problem

$$\begin{cases} \sum\limits_{j=1}^{n} \lambda_j K_{ij}\varphi_i = \varphi_i, \qquad \varphi_i \in H_i \\ \qquad\qquad i = 1;2;...;n; \end{cases}$$

i.e. for the operators $\Delta_i > 0$, $i = 0;1;...;n$ that are defined in the same way.

References

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Received May 21, 2012; Revised September 20, 2012.