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ON PERTURBATION OF b-BASIS IN BANACH SPACES WITH RESPECT TO CB-SPACE

Abstract

In the paper, the perturbation of b-basis in Banach spaces is studied. The notain of CB-spaces and b-Bessel basis in Banach space with respect to CB-space are introduced. Theorems on pertirbation of b-Bessel basis in Banach space with respect to CB-space are set up. In particular, the known theorem on the Riesz basis is generalized.

Introduction. Sometimes the basicity of the system in this or other space is established by means of the known theorems on perturbation of the basis of this space. The Bari theorem about perturbation of orthonormed basis in Hilbert space is known.

Theorem ([1]). *Let $\{\varphi_n\}_{n\in N}$ be an orthonormed basis in Hilbert space H, the system $\{\psi_n\}_{n\in N} \subset H$ be ω-linearly independent in H, and*

$$\sum_{n=1}^{\infty} \|\psi_n - \varphi_n\|_H^2 < +\infty$$

Then $\{\varphi_n\}_{n\in N}$ forms a basis in H isomorphic to $\{\varphi_n\}$, i.e. the Riesz basis.

Under some weaker condition it is valid

Theorem ([2]). *Let $\{\psi_n\}_{n\in N} \subset H$ be an orthonormed basis. Then the system $\{\psi_n\}_{n\in N} \subset H$ is a basis in H if*

$$\sum_{n=1}^{\infty} \left(\|\psi_n - \varphi_n\|_H^2 - \frac{|(\varphi_n - \psi_n,\ \psi_n)|^2}{\|\psi_n\|_H^2} \right) < 1.$$

Paley-Wiener, Krein-Rootman-Milman, Birkhoff-Rota theorems, Kadets theorem-1/4 for the theorems on perturbation in Banach spaces are known.

Note that Kadets theorem -$\frac{1}{4}$ on Riesz basicity of the system $\left\{e^{i\lambda_n t}\right\}_{n\in Z}$, $\{\lambda_n\} \subset R$ in the space $L_2(-\pi,\pi)$ is obtained from Paley-Wiener criterion provided $\sup\limits_n |\lambda_n - n| < \frac{1}{4}$. The accuracy of the constant 1/4 is illustrated by the Levinson example ([3]). One can be acquainted with these or other facts of theory of bases in the monographs [4-8].

In Hilbert space H, each orthonormed system $\{\varphi_n\}_{n\in N}$ is a Bessel system, i.e. there exists a constant $B > 0$ such that $\sum\limits_{n=1}^{\infty} |(h,\varphi_n)|^2 \le B\|h\|_H^2$ for any $h \in H$. The Bessel system in H is a particular case of frame [9-11]. The frames find numerous applications in many fields such as signal processes, in image processing, data compression and etc.

The present paper is devoted to studying the perturbations of b-basis ([8;12]) in Banach spaces. The notion of CB-space, b-Bessel basis in Banach space with respect to CB-space are introduced, the proximity conditions, in definite sense, of the system to b- Bessel basis under which the system forms b-isomorphic b-basis are found. In particular, Bari's known theorem on Riesz bases is generalized.

1. Some denotation and auxiliary facts. Cite some notion and facts from [8] and [13] used in the course of the paper.

Let X, Y, Z and be Banach spaces, $b(x,y) : X \times Y \to Z$ be a bounded bilinear mapping:

$$\|b(x,y)\|_Z \le M \|x\|_X \|x\|_Y \quad \text{for any} \quad x \in X \text{ and } y \in Y.$$

The system $\{\varphi_n\}_{n\in N} \subset Y$ is called b-basis in Z if for any $z \in Z$ there exists a unique sequence $\{x_n\} \subset X$ such that $z = \sum_{n=1}^{\infty} b(x_n, \varphi_n)$.

The systems $\{\varphi_n\}_{n\in N} \subset Y$ and $\{\varphi_n^*\}_{n\in N} \subset L(Z,X)$ are called b-biorthogonal if $\varphi_n^*(b(x,\varphi_k)) = \delta_{nk}x$ for each $x \in X$.

The system $\{\varphi_n\}_{n\in N} \subset Y$ is called b-complete if the closure of the totality $L_b\left(\{\varphi_n\}_{n\in N}\right)$ of all possible finite sums $\sum b(x_n,\varphi_n)$, $x_n \in X$ coincides with Z.

The systems $\{\varphi_n\}_{n\in N}$ and $\{\psi_n\}_{n\in N} \subset Y$ are said to be b-isomorphic if there exists a bounded inversible operator $T \in L(Z) : T(b(x,\varphi_n)) = b(x,\psi_n)$ for any $x \in X$ and $n \in N$.

Let $\widetilde{X}$ be B-space of sequences $\{x_n\} \subset X$ with coordinate wise linear operations and $\lim_{i\to\infty} \left\| \left\{ \chi_{J(i)}(n)\, x_n \right\}_{n\in N} \right\|_{\widetilde{X}} = 0$, where $J(i) = \{k \in N : k \ge i\}$, and $\chi_{J(i)}$ is a characteristic function of the set $J(i)$ (abbreviated KB-space).

The system $\{\varphi_n\}_{n\in N} \subset Y$ is called ω_b-linearly independent in Z with respect to $\widetilde{X}$, if from $\sum_{n=1}^{\infty} b(x_n,\varphi_n) = 0$, $\{x_n\}_{n\in N} \subset \widetilde{X}$ it follows that $x_n = 0$ for any $n \in N$.

In what follows we'll use the following lemma.

Lemma ([13]). *Let X, Y and Z be B-spaces, the system $\{\varphi_n\}_{n\in N} \subset Y$ form a b-basis in Z with b-biorthogonal system, $\{\varphi_n^*\}_{n\in N}$, $F \in L(Z)$ be a Fredholm operator, the system $\{\varphi_n\}_{n\in N} \subset Y$ and $F(b(x,\varphi_n)) = b(x,\psi_n)$ $\forall x \in X,\ n \in N$. Then the following properties are equivalent.*

a) $\{\psi_n\}_{n\in N}$ is b-complete in Z;

b) $\{\psi_n\}_{n\in N}$ has b-biorthogonal system;

c) $\{\psi_n\}_{n\in N}$ is ω_b-linearly independent in Z;

d) $\{\psi_n\}_{n\in N}$ forms a b-basis in Z, b-isomorphic to $\{\varphi_n\}_{n\in N}$.

Perturbation of b-Bessel basis in Banach space with respect to CB-space.

Let $\widetilde{X}$ be a KB-space, the systems $\{\varphi_n\} \subset Y$ and $\{\varphi_n^*\}_{n\in N} \subset L(Z,X)$ be b-biorthogonal. The pair $\left(\{\varphi_n^*\}_{n\in N}, \{\varphi_n\}_{n\in N}\right)$ is called b-Bessel in Z with respect to $\widetilde{X}$ if

1) $\{\varphi_n^*(z)\}_{n\in N} \in \widetilde{X}$ for any $z \in Z$;

2) there exists a number $B > 0$ such that $\left\|\{\varphi_n^*(z)\}_{n\in N}\right\|_{\widetilde{X}} \le B\|z\|_Z$ for any $z \in Z$.

In the case of b-basicity $\{\varphi_n\}_{n\in N}$ the b-Bessel pair $\left(\{\varphi_n^*\}_{n\in N}, \{\varphi_n\}_{n\in N}\right)$ is caled b-Bessel basis in Z with respect to $\widetilde{X}$.

$\widetilde{X}$ is said to be a CB-space if

$$\widetilde{X}^{\,*} = \left\{\{t_n\} \subset X^* : (\{t_n\}, \{x_n\}) = \sum_{n=1}^{\infty} t_n(x_n),\ \{x_n\} \in \widetilde{X}\right\}.$$

$\widetilde{Y}$ is a KB space over Y. Say that $\widetilde{X}$ is normally subjected to $\widetilde{Y}$ if from $\{x_n\} \subset X$, $\{\varphi_n\} \subset Y$, $\|x_n\|_X \le \|\varphi_n\|_Y$ and $\{\varphi_n\} \in \widetilde{Y}$ it follows that $\{x_n\} \in \widetilde{X}$ and $\|\{x_n\}\|_{\widetilde{X}} \le \|\{\varphi_n\}\|_{\widetilde{Y}}$.

It is valid

Theorem 1. *Let $\widetilde{X}$ be CB space, $\widetilde{Y}$ be KB-space and $\widetilde{X}^*$ be normally subjected to $\widetilde{Y}$, the systems $\{\varphi_n\} \subset Y$ and $\{\varphi_n^*\}_{n\in N} \subset \sigma(Z, X)$ be b-biorthogonal and $\left(\{\varphi_n^*\}_{n\in N}, \{\varphi_n\}_{n\in N}\right)$ be b-Bessel basis in Z with respect to $\widetilde{X}$, the system $\{\psi_n\} \subset Y$ be sich that $\{\varphi_n - \psi_n\}_{n\in N} \subset \widetilde{Y}$. Then the following properties are equivalent:*

a) $\{\psi_n\}_{n\in N}$ b is complete in Z:

b) $\{\psi_n\}_{n\in N}$ has a b-biorthogonal system;

c) $\{\psi_n\}_{n\in N}$ is ω_b-linearly dependent in Z;

d) $\{\psi_n\}_{n\in N}$ forms a b-basis in Z, b-isomorphic to $\{\varphi_n\}_{n\in N}$.

Proof. Define for the fixed $f \in Z^*$ and $y \in Y$ the mapping $\langle f, y\rangle : X \to C$ by the formula $\langle f, y\rangle : (x) = f(b(x, y))$. It is clear that

$$\|\langle f, y\rangle\| \le M\,\|f\|\,\||y|\|_Y\,. \tag{1}$$

Since $\widetilde{X}^*$is normally subjected to $\widetilde{Y}$, then from (1) we get

$$\left\|\{\langle f, y_n\rangle\}_{n\in N}\right\|_{\widetilde{X}^*} \le M\,\|f\|\,\left\|\{y_n\}_{n\in N}\right\|_{\widetilde{Y}} \quad \text{as any} \quad f \in Z^* \text{ and } \{y_n\} \in \widetilde{Y} \tag{2}$$

Show that the series $\sum_{n=1}^{\infty} b(\varphi_n^*(z), \varphi_n - \psi_n)$ converges for any $z \in Z$.

For any $m, p \in N$ and $z \in Z$ from the corollary of Hahn-Banach theorem there exists $f_{m,p} \in Z^*\ \|f_{m,p}\| = 1$, that

$$f_{m,p}\left(\sum_{n=m+1}^{m+p} b(\varphi_n^*(z), \varphi_n - \psi_n)\right) = \left\|\sum_{n=m+1}^{m+p} b(\varphi_n^*(z), \varphi_n - \psi_n)\right\|_Z.$$

We have

$$\left\|\sum_{n=m+1}^{m+p} b(\varphi_n^*(z), \varphi_n - \psi_n)\right\|_Z = \left|\sum_{n=m+1}^{m+p} f_{m,p}\left(b(\varphi_n^*(z), \varphi_n - \psi_n)\right)\right| =$$

$$= \left|\sum_{n=m+1}^{m+p} \langle f_{m,p}, \varphi_n - \psi_n\rangle(\varphi_n^*(z))\right| =$$

$$= \left|\left(\left\{\chi_{J(m+1,m+p)}(n) < f_{m,p},\ \varphi_n - \psi_n >\right\}_{n\in N},\ \{\varphi_n^*(z)\}_{n\in N}\right)\right|, \tag{3}$$

where $J(i, j) = \{k \in N : i \le k \le j\}$ for any $i, j \in N$.

From (3), taking into account (2), we get

$$\left\|\sum_{n=m+1}^{m+p} b\left(\varphi_n^*\left(z\right),\varphi_n-\psi_n\right)\right\|_Z = \left\|\left\{\left\langle \chi_{J(m+1,m+p)}\left(n\right) f_{m,p},\ \varphi_n-\psi_n\right\rangle\right\}_{n\in N}\right\|_{\widetilde{X}}\times$$

$$\times\left\|\left\{\varphi_n^*\left(z\right)\right\}_{n\in N}\right\|_{\widetilde{X}}\leq$$

$$\leq M\left\|f_{m,p}\right\|\left\|\left\{\chi_{J(m+1,m+p)}\left(n\right)\left(\varphi_n-\psi_n\right)\right\}_{n\in N}\right\|_{\widetilde{Y}}\left\|\left\{\varphi_n^*\left(z\right)\right\}_{n\in N}\right\|_{\widetilde{X}}\leq$$

$$\leq M\ \ B\left\|\left\{\chi_{J(m+1,m+p)}\left(n\right)\left(\varphi_n-\psi_n\right)\right\}_{n\in N}\right\|_{\widetilde{Y}}\left\|z\right\|_Z. \tag{4}$$

Consequentely the series $\sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right),\varphi_n-\psi_n\right)$ converges for any $z\in Z$. Then we can define the operator $T\left(z\right)=\sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right),\varphi_n-\psi_n\right)$, $z\in Z$. Consider for each $m\in N$ the operator $T_m: Z\to Z$ by the formula $T_m\left(z\right)=\sum_{n=1}^{m} b\left(\varphi_n^*\left(z\right),\varphi_n-\psi_n\right)$, $z\in Z$. Obviously, it follows form $\left\{\varphi_n^*\left(z\right)\right\}_{n\in N}\subset\sigma\left(Z,X\right)$ that $T_m\in\sigma\left(Z,X\right)$. Tending $p\to\infty$, from (4) we get

$$\left\|T\left(z\right)-T_m\left(z\right)\right\|_Z\leq M\ \ B\left\|\left\{\chi_{J(m+1)}\left(n\right)\left(\varphi_n-\psi_n\right)\right\}_{n\in N}\right\|_{\widetilde{Y}}\left\|z\right\|_Z.$$

Thus,

$$\left\|T-T_m\right\|\leq M\ \ B\left\|\left\{\chi_{J(m+1)}\left(n\right)\left(\varphi_n-\psi_n\right)\right\}_{n\in N}\right\|_{\widetilde{Y}}.$$

Therefore $T=\lim_{m\to\infty} T_m\in\sigma\left(Z,X\right)$. Then the operator F given by the equality $F=I-T$, is a Fredholm operator, there I is an identity operator in Z, and $F\left(z\right)=\sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right),\psi_n\right)$, $z\in Z$. Obviously, $F\left(b\left(x,\varphi_n\right)\right)=b\left(x,\psi_n\right)\ \forall x\in X,\ n\in N$. By lemma 1 the properties a)-d) are equivalent for the system $\left\{\psi_n\right\}_{n\in N}$. The theorem is proved.

Theorem 2. *Let $\widetilde{X}$ be CB-space, $\widetilde{Y}$ be KB-space, and $\widetilde{X}^*$be normally subjected to $\widetilde{Y}$, the systems $\left\{\varphi_n\right\}_{n\in N}\subset Y$ and $\left\{\varphi_n^*\right\}_{n\in N}\subset L\left(Z,X\right)$ be b-biorthogonal $\left(\left\{\varphi_n^*\right\}_{n\in N},\left\{\varphi_n\right\}_{n\in N}\right)$ and be b-Bessel basis in Z with respect to $\widetilde{X}$, the system $\left\{\psi_n\right\}_{n\in N}\subset Y$ be such that $\left\{\varphi_n-\psi_n\right\}_{n\in N}\in\widetilde{Y}$ and $\left\|\left\{\varphi_n-\psi_n\right\}_{n\in N}\right\|_{\widetilde{Y}}<\frac{1}{MB}$. Then $\left\{\psi_n\right\}_{n\in N}$ is b-isomorphic to $\left\{\varphi_n\right\}_{n\in N}$.*

Proof. From the proof of theorem 1 we get that the series $\sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right),\varphi_n-\psi_n\right)$ converges for any $z\in Z$. Define the operator T by formula $T\left(z\right)=\sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right),\psi_n\right)$, $z\in Z$. For any $z\in Z$ by the corollary of Hahn-Banach theorem there exists $f_z\in Z^*$ such that $\left\|f_z\right\|=1$ and $f_z\left(\sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right),\varphi_n-\psi_n\right)\right)=\left\|\sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right),\varphi_n-\psi_n\right)\right\|_Z$.

Then similar to (4) we have

$$\|(I-T)(z)\|_Z = \left\|\sum_{n=1}^{\infty} b\left(\varphi_n^*(z), \varphi_n - \psi_n\right)\right\|_Z = \left|\sum_{n=1}^{\infty} f_z\left(b\left(\varphi_n^*(z), \varphi_n - \psi_n\right)\right)\right| =$$

$$= \left|\sum_{n=1}^{\infty} \langle f_z, \varphi_n - \psi_n\rangle \left(\varphi_n^*(z)\right)\right| = \left|\left(\{\langle f_z, \ \varphi_n - \psi_n\rangle\}_{n\in N}, \ \{\varphi_n^*(z)\}_{n\in N}\right)\right| \le$$

$$\le \left\|\{\langle f_z, \ \varphi_n - \psi_n\rangle\}_{n\in N}\right\|_{\widetilde{X}^*} \left\|\{\varphi_n^*(z)\}_{n\in N}\right\|_{\widetilde{X}} \le$$

$$\le M\|f_z\| \left\|\{\varphi_n - \psi_n\}_{n\in N}\right\|_{\widetilde{Y}} \left\|\{\varphi_n^*(z)\}_{n\in N}\right\|_{\widetilde{X}} \le M \quad B\left\|\{\varphi_n - \psi_n\}_{n\in N}\right\|_{\widetilde{Y}} \|z\|_Z .$$

Consequently, $\|I-T\| \le M \quad B\left\|\{\varphi_n - \psi_n\}_{n\in N}\right\|_{\widetilde{Y}}$ and by the same token $\|I-T\| < 1$. Therefore, the operator T is boundedly invertible. It is clear that $T\left(b\left(x, \varphi_n\right)\right) = b\left(x, \psi_n\right) \quad \forall x \in X, \ n \in N$. The theorem is proved.

Remark. Note that from theorem 1 we get the generalization of N.K. Bari theorem, more exactly: let $\widetilde{X}$ be CB-space, $\widetilde{Y}$ be KB space, and $\widetilde{X}^*$ be normally subjected to $\widetilde{Y}$, the systems $\{\varphi_n\}_{n\in N} \subset Y$ and $\{\varphi_n^*\}_{n\in N} \subset \sigma(Z, X)$ be b-biorthogonal and $\left(\{\varphi_n^*\}_{n\in N}, \ \{\varphi_n\}_{n\in N}\right)$ be b-Bessel basis in Z with respect to $\widetilde{X}$, the system $\{\psi_n\}_{n\in N} \subset Y$ be ω_b-linearly independent in Z and such that $\{\varphi_n - \psi_n\}_{n\in N} \in \widetilde{Y}$, then $\{\psi_n\}_{n\in N}$ is b-isomorphic to $\{\varphi_n\}_{n\in N}$.

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