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THE EXISTENCE OF AN ABSORBING SET FOR ONE MIXED PROBLEM WITH HYSTERESIS

Abstract

In this work we consider the mixed problem for one quasilinear parabolic equation with hysteresis (generalized play) nonlinearity. We prove the existence of a unique solution and the existence of a bounded absorbing set for this problem.

Let $\Omega \subset R^N$ $(N \geq 1)$ be a bounded, connected set with a smooth boundary Γ. We consider the following problem

$$\frac{\partial}{\partial t}\left[u + F\left(u\right)\right] - \Delta u + \left|u\right|^p u = h \quad \text{in} \quad Q = \Omega \times \left(0, T\right), \tag{1}$$

$$u = 0 \quad \text{on} \quad \Gamma \times \left[0, T\right], \tag{2}$$

$$\left.\left[u + F\left(u\right)\right]\right|_{t=0} = u^0 + w^0 \quad \text{in} \quad \Omega, \tag{3}$$

where $0 < p \leq \frac{2}{2-N}$ if $N \geq 3$ and $p > 0$ if $N = 1, 2, ;$ F is a memory operator (at any instant t, $F\left(u\right)$ may depend not only on $u\left(t\right)$ but also on the previous evolution of u), which acts from $M\left(\Omega; C^0\left(\left[0, T\right]\right)\right)$ to $M\left(\Omega; C^0\left(\left[0, T\right]\right)\right)$. Here $M\left(\Omega; C^0\left[0, T\right]\right)$ is a space of strongly measurable functions $\Omega \to C^0\left(\left[0, T\right]\right)$. We assume that the operator F is applied at each point $x \in \Omega$ independently: the output $\left[F\left(u\right)\right]\left(x, t\right)$ depends on $\left.u\left(x, \cdot\right)\right|_{[0,t]}$, but not on $\left.u\left(y, \cdot\right)\right|_{[0,t]}$ for any $y \neq x$.

We assume that

$$\left\{\begin{array}{l} \forall v_1, v_2 \in M\left(\Omega; C^0\left(\left[0, T\right]\right)\right), \ \forall t \in \left[0, T\right], \ \text{ if } \ v_1 = v_2 \ \text{ in } \ \left[0, t\right], \text{ a.e. in } \Omega, \\ \text{then } \left[F\left(v_1\right)\right]\left(\cdot, t\right) = \left[F\left(v_2\right)\right]\left(\cdot, t\right) \ \text{ a.e. in } \ \Omega, \end{array}\right. \tag{4}$$

$$\left\{\begin{array}{l} \forall \left\{v_n \in M\left(\Omega; C^0\left(\left[0, T\right]\right)\right)\right\}_{n \in N}, \ \text{ if } \ v_n \to v \ \text{ uniformly in } \left[0, T\right] \text{ a.e. in } \Omega, \\ \text{then } F\left(v_n\right) - F\left(v\right) \ \text{ uniformly in } \left[0, T\right], \ \text{ a.e. in } \ \Omega. \end{array}\right. \tag{5}$$

Let $V = H_0^1\left(\Omega\right)$ and

$$u^0 \in V, \quad w^0 \in L^2\left(\Omega\right), \quad h \in L^2\left(\Omega\right). \tag{6}$$

Definition. *A function $u \in M\left(\Omega; C^0\left(\left[0, T\right]\right)\right) \cap L^2\left(0, T; V\right)$ is said to be a solution of problem (1)-(3) if $F\left(u\right) \in L^2\left(Q\right)$ and*

$$\int_0^T \int_\Omega \left\{ -\left[u + F\left(u\right)\right] \frac{\partial v}{\partial t} + \nabla u \cdot \nabla v + \left|u\right|^p u v \right\} dx dt =$$

$$= \int_0^T \int_\Omega h v dx dt + \int_\Omega \left[u^0\left(x\right) + w^0\left(x\right)\right] v\left(x, 0\right) dx$$

for any $v \in L^2\left(0, T; V\right) \cap H^1\left(0, T; L^2\left(\Omega\right)\right)$ $\left(v\left(\cdot, T\right) = 0 \ \text{ a.e. in } \Omega\right)$.

We also assume that

$$\begin{cases} \exists \overline{L} \in R^{+}, \quad \exists g \in L^{2}(\Omega) : \forall v \in M\left(\Omega; C^{0}([0,T])\right), \\ \left\| F(v)(x,\cdot)\right\|_{C^{0}([0,T])} \leq \overline{L} \left\| v(x,\cdot)\right\|_{C^{0}([0,T])} + g(x) \quad a.e.\ in\ \ \Omega, \end{cases} \tag{7}$$

$$\begin{cases} \forall v \in M\left(\Omega; C^{0}([0,T])\right), \quad \forall [t_1,t_2] \subset [0,T], \\ if\ \ v(x,\cdot)\ \text{ is affine in } [t_1,t_2] \quad a.e.\ in\ \ \Omega,\ then \\ \left\{[F(v)](x,t_2) - [F(v)](x,t_1)\right\} \cdot [v(x,t_2) - v(x,t_1)] \geq 0 \quad a.e.\ in\ \ \Omega. \end{cases} \tag{8}$$

The following theorem (existence of solutions of the problem (1)-(3)) was proved in [3].

Theorem 1. *Assume that (4)-(8) hold. Then problem (1)-(3) has at least one solution such that*

$$u \in H^{1}\left(0,T;L^{2}(\Omega)\right) \cap L^{\infty}(0,T;V), \ \ F(u) \in L^{2}\left(\Omega; C^{0}([0,T])\right). \tag{9}$$

In this work we consider the problem (1)-(3) under the additional condition that F is a generalized play operator (see [1]). Note that similar mixed problems with a generalized play operator was studied in the works, for exapmle [3] , [4] . Well posedness of problem (1)-(3) without F was studied in the works of different authors (see, for example [2]). The corresponding problem for the parabolic equation without nonlinear term $|u|^{p} u$ was studied in [1]. In this work we prove the existence of a unique solution and the existence of a bounded absorbing set for problem (1)-(3) under the additional condition that F is a generalized play operator.

Assume that we are given two functions $\gamma_l(\sigma),\ \gamma_r(\sigma) \in C^{0}(R)$ such that for any $\sigma \in R$ it holds

$$\gamma_r(\sigma) \leq \gamma_l(\sigma). \tag{10}$$

We denote by E a generalized play operator (hysteresis operator) (see [1], chapter III). We fix any $\xi^{0} \in L^{1}(\Omega)$ and set for any $v \in M\left(\Omega; C^{0}([0,T])\right)$

$$[F(v)](x,t) = \left[E\left(v(x,\cdot), \xi^{0}(x)\right)\right](t) \quad \forall t \in (0,T), \quad a.e.\ in\ \ \Omega. \tag{11}$$

The operator E satisfies the conditions (4),(5),(7),(8), the inequality

$$\left|\frac{\partial}{\partial t} F(u)\right| \leq L \left|\frac{\partial u}{\partial t}\right| \quad a.e.\ \ on \quad (0,T) \tag{12}$$

and the Hilpert inequality ([1], chapter III).

Hilpert's inequality. *Let* $\left(\sigma_i, \varepsilon_i^{0}\right) \in W^{1,1}(0,T) \times R \ \ (i=1,2)$ *and* $g : [0,T] \to R$ *be a measurable function such that* $g \in H(\sigma_1 - \sigma_2)$ *a.e. in* $(0,T)$. *Set* $\varepsilon_i = E\left(\sigma_i, \varepsilon_i^{0}\right)(i=1,2)$, $\overline{\varepsilon} = \varepsilon_1 - \varepsilon_2$ *and* $\overline{\varepsilon}^{+} = \max\{\overline{\varepsilon}, 0\}$. *Then*

$$\frac{d\overline{\varepsilon}}{dt} g \geq \frac{d}{dt}\left(\overline{\varepsilon}^{+}\right) \quad \text{a.e. in } (0,T).$$

Theorem 2. *For* $i = 1,2$ *let* $u_i^{0},\ \xi_i^{0} \in L^{2}(\Omega)$, $h_i(x) \in L^{2}(\Omega)$ *and* $h_1 - h_2 \in L^{2}(\Omega)$. *Let* $\gamma_l(\sigma), \gamma_r(\sigma) \in C^{0}(R)$ *be as in (10), locally Lipschitz continuous and affinely bounded. Define* F *as in (11) and set*

$$w_i^{0} = \min\left\{\max\left\{\xi_i^{0}, \gamma_r\left(u_i^{0}\right)\right\}, \gamma_l\left(u_i^{0}\right)\right\} \quad \text{a.e. in } \Omega\,(i=1,2).$$

Let $u_i \in W^{1,1}\left(0,T;L^1(\Omega)\right) \cap L^2(0,T;V)$ *be any solution of the corresponding problem (1)-(3) with* $h = h_i$ *and* $w_i = F(u_i)$ $(i = 1,2,)$. *Then for any* $t \in [0,T]$

$$\int\limits_{\Omega} \left[(u_1 - u_2)^+ (x,t) + (w_1 - w_2)^+ (x,t)\right] dx \leq$$

$$\leq \int\limits_{\Omega} \left[\left(u_1^0 - u_2^0\right)^+ (x) + \left(w_1^0 - w_2^0\right)^+ (x)\right] dx + T \int\limits_{\Omega} (h_1 - h_2)^+ dx. \qquad (13)$$

Proof. We set for any $m \in N$

$$H_m(\eta) = \begin{cases} 1, & \text{if } \ \eta > \dfrac{1}{m}, \\ m\eta, & \text{if } \ 0 \leq \eta > \dfrac{1}{m}, \\ 0, & \text{if } \ \eta < 0. \end{cases}$$

By Theorem III. 2.3 (see [1]), $w_i \in W^{1,1}\left(0,T;L^1(\Omega)\right)$, $i = 1,2$.

Since

$$\frac{\partial}{\partial t}\left[u_1 + F(u_1)\right] - \Delta u_1 + |u_1|^p u_1 = h_1,$$

$$\frac{\partial}{\partial t}\left[u_2 + F(u_2)\right] - \Delta u_2 + |u_2|^p u_2 = h_2,$$

then

$$\frac{\partial}{\partial t}\left[(u_1 - u_2) + F(u_1) - F(u_2)\right] - \Delta(u_1 - u_2) + |u_1|^p u_1 - |u_2|^p u_2 = h_1 - h_2,$$

whence multiplying by $H_m(u_1 - u_2)$ and integrating in Ω,we have

$$\int\limits_{\Omega} \left[\frac{\partial}{\partial t}(u_1 - u_2) + \frac{\partial}{\partial t}(w_1 - w_2)\right] H_m(u_1 - u_2)\, dx + \int\limits_{\Omega} \nabla(u_1 - u_2) \nabla H_m(u_1 - u_2)\, dx +$$

$$+ \int\limits_{\Omega} \left[|u_1|^p u_1 - |u_2|^p u_2\right] H_m(u_1 - u_2)\, dx = \int\limits_{\Omega} (u_1 - u_2) H_m(u_1 - u_2)\, dx.$$

Since

$$\int\limits_{\Omega} \nabla(u_1 - u_2) \nabla H_m(u_1 - u_2)\, dx = \int\limits_{\Omega} H_m'(u_1 - u_2) \left|\nabla(u_1 - u_2)\right|^2 dx \geq 0$$

a.e. in $(0,T)$ and

$$\int\limits_{\Omega} \left[|u_1|^p u_1 - |u_2|^p u_2\right] H_m(u_1 - u_2)\, dx \geq 0,$$

we get

$$\int\limits_{\Omega} \left[\frac{\partial}{\partial t}(u_1 - u_2) + \frac{\partial}{\partial t}(w_1 - w_2)\right] H_m(u_1 - u_2)\, dx \leq$$

$$\leq \int_{\Omega} (h_1 - h_2) H_m (u_1 - u_2) dx. \qquad (14)$$

Now we pass to the limit as $m \to \infty$. There exists $\psi \in L^{\infty}(\Omega)$ such that $H_m(u_1 - u_2) \to \psi$ a.e. in Q. Moreover $\psi \in H(u_1 - u_2)$ a.e. in Q, where

$$H(y) = \begin{cases} \{0\}, & \text{if } y < 0, \\ [0,1], & \text{if } y = 0, \\ \{1\}, & \text{if } y > 0. \end{cases}$$

Then from (14) we get that

$$\int_{\Omega} \left[\frac{\partial}{\partial t}(u_1 - u_2) + \frac{\partial}{\partial t}(w_1 - w_2) \right] \psi dx \leq \int_{\Omega} (h_1 - h_2) \psi dx \leq \int_{\Omega} (h_1 - h_2)^{+} dx. \qquad (15)$$

Since by the Hilpert inequality it holds

$$\frac{\partial}{\partial t}(w_1 - w_2)\psi \geq \frac{\partial}{\partial t}\left[(w_1 - w_2)^{+}\right] \quad a.e. \ in \ Q,$$

then from (15) we have

$$\frac{\partial}{\partial t} \int_{\Omega} \left[(u_1 - u_2)^{+} + (w_1 - w_2)^{+} \right] dx \leq \int_{\Omega} (h_1 - h_2)^{+} dx \quad a.e. \ in \ Q,$$

whence it is obtained (13).

Theorem 2 is proved.

Theorem 3. *Assume that*

$$u^0 \in V, \quad \xi \in L^2(\Omega), \ h \in L^2(\Omega). \qquad (16)$$

Let $\gamma_l(\sigma), \gamma_r(\sigma) \in C^0(R)$ be as in (10), locally Lipschits continuous and affinely bounded and F be as in (11). Then the problem (1)-(3) has one and only one solution with the regularity (9).

Proof. Straightforward consequence of the theorems 1 and 2.

The problem (1)-(3) generates a semigroup $\{S(t)\}_{t \geq 0}$ in V by the formula

$$S(t)\left(u^{(0)}\right) = u,$$

where u is a unique solution of this problem.

Theorem 4. *Under the conditions of theorem 3 there exists an absorbing set $B_0 \subset V$ for the problem (1)-(3).*

Note that a bounded set $B_0 \subset V$ is said to be absorbing, if for arbitrary bounded set $B \subset V$ there exists $t_1(B)$ such that $S(t)B \subset B_0$ for all $t \geq t_1(B)$.

Proof. We introduce the following functional

$$\Phi(u) = \int_{\Omega} \left(|u|^2 + |\nabla u|^2 + |u|^{p+2} \right) dx.$$

We prove at first that if u is a solution of the problem (1)-(3) then for arbitrary $t \geq 0$ is hold the inequality

$$\frac{d}{dt}\Phi(u) + \delta\Phi(u) \leq C, \tag{17}$$

where C is some positive constant.

By multiplying (1) by u_t and by integrating in Ω we obtain

$$\int_\Omega |u_t|^2 dx + \int_\Omega \frac{\partial}{\partial t}F(u)\, u_t dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \frac{1}{p+2}\cdot\frac{d}{dt}\int_\Omega |u|^{p+2} dx \leq$$

$$\leq \frac{1}{2}\int_\Omega h^2 dx + \frac{1}{2}\int_\Omega u_t^2 dx.$$

Using at last relation the inequality

$$\int_\Omega \frac{\partial}{\partial t}F(u)\, u_t dx \geq 0,$$

which is obtained from the condition (8), we have

$$\frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \frac{1}{p+2}\cdot\frac{d}{dt}\int_\Omega |u|^{p+2} dx \leq \frac{1}{2}\int_\Omega h^2 dx. \tag{18}$$

Now by multiplying (1) by u and integrating in Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^{p+2} dx \leq \int_\Omega \left|\frac{\partial}{\partial t}F(u)\right| |u|\, dx + \int_\Omega hu dx.$$

Using the inequality (12) at last relation, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^{p+2} dx \leq L\int_\Omega |u_t|\,|u|\, dx + \int_\Omega hu dx \leq$$

$$\leq L\left[\frac{L}{4\nu}\int_\Omega |u_t|^2 dx + \frac{\nu}{L}\int_\Omega |u|^2 dx\right] + \frac{1}{2}\int_\Omega h^2 dx + \frac{1}{2}\int_\Omega u^2 dx$$

or

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^{p+2} dx \leq$$

$$\leq \frac{L^2}{4\nu}\int_\Omega |u_t|^2 dx + \left(\nu + \frac{1}{2}\right)\int_\Omega |u|^2 dx + \frac{1}{2}\int_\Omega h^2 dx. \tag{19}$$

By multiplying (18) by $\dfrac{L^2}{4\nu}$ and by adding with (19), we obtain that

$$\frac{L^2}{4\nu}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \frac{L^2}{2\nu(p+2)}\frac{d}{dt}\int_\Omega |u|^{p+2} dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx +$$

$$+\int_\Omega |\nabla u|^2\,dx+\int_\Omega |u|^{p+2}\,dx\le \frac{L^2}{4\nu}\int_\Omega h^2dx+\left(\nu+\frac{1}{2}\right)\int_\Omega |u|^2\,dx+\frac{1}{2}\int_\Omega h^2dx$$

or

$$\frac{L^2}{4\nu}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx+\frac{L^2}{2\nu\,(p+2)}\frac{d}{dt}\int_\Omega |u|^{p+2}\,dx+\int_\Omega |\nabla u|^2\,dx+$$

$$+\int_\Omega |u|^{p+2}\,dx-\left(\nu+\frac{1}{2}\right)\int_\Omega |u|^2\,dx\le\left(\frac{L^2}{4\nu}+\frac{1}{2}\right)\int_\Omega h^2dx. \qquad (20)$$

Let

$$L_1=\min\left(\frac{L^2}{4\nu},\frac{1}{2},\frac{L^2}{2\nu\,(p+2)}\right).$$

Since

$$\int_\Omega |\nabla u|^2\,dx=\frac{1}{2}\int_\Omega |\nabla u|^2\,dx+\frac{1}{2}\int_\Omega |\nabla u|^2\,dx\ge\frac{1}{2}\int_\Omega |\nabla u|^2\,dx+\frac{1}{2C_\Omega^2}\int_\Omega |u|^2\,dx,$$

then from (20) we have

$$L_1\frac{d}{dt}\int_\Omega \left(|\nabla u|^2+|u|^2+|\nabla u|^{p+2}\right)dx+\frac{1}{2}\int_\Omega |\nabla u|^2\,dx+$$

$$+\left(\frac{1}{2C_\Omega^2}-\nu-\frac{1}{2}\right)\int_\Omega |u|^2\,dx+\int_\Omega |u|^{p+2}\,dx\le\left(\frac{L^2}{4\nu}+\frac{1}{2}\right)\int_\Omega h^2dx.$$

By dividing by L_1 this inequality, by setting

$$\delta=\frac{1}{L_1}\min\left(\frac{1}{2},\frac{1}{2C_\Omega^2}-\nu-\frac{1}{2}\right)$$

and using

$$\|h\|^2_{L^2(\Omega)}\le m$$

we obtain

$$\frac{d}{dt}\int_\Omega \left(|\nabla u|^2+|u|^2+|\nabla u|^{p+2}\right)dx+$$

$$+\delta\frac{d}{dt}\int_\Omega \left(|\nabla u|^2+|u|^2+|\nabla u|^{p+2}\right)dx\le\left(\frac{L^2}{4\nu}+\frac{1}{2}\right)m=C,$$

that is the inequality (17).

Let

$$B_0=\left[\left\{y\in V:\Phi\left(y\right)\le\frac{2C}{\delta}\right\}\right],$$

where $[M]$ denotes a closure of set M.

It is easy to see that B_0 is bounded.

We must prove that B_0 is absorbing. For an arbitrary bounded set $B \subset V$ $(B = \{y \in V : \|y\|_V \le \chi\})$ we must find $t_1(B) = t_1(\chi)$ such that $S(t)B \subset B_0$ for any $t \ge t_1(\chi)$, that is for arbitrary $u^0 \in B$ it holds $u = S(t)u^0 \in B_0$. Since u is a solution of problem (1)-(3) with initial data u^0, then it holds the inequality (17). By multiplying (17) by $e^{\delta t}$ we have

$$\frac{d}{dt}\left(\Phi(u)e^{\delta t}\right) \le Ce^{\delta t}$$

or

$$\Phi(u)e^{\delta t} - \Phi(u)\Big|_{t=0} \le \frac{C}{\delta}\left(e^{\delta t}-1\right)$$

for any $t \ge 0$, whence

$$\Phi(u) \le \frac{C}{\delta} + \left(-\frac{C}{\delta} + \Phi(u)\Big|_{t=0}\right)e^{-\delta t}. \tag{21}$$

Since $o < p \le \dfrac{2}{2-N}$ if $N \ge 3$ and $p > 0$ if $N = 1, 2$, then

$$\Phi(u)|_{t=0} = \int\limits_{\Omega}\left(\left|u^0\right|^2 + \left|\nabla u^0\right|^2 + \left|u^0\right|^{p+2}\right)dx = \left\|u^0\right\|_V^2 + \left\|u^0\right\|_{p+2}^{p+2} \le \left\|u^0\right\|_V^2 + \left\|u^0\right\|_V^{p+2}.$$

Since

$$u^0 \in B = \{y \in V : \|y\|_V \le \chi\},$$

then the right part of the last inequality is bounded by a constant which depends on χ. We denote this constant by $C(\chi)$. Then from (21) we obtain

$$\Phi(u) \le \frac{C}{\delta} + \left(-\frac{C}{\delta} + C(\chi)\right)e^{-\delta t}. \tag{22}$$

We choose t such that

$$\left(-\frac{C}{\delta} + C(\chi)\right)e^{-\delta t} \le \frac{C}{\delta},$$

that is

$$t \ge \frac{1}{\delta}\ln\frac{-\frac{C}{\delta} + C(\chi)}{\frac{C}{\delta}} = \frac{1}{\delta}\ln\left(\frac{\delta C(\chi)}{C} - 1\right) = t_1(\chi)$$

$$\left(t > 0,\ \text{ if } \frac{\delta C(\chi)}{C} - 1 \le 0\right).$$

Therefore from (22) we obtain that

$$\Phi(u) \le \frac{2C}{\delta},$$

that is

$$u = S(t)u^0 \in B^0$$

for any $t \ge t_1(\chi)$.

Theorem 4 is proved.

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