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# SPECTRAL ANALYSIS OF STURM-LIOUVILLE OPERATOR ON A SEMI-AXIS WITH $\delta'$-POTENTIAL

**Abstract**

*In the paper, the self-adjointness of the operator* $A = -\dfrac{d^2}{dx^2} + \beta\delta'(x - x_0)$ *in the space* $L_2(0, +\infty)$ *is proved. It is shown that the limit spectrum of the operator* $A$ *coincides with the absolutely continuous part of its spectrum. The integral property of the resolvent of the operator* $A$ *is proved and an explicit expression for the integral kernel is obtained.*

In the present paper we prove the self-adjointness and investigate the spectrum of the operator $-\dfrac{d^2}{dx^2} + \delta'(x - x_0)$ in the space $L_2(0, +\infty)$, where $\delta(x - x_0)$ is Dirac's function at the point $x_0 \in (0, +\infty)$, $\delta'(x - x_0)$, is a generalized derivative $\beta \in (-\infty, +\infty)$.

Spectral analysis of Sturm-Liouville operator with $\delta'$-potential has a great importance for the problems of quantum mechanics [1].

In [2-4], some spectral properties of Sturm-Liouville operator on a semi-axis with pointwise $\delta$ and $\delta'$-interactions are studied.

Our approach on definition of the operator $-\dfrac{d^2}{dx^2} + \beta\delta'(x - x_0)$ in the space $L_2(0, +\infty)$ is based on the formula of multiplication of $\delta'(x - x_0)$ by discontinuous functions $f(x)$ for which $f(x)$ and its classic derivative $f'(x)$ have at the point $x_0$ discontinuities of first kind. This multiplication formula has the following form [5]:

$$\delta'(x - x_0)f = -\frac{1}{2}\left[f'(x_0 + 0) + f'(x_0 - 0)\right]\delta(x - x_0)+$$

$$+\frac{1}{2}\left[f(x_0 + 0) + f(x_0 - 0)\right]\delta'(x - x_0). \tag{1}$$

Denote by $D(A)$ the set of functions $f \in W_2^2((0, +\infty)\setminus\{x_0\})$ satisfying the boundary conditions

$$f(0) = 0, \quad (\beta - 2)f(x_0 + 0) + (\beta + 2)\cdot f(x_0 - 0) = 0,$$

$$(\beta + 2)\, f'(x_0 + 0) + (\beta - 2)\cdot f'(x_0 - 0) = 0. \tag{2}$$

Determine in the space $L_2(0, +\infty)$ the operator $A$ :

$$Af = -\frac{d^2 f}{dx^2} + \beta\delta'(x - x_0)\cdot f, \qquad f \in D(A),$$

where $\dfrac{d^2 f}{dx^2}$ is a generalized derivative of first order of the function $f \in D(A)$, the product $\delta'(x - x_0)\cdot f$ is determined by formula (1).

By conditions (2), the operator $A$ is a closed symmetric operator in the space $L_2(0, +\infty)$.

**Theorem 1.** *The operator $A$ is self-adjoint operator in the space $L_2(0,+\infty)$. The resolvent $R_z(A)=(A-zE)^{-1}$ of the operator $A$ is an integral operator in $L_2(0,+\infty)$ and the integral kernel $K(x,y;z)$ for $z=-\lambda^2\in\rho(A)$ has the representation*

$$K(x,y;-\lambda^2)=G(x,y;-\lambda^2)+\frac{\beta}{2\Delta}\times$$

$$\times\left\{\left[\left(2-\beta e^{-2\lambda x_0}\right)G_1(x,y;-\lambda^2)-\beta\lambda(e^{-2\lambda x_0}+1)G(x_0,y;-\lambda^2)\right]-\right.$$

$$-G(x,x_0;-\lambda^2)+\left[\left(2-\beta e^{-2\lambda x_0}\right)G(x_0,y;-\lambda^2)-\right.$$

$$\left.\left.-\frac{\beta}{\lambda}(e^{-2\lambda x_0}-1)G_1(x_0,y;-\lambda^2)\right]G_2(x,x_0;-\lambda^2)\right\}, \tag{3}$$

*where*

$$G(x,y;-\lambda^2)=\begin{cases}\frac{1}{\lambda}sh\ \lambda\ x\cdot e^{-\lambda y}, & 0<x<y,\\ \frac{1}{\lambda}sh\ \lambda\ y\cdot e^{-\lambda x}, & y<x,\end{cases}$$

$$G_1(x,y;-\lambda^2)=\begin{cases}ch\ \lambda\ x\ \cdot e^{-\lambda y}, & 0<x<y,\\ -sh\ \lambda\ y\cdot e^{-\lambda x}, & y<x,\end{cases}$$

$$G_2(x,y;-\lambda^2)=\begin{cases}-sh\ \lambda\ x\cdot e^{-\lambda y}, & 0<x<y,\\ ch\ \lambda\ y\cdot e^{-\lambda x}, & y<x,\end{cases}$$

$$\Delta=1+\frac{\beta^2}{4}-\beta e^{-2\lambda\ x_0}.$$

**Proof.** Since $A$ is a closed symmetric operator, then for proving the self-adjointness of the operator $A$ it suffies to show that the resolvent set of this operator contains real numbers ( [6] Corollary of theorem X.1).

Find the resolvent of the operator $A$. To this end we solve in the space $L_2(0,+\infty)$ the equaton

$$-\frac{d^2f}{dx^2}+\beta\delta'(x-x_0)\cdot f+\lambda^2 f=g\quad(\lambda>0,\ g\in L_2(0,+\infty)). \tag{4}$$

Taking into account formula (1), we write equation (4) in the following form

$$-\frac{d^2f}{dx^2}+\lambda^2 f=\frac{\beta}{2}\left[f'(x_0+0)+f'(x_0-0)\right]\delta(x-x_0)-$$

$$-\frac{\beta}{2}\left[f(x_0+0)+f(x_0-0)\right]\delta'(x-x_0)+g(x). \tag{5}$$

Let $G(x,y;-\lambda^2)$ be a fundamental solution of the operator $-\frac{d^2}{dx^2}+\lambda^2$ in $(0,+\infty)$. It is known that

$$G(x,y;-\lambda^2)=\begin{cases}\frac{1}{\lambda}sh\ \lambda\ x\cdot e^{-\lambda y}, & 0<x<y,\\ \frac{1}{\lambda}sh\ \lambda\ y\cdot e^{-\lambda x}, & y<x.\end{cases} \tag{6}$$

Applying $\left(-\frac{d^2}{dx^2}+\lambda^2\right)^{-1}$ to both sides of equation (5), we get

$$f(x)=\frac{\beta}{2}\left[f'(x_0+0)+f'(x_0-0)\right]G(x,x_0;-\lambda^2)+$$

$$+\frac{\beta}{2}\left[f(x_0+0)+f(x_0-0)\right]G'_y(x,x_0;-\lambda^2)+\int\limits_0^{+\infty}G(x,y;-\lambda^2)g(y)dy. \qquad (7)$$

Here we used the equalities

$$G(x,y;-\lambda^2)^*\delta(x-x_0)=G(x,x_0;-\lambda^2)$$

$$G(x,y;-\lambda^2)^*\delta'(x-x_0)=-G'_y(x,x_0;-\lambda^2).$$

From (6) we find

$$G(x,x_0;-\lambda^2)=\begin{cases}\frac{1}{\lambda}sh\ \lambda\ x\cdot e^{-\lambda x_0}, & 0<x<x_0,\\ \frac{1}{\lambda}sh\ \lambda\ x_0\cdot e^{-\lambda x}, & x_0<x,\end{cases} \qquad (8)$$

$$G'_y(x,x_0;-\lambda^2)=\begin{cases}-sh\ \lambda\ x\cdot e^{-\lambda x_0}, & 0<x<x_0,\\ ch\ \lambda\ x_0\cdot e^{-\lambda x}, & x_0<x.\end{cases} \qquad (9)$$

Find the quantities $f(x_0+0)+f(x_0-0)$ and $f'(x_0+0)+f'(x_0-0)$. Passing to limit in (7) as $x\to x_0+0$ and $x\to x_0-0$, and putting together the obtained equalities, using relations (8) and (9), we get:

$$\left(1-\frac{\beta}{2}e^{-2\lambda x_0}\right)\left[f(x_0+0)+f(x_0-0)\right]+\frac{\beta}{2\lambda}\left(e^{-2\lambda x_0}-1\right)\times$$

$$\times\left[f'(x_0+0)+f'(x_0-0)\right]=2\int\limits_0^{+\infty}G(x_0,y;-\lambda^2)g(y)dy. \qquad (10)$$

Calculate the derivative $f'(x)$ for $x\neq x_0$. From (7) we find

$$f'(x)=\frac{\beta}{2}\left[f'(x_0+0)+f'(x_0-0)\right]G'_x(x,x_0;-\lambda^2)+$$

$$+\frac{\beta}{2}\left[f(x_0+0)+f(x_0-0)\right]G''_{yx}(x,x_0;-\lambda^2)+\int\limits_0^{+\infty}G'_x(x,y;-\lambda^2)g(y)dy. \qquad (11)$$

Obviously,

$$G'_x(x,y;-\lambda^2)=\begin{cases}ch\ \lambda\ x\ \cdot e^{-\lambda y}, & 0<x<y,\\ -sh\ \lambda\ y\cdot e^{-\lambda x}, & y<x,\end{cases} \qquad (12)$$

$$G'_x(x,x_0;-\lambda^2)=\begin{cases}ch\ \lambda\ x\ \cdot e^{-\lambda x_0}, & 0<x<x_0,\\ -sh\ \lambda\ x_0\cdot e^{-\lambda x}, & x_0<x,\end{cases} \qquad (13)$$

$$G''_{yx}(x,x_0;-\lambda^2)=\begin{cases} -\lambda ch\ \lambda\ x\ \cdot e^{-\lambda x_0}, & 0<x<x_0, \\ -\lambda\ ch\lambda\ x_0\cdot e^{-\lambda x}, & x_0<x. \end{cases} \tag{14}$$

Pass in (11) to limit as $x\to x_0+0$ and $x\to x_0-0$, put together the obtained equalities. Then using (13) and (14), after simple transformations we get

$$\frac{\beta\lambda}{2}\left(e^{-2\lambda x_0}+1\right)\left[f(x_0+0)+f(x_0-0)\right]+\left(1-\frac{\beta}{2}e^{-2\lambda x_0}\right)\times$$

$$\times\left[f'(x_0+0)+f'(x_0-0)\right]=2\int\limits_0^{+\infty}G'_x(x_0,y;-\lambda^2)g(y)dy. \tag{15}$$

Then, we solve the system of linear equations (10), (15) by the Cramer principe and find the quantities $f(x_0+0)+f(x_0-0)$ and $f'(x_0+0)+f'(x_0-0)$. Substituting the found expressions in (7), after simple transformations we get

$$f(x)=\int\limits_0^{+\infty}K(x,y;-\lambda^2)g(y)dy, \tag{16}$$

where

$$K(x,y;-\lambda^2)=G(x,y;-\lambda^2)+$$

$$+\frac{\beta}{2\Delta}\left\{\left[\left(2-\beta e^{-2\lambda x_0}\right)G_1(x,y;-\lambda^2)-\beta\lambda\left(e^{-2\lambda x_0}+1\right)G(x_0,y;-\lambda^2)\right]-\right.$$

$$-G(x,x_0;-\lambda^2)+\left[\left(2-\beta e^{-2\lambda x_0}\right)G(x_0,y;-\lambda^2)-\right.$$

$$\left.\left.-\frac{\beta}{2}\left(e^{-2\lambda x_0}-1\right)G_1(x_0,y;-\lambda^2)\right]G_2(x,x_0;-\lambda^2)\right\}$$

$$G_1(x,y;-\lambda^2)\equiv G'_x(x,y;-\lambda^2)=\begin{cases} ch\ \lambda\ x\ \cdot e^{-\lambda y}, & 0<x<y, \\ -sh\ \lambda y\ \cdot e^{-\lambda x}, & y<x, \end{cases}$$

$$G_2(x,y;-\lambda^2)\equiv G'_y(x,y;-\lambda^2)=\begin{cases} -sh\ \lambda\ x\ \cdot e^{-\lambda y}, & 0<x<y, \\ ch\ \lambda y\ \cdot e^{-\lambda x}, & y<x. \end{cases}$$

$\Delta$ is the determinant of the considered system of linear equations

$$\Delta=\Delta(\lambda)=\begin{vmatrix} 1-\frac{\beta}{2}e^{-2\lambda x_0} & \frac{\beta}{2\lambda}(e^{-2\lambda x_0}-1) \\ \frac{\beta\lambda}{2}(e^{-2\lambda x_0}+1) & 1-\frac{\beta}{2}e^{-2\lambda x_0} \end{vmatrix}=1+\frac{\beta^2}{4}-\beta e^{-2\lambda x_0}. \tag{17}$$

It is easy to show that $\Delta(\lambda)\neq 0$ for $\lambda\in(0,+\infty)$, $\beta\in(-\infty,+\infty)$. From (13) it follows that for the fixed $\lambda=\lambda_0>0$ and $\beta=\beta_0\in(-\infty,+\infty)$ it is valid the estimation

$$\sup_{x\in(0,+\infty)}\int\limits_0^{+\infty}\left|K(x,y;-\lambda_0^2)\right|dy=M<+\infty.$$

uniform with respect to $y\in(0,+\infty)$.

Hence and from (16) we get

$$\|f\|_{L_2} \leq M \|g\|_{L_2},$$

where $\|\cdot\|_{L_2}$ is a norm in the space $L_2(0,+\infty)$.

Consequently,

$$R_{-\lambda_0^2}(A)f = (A+\lambda_0^2)^{-1}g(x) = \int\limits_0^{+\infty} K(x,y;-\lambda_0^2)g(y)dy. \tag{18}$$

Thus, the real number $-\lambda_0^2$ belongs to the resolvent operator $A$, i.e. $-\lambda_0^2 \in$ $\in \rho(A)$.

Therefore, $A$ is a self-adjoint operator in $L_2(0,+\infty)$ ([6], Corollary of theorem X.1). By (19), the resolvent $R_z(A)$ for $z = -\lambda^2 \in \rho(A)$ is an integral operator in the space $L_2(0,+\infty)$. Theorem 1 is proved.

The structure of the spectrum of the operator $A$ is described by the following theorem.

**Theorem 2.** *The limit spectrum of the operator $A$ coincides with the absolutely continuous part of its spectrum, more over*

$$\sigma(A) = \sigma_{ess}(A) = \sigma_{ac}(A) = [0,+\infty]. \tag{19}$$

*The resolven set $\rho(A) = \{z : z \in C \backslash [0,+\infty]\}$.*

*The operator $A$ has no eigen values.*

**Proof.** Equality (19) is proved by means of the standard procedure that is usually used by investigating such problems. Weyl's theorem on limit (essential) spectrum ([7], theorem XIII.14] and the theorem on preservation of absolutely continuous parts of the spectra of perturbed and unperturbed operators ([8], ch.X, theorem 42) are used. Application of these theorems reduce to equalities (19).

Show that operator $A$ has no eigen values. For that it suffices to show that the equation

$$1 + \frac{\beta^2}{4} - \beta e^{-2\lambda x_0} = 0 \tag{20}$$

has no positive solutions.

Obviously, for $\beta \leq 0$ equation (20) has no solutions. Let $\beta > 0$. In this case equation (20) has the solution

$$\lambda = -\frac{1}{2x_0} In \left(\frac{1}{\beta} + \frac{\beta}{4}\right).$$

Since $x_0 > 0$, then condition $\lambda > 0$ is equivalent to the inequality $ln \left(\frac{1}{\beta} + \frac{\beta}{4}\right) < 0$. The last inequality is contradictary, since for any $\beta > 0$ it holds the inequality $\frac{1}{\beta} + \frac{\beta}{4} > 1$. Consequently, equation (20) has no positive solutions. This means that the operator has no eigen values.

Obviously, $\rho(A) = \{z : z \in \backslash [0,+\infty)\}$. Theorem 2 is proved.

The main results of this paper were announced by the authors in [9].

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