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ELLIPTIC EQUATION RELATIVE TO DOMAIN EVOLUTION

Abstract

In the paper we consider the Dirichlet problem for the Poisson equation relative to domain evolution. For studying this problem, at first we introduce a space of a pair of convenx domains and define a scalar product in this space. Using this we prove the existence and uniqueness of the stated problem and get an analogy of the maximum principle.

1. Introduction

A wide class of problems of practice reduces to studying the change of the shape of the object or body under consideration relative to some parameters [1-3]. Diffusion processes, the problems on thermal extension and straightening of the body, the elasticity theory problems, ecological problems, the problem on oil spot distribution on sea surface, biological processes and etc. are the examples of such problems.

As a rule, by investigating such problems, variations of the points of the body are studied. However, very often, not the alternation of the points of the body but variation of its shape represents a great interest.

Study of the problem in such a statement is connected with some mathematical difficulties [4-6, 10]. This in the first place is connected with definition of rate of change of domain characterizing the body's shape [7].

In the present paper, we consider a boundary value problem relative to domain evalution. As first we study solvability of this problem and then prove the maximum pronciple for the problem under consideration.

For investigating such problems, we define the rate of change of the shape of domain in linear space of a pair of convex sets. Such a definition of domain evolution enables to investigase a wide class of such practical problems such as optimal control problems.

2. Space of convex sets

Let M be a totality of convex closed bounded sets in R^m. The function

$$P_D(x) = \sup_{l \in D} (l, x), \ x \in D, \tag{1}$$

is said to be a support function of the set $D \in M$, where $P_D(x)$ is continuously convex and positive homogeneous ([8]). The latter means that $P_D(\lambda x) = \lambda P_D(x), \ \lambda > 0$. Note that $P_D(0) = 0$. To each convex closed bounded $D \in M$, formula (1) assigns a convex, continuous, positive-homogeneous function $P_D(x)$. The inverse is also true: for each contiunously-convex, positive-homogeneous function $P(x)$ there exists a unique closed convex bounded set $D \in M$ such that $P(x) = P_D(x)$. The set D coincides with the subdifferential of the function $P(x)$ at the point $0 \in R^m$ ([8]).

Consider the direct product $M \times M$, i.e. the totality of pairs (A, B), where $A, B \in M$. Define in $M \times M$ the operations of addition and multiplication by a real number:

$$(A, B) + (C, D) = (A + C, B + D)$$

$$\lambda (A, B) = (\lambda A, \lambda B) \text{ if } \lambda \geq 0$$

$$\lambda (A, B) = (|\lambda| B, |\lambda| A) \text{ if } \lambda < 0$$

Introduce in $M \times M$ the equivalence ratio: the pairs (A, B) and (C, D) are equivalent if $A + D = B + C$. We denote it as $(A, B) \approx (C, D)$ or $(A, B) = (C, D)$. In [8] it is shown that the set $M \times M$ together with the above defined algebraic operations is a linear space.

Let $a = (A_1, A_2)$, $b = (B_1, B_2)$, A_i, $B_i \in M$, $i = 1, 2$, B be a unique ball, $S_B = \partial B$ be a unique sphere. Define the scalar product $a \bullet b$ in $M \times M$ as follows

$$a \bullet b = \int_{S_B} p(x)\, q(x)\, ds, \tag{2}$$

here $P(x) = P_{A_1}(x) - P_{A_2}(x)$, $q(x) = P_{B_1}(x) - P_{B_2}(x)$, $P_{A_i}(x)$, $P_{B_i}(x)$ are support functions of the sets A_i and B_i $i = 1, 2$, respectively.

It is shown that $a \bullet b$ satisfies all the axioms of a sclar product.

Denote the space $M \times M$ with scalar product (2) by ML_2. The distance in this space between the sets $A \in M$ and $B \in M$ is determined as the norm of the element $a = (A, 0) - (B, 0) = (A, B)$,

$$\|a\|_{ML_2} = \sqrt{a \bullet a} = \left(\int_{S_B} [P_A(x) - P_B(x)]^2\, ds \right)^{1/2}. \tag{3}$$

Let $z = [z_1, z_2, ..., z_n]$, $z = [y_1, y_2, ..., y_n]$ be vectors, where $x_i, y_i \in M \times M$. In this case, the scalar product and norm are determined in the following way:

$$z \bullet y = z_1 \bullet y_1 + z_2 \bullet y_2 + ... + z_n \bullet y_n,$$

$$\|z\|^2 = \|z_1\|^2 + \|z_2\|^2 + ... + \|z_n\|^2.$$

For simplicity, instead of $z \in ML_2^{(n)}$, write $z \in ML_2$.

3. Derivative domain of the function

Assume that at time $t \in [0,T]$ the domain under consideration has the shape $U(t)$. By changing t the domain $U(t)$ also changes. The evolution rate of domain $U(t)$ is characterized by the variable

$$\frac{\partial P_{U(t)}(x)}{\partial t} = \lim_{\Delta t \to 0} \frac{P_{U(t+\Delta t)}(x) - P_{U(t)}(x)}{\Delta t}, \ x \in S_B. \tag{4}$$

If there exist the domains $V_1(t), V_2(t) \in M, \ \ t \in [0,T]$ such that

$$\frac{\partial P_{U(t)}(x)}{\partial t} = P_{V_1(t)}(x) - P_{V_2(t)}(x),$$

we call the variable $v(t) = (V_1(t), V_2(t)) \in M \times M$ the rate of evolution of domain $U(t)$.

Example 1. Let $U(t) = B_t$ be a ball of radius $t > 0$ with the center at the origin. It is known [9] that in this case $P_{U(t)}(x) = t \cdot \|x\|_{R^m}$. Then $v(t) = (B_1, 0)$. If $U(t)$ is a rectangle

$$U(t) = \{(x_1, x_2) : -t \le x_1 \le 2t, \ \ t \le x_2 \le 3t\},$$

then $v(t) = (U(1), 0)$.

For any t consider the pair $u(t) = (U_1(t), U_2(t)) \in M \times M$. Writing

$$u(t) = u_1(t) - u_2(t) = (U_1(t), 0) - (U_2(t), 0)$$

and assuming that $\dot{u}_1(t), \dot{u}_2(t) \in M \times M$ we similarly determine $\dot{u}(t) = \dot{u}_1(t) - \dot{u}_2(t) \in M \times M$.

We can show that for any $u(t), \eta(t) \in M \times M$, where $\left\|\dot{u}(t)\right\| \in L_2(t_0, T)$, $\left\|\dot{\eta}(t)\right\| \in L_2(t_0, T)$ the following relation is valid:

$$\int_{t_0}^{T} \dot{u}(t) \bullet \eta(t)\, dt = u(T) \bullet \eta(T) - u(t_0) \bullet \eta(t_0) + \int_{t_0}^{T} u(t) \bullet \dot{\eta}(t)\, dt. \tag{5}$$

Now, let the domain U depend on the parameter $y \in R^n$. We can similarly determine the partial derivatives of $y_1, y_2, ..., y_n$.

Example 2. Let $U(y_1, y_2) = y_1^2 V_1 + y_2^2 V_2$, where V_1, V_2 are some convex bounded sets. Then, by checking we can see that if $y_i \ge 0, \ i = 1, 2$, then

$$\frac{\partial U}{\partial y_i} = (2 y_i V_i, 0).$$

Otherwise

$$\frac{\partial U}{\partial y_i} = (0, 2|y_i| V).$$

It is also clear that

$$\frac{\partial^2 U}{\partial y_i^2} = (2V_i, 0).$$

4. Problem statement

Let $D \subset R^n$ be a given bounded domain with rather smooth boundary S, and the set $U \subset R^m$ depend on the parameter $y = (y_1, y_2, ..., y_n) \in D$ i.e. $U = U(y)$. Write that $U \in C(D)$ if the support function $P_{U(y)}(x)$ of the set $U(y)$ is continuous with respect to y in D. $U \in C^1(D)$ is determined in the same way.

Consider the following boundary value problem for the Poisson equation

$$\Delta U = -F(y), \quad y \in D, \tag{6}$$

$$U(\xi) = G(\xi), \quad \xi \in S. \tag{7}$$

Let

$$F(y) = (F_1(y), F_2(y)) \in M \times M, \ y \in D, \ \ G(\xi) = (G_1(\xi), G_2(\xi)) \in M \times M, \ \xi \in S$$

Unlike the traditional boundary value problems, here the solution of problem (6),(7) is the set $U = U(y) \in M$ or the pair of convex sets $U(y) = (U_1(y), U_2(y)) \in M \times M$.

Without loss of generality later on we'll call such type functions the domain of the function. We'll understand equation (6) and boundary condition (7) as equality of a pair of convex sets.

Theorem 1. *Let $F_i \in C^1(D) \cap C(\overline{D})$ and $G_i \in C(S)$, $i = 1, 2$. Then there exists a unique solution of problem (6), (7) $U(y) = (U_1(y), U_2(y)) \in M \times M$, $y \in D$.*

Proof. For any $x \in R^m$ consider the following boundary value problem

$$\Delta P(y; x) = P_{F_1(y)}(x) - P_{F_2(y)}(x), \ y \in D, \tag{8}$$

$$P(\xi; x) = P_{G_1(\xi)}(x) - P_{G_2(\xi)}(x), \ \xi \in S. \tag{9}$$

By $L(y, z)$ denote the Green function of the first boundary value problem (the Dirichlet problem) for the Laplace operator in domain D. Then for any $x \in R^m$ the solution of problem (6),(7) may be represented by the formula [11].

$$P(y; x) = \int_S \frac{\partial L(y, z)}{\partial n_z} \left[P_{G_2(z)}(x) - P_{G_1(z)}(x)\right] dS_z +$$

$$+ \int_D L(y, z) \left[P_{F_2(z)}(x) - P_{F_1(z)}(x)\right] dz. \tag{10}$$

Obviously, there exist positive functions $L_1(y, z), L_2(y, z)$ such that

$$\frac{\partial L(y, z)}{\partial n_z} = L_2(y, z) - L_1(y, z).$$

Then we can write (10) in the form

$$P(y; x) = \int_S \left[L_1(y, z) P_{G_1(z)}(x) + L_2(y, z) P_{G_2(z)}(x)\right] dS_z + \int_D L(y, z) P_{F_2(z)}(x) dz -$$

$$-\int\limits_S \left[L_1(y,z)\,P_{G_2(z)}(x)+L_2(y,z)\,P_{G_1(z)}(x)\right]dS_z-\int\limits_D L(y,z)\,P_{F_1(z)}(x)\,dz$$

Since the sets $F_i(x)$, $G_i(x)$, $i=1,2$, are convex, their support functions $P_{F_i(y)}(x)$, $P_{G_i(y)}(x)$ are convex and positive homogeneous in $x\in R^m$.

Taking into account that the Green's function is nonnegative, all subintegrand functions are convex with respect to $x\in R^m$. So,

$$P(y;x)=P_1(y;x)-P_2(y;x),$$

where $P_i(y;x)$, $i=1,2$, is convex and positive homogeneous with respect to $x\in R^m$. Then for any $y\in D$ there exist unique convex bounded sets $U_1(y), U_2(y)\subset R^m$ that

$$P_i(y;x)=P_{U_i(y)}(x),\quad x\in R^m,\ y\in D,\ i=1,2.$$

Thus, we get that the function

$$P(y;x)=P_{U_1(y)}(x)-P_{U_2(y)}(x)$$

is the solution of problem (8),(9). And this in its turn shows that $U(y)=(U_1(y),U_2(y))\in M\times M$, $y\in D$ is the solution of problem (6),(7) and this solution is unique.

The theorem is proved.

It is interesting to investigate problem (6),(7) when $F(y), G(\xi)$ are convex sets from the space R^m. It turns out to be that in this case the solution of problem (6),(7) is also a convex set from R^m.

Theorem 2. *Let for any $y\in D$ and $\xi\in S$ $F(y), G(\xi)$ be closed convex bounded sets, and $F\in C^1(D)\cap C(\overline{D})$, $G\in C(S)$. Then there exists a unique domain of the function $U=U(y)\subset R^m$ of the solution of problem (6),(7) and this solution is a closed, convex, bounded set.*

Proof. At first consider the case

$$\Delta U=0,\ y\in D, \tag{11}$$

$$U(\xi)=G(\xi),\ \xi\in S. \tag{12}$$

Show that if $G(\xi)\in M$, $\xi\in S$, the solution of problem (11), (12) also belongs to M. For any $x\in R^m$ consider the following boundary value problem

$$\Delta P(y;x)=0,\quad y\in D, \tag{13}$$

$$P(\xi;x)=P_{G(\xi)}(x),\ \xi\in S. \tag{14}$$

It is clear that under the imposed conditions, for any $x\in R^m$ there exists a unique solution $P(y;x)$ of problem (13),(14) [11]. Show that $P(y;x)$ is positive-homogeneous and convex with respect to $x \in R^m$. Taking into account that the support function of the set $G(\xi)$ is positive-homogebeous, we have

$$P_{\lambda G(\xi)}(x) = \lambda P_{G(\xi)}(x) = P_{G(\xi)}(\lambda x), \ \lambda \geq 0,$$

Then from (13),(14) we get $P(y;\lambda x) = \lambda P(y;x)$ i.e. $P(y;x)$ is positive-homogeneous with respect to $x \in R^m$. Now show the convexity of $P(y;x)$ with respect to $x \in R^m$. Take any $x_1, x_2 \in R^m$. It is clear that $P\left(y;\frac{x_1+x_2}{2}\right)$ is the solution of the following problem

$$\Delta P\left(y;\frac{x_1+x_2}{2}\right) = 0, \ y \in D, \tag{15}$$

$$P\left(\xi;\frac{x_1+x_2}{2}\right) = P_{G(\xi)}\left(\frac{x_1+x_2}{2}\right), \ \xi \in S. \tag{16}$$

From the convexity condition of the set $G(\xi)$ it follows that $P_{G(\xi)}(x)$ is convex with respect to $x \in R^m$ [9,13], i.e.

$$P_{G(\xi)}\left(\xi;\frac{x_1+x_2}{2}\right) \leq \frac{1}{2}P_{G(\xi)}(x_1) + \frac{1}{2}P_{G(\xi)}(x_2).$$

Using the maximum principle [11], it is easy to show

$$P_{G(\xi)}\left(\xi;\frac{x_1+x_2}{2}\right) \leq \overline{P}(y). \tag{17}$$

Here $\overline{P}(y)$ denotes the solution of the problem

$$\Delta\overline{P}(y;x) = 0, \quad y \in D$$

$$\overline{P}(\xi) = \frac{1}{2}P_{G(\xi)}(x_1) + \frac{1}{2}P_{G(\xi)}(x_2), \quad \xi \in S.$$

It is clear, that

$$\overline{P}(y) = \frac{1}{2}P(y;x_1) + \frac{1}{2}P(y;x_2).$$

Then from (17) we get the convexity of $P(y;x)$. So, $P(y;x)$ is positive homogeneous, convex and continuous with respect to $x \in R^m$. Then for any $y \in D$ there exists a unique convex bounded set $U = U(y) \subset R^m$ such that

$$P(y;x) = P_{U(y)}(x), \ x \in R^m.$$

Then from (13),(14) we get

$$\Delta P_{U(y)}(x) = 0, \ y \in D,$$

$$P_{U(\xi)}(x) = P_{G(\xi)}(x), \quad \xi \in S, \ x \in R^m.$$

Taking into account that for convex closed bounded sets the condition $P_A(x) = P_B(x)$ is equivalent on $A = B$, it follows from the latter that $U = U(y) \in M$ is the solution of problem (11),(12).

Now come back to problem (6),(7). We look for the solution of problem (6),(7) in the form

$$U(y) = V(y) + W(y), \tag{18}$$

where $V = V(y)$ is the solution of problem (11),(13) and $W = W(y)$ is the solution of the following problem

$$\Delta W(y) = -F(y), \ y \in D, \tag{19}$$

$$W(\xi) = 0, \ \xi \in S. \tag{20}$$

We showed that for any $y \in D$ $V = V(y) \in M$. Show that $W(y) \in M$. Instead of problem (19),(20) consider the problem

$$\Delta P(y; x) = -P_{F(y)}(x) \quad y \in D, \tag{21}$$

$$P_{U(\xi)}(x) = 0, \quad \xi \in S, \ x \in R^m. \tag{22}$$

Denote by $L(y; z)$ the Green function of the first boundary value problem for the Laplace operator in domain D. Then we can represent the solution of problem (21),(22) in the form

$$P(y; x) = \int_D L(y; z) P_{F(z)}(x) dz. \tag{23}$$

It is clear that $P(y; x)$ is positive-homogeneous with respect to $x \in R^m$. Show the convexity of $P(y; x)$ with respect to $x \in R^m$. Taking into account that the Green function is non-negative [11], from (23) we have

$$P\left(y; \frac{x_1 + x_2}{2}\right) = \int_D L(y; z) P_{F(z)}\left(\frac{x_1 + x_2}{2}\right) dz \leq$$

$$\leq \frac{1}{2} \int_D L(y; z) P_{F(z)}(x_1) dz + \frac{1}{2} \int_D L(y; z) P_{F(z)}(x_2) dz =$$

$$= \frac{1}{2} P(y; x_1) + \frac{1}{2} P(y; x_2), \quad \forall x_1, x_2 \in R^m.$$

So, $P(y; x)$ is positive-homogeneous convex and continuous with respect to $x \in R^m$. Then for any $y \in D$ there exists a unique convex bounded set $W = W(y) \subset R^m$ such that $P(y; x) = P_{W(y)}(x)$, $x \in R^m$ i.e. $W = W(y)$ is the solution of problem (21),(22). Then from (18) it is clear that the convex closed, bounded, set $U = U(y)$ is the solution of problem (6),(7). The theorem is proved.

The domain of the function (or of the function of the set) $U = U(y)$ is called harmonic in domain $D \subset R^n$, if $U \in C^2(D)$ and at each point $y \in D$ satisfies the Laplace equation

$$\Delta U(x) = 0, \ x \in D.$$

Theorem 3. (Maximum principle). *Let $U = U(y)$ be a harmonic function in D, $U \in C(\overline{D})$, and $U(\xi)$, $\xi \in S$ be a convex, closed, bounded set. If there exist convex sets G_0, $G_1 \subset R^m$ such that*

$$G_0 \subset U(\xi) \subset G_1, \quad \forall \xi \in S, \tag{24}$$

then for any $y \in D$

$$G_0 \subset U(y) \subset G_1.$$

Proof. By condition of theorem 3, $U(\xi) = G(\xi) \in M$, $\xi \in S$ i.e. $U = U(y)$ is the solution of problem (11),(12). Then by theorem 2, $U(y) \in M$, $\forall y \in D$. So, the support function $P_{U(y)}(x)$ of the set $U(y)$ is the solution of problem (13),(14). By condition (24) [8,13]

$$P_0(x) \le P_{U(\xi)}(x) \le P_1(x), \ \forall x \in R^m, \ \xi \in S.$$

Here $P_0(x), P_1(x)$ is a support function of the set $G_0, G_1 \subset R^m$. Using the maximum principle [11], we see that for any $x \in R^m$

$$P_0(x) \le P_{U(y)}(x) \le P_1(x), \quad y \in D.$$

And this shows that [8,13]

$$G_0 \subset U(y) \subset G_1.$$

The theorem is proved.

5. Generalized solution of the first boundary value problem. Now let $F \in L_2(D)$, $G \in L_2(S)$ and domain $F(y)$, $y \in D$ and $G(\xi), \xi \in S$ be convex and bounded. The condition $F \in L_2(D)$ means that the support function $P_{F(y)}(x)$ of the set $F(y)$ belongs to $L_2(D)$, i.e. $P_{F(y)}(x) \in L_2(D)$, $\forall x \in R^m$.

Theorem 4. *Let for any $y \in D$ and $\xi \in S$ $F(y), G(\xi)$ be closed convex bounded sets, R^m and $F \in L_2(D)$, $G \in L_2(S)$. Then there exists a unique domain of the function $U \in W_2^1(D)$ of solution of problem (6),(7) and almost for all $y \in D$ this solution is a closed convex bounded set.*

Proof. As we have noted above, we can write problem (6),(7) in the equivalent form

$$\Delta P(y;x) = P_{F(y)}(x), \ y \in D, \tag{25}$$

$$P(\xi;x) = P_{G(\xi)}(x) \quad \xi \in S. \tag{26}$$

It is known that for any $x \in R^m$ the solution of problem is (25),(26) $P(\cdot;x) \in W_2^1(D)$. It is clear that $P(y;x)$ is positive-homogeneous and continuous with respect to $x \in R^m$. Then we can show that $P(y;x)$ is a convex function with respect to $x \in R^m$. For that it suffices to take the sequence of closed bounded sets $F_n(y)$, $y \in D$ and $G_n(\xi), \xi \in S$ satisfying the conditions of theorem 2 such that

$$\max_{x \in B} \left\| P_{F_n(y)}(x) - P_{F(y)}(x) \right\|_{L_2(D)} \to 0,$$

$$\max_{x\in B}\left\|P_{G_n(\xi)}(x)-P_{G(\xi)}(x)\right\|_{L_2(S)}\to 0,\quad \text{as}\quad n\to\infty$$

Then the solution of problem (6), (7) the sequence of the domain of the function $U_n = U_n(y)$, for all $y \in D$ is a closed convex bounded set and

$$\max_{x\in B}\left\|P_{U_n(\cdot)}(x)-P(\cdot,x)\right\|_{W_2^1(D)}\to 0.$$

Hence it follows that $P(y;x)$ is a convex function with respect to $x \in R^m$. Then for any $y \in D$ there exists a unique convex bounded set $U = U(y) \subset R^m$ such that $P_{U(y)}(x) = P(y;x) \in W_2^1(D)$, $x \in R^m$ i.e. $U \in W_2^1(D)$ is the solution of problem (6),(7).

The theorem is proved.

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