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ON REMOVABLE SETS FOR DEGENERATED ELLIPTIC EQUATIONS

Abstract

The questions of compact removability for degenerated elliptic equations in classes bounded functions is considered.

Let E_n be n dimensional Euclidean space of the points $x = (x_{1...,}x_n)$. Dennote the ball $B_R(x^0) = \{x : |x - x^0| < R\}$ for $R > 0$ and the cylinder $Q_T^R(x^0) = B(x_0) \cup (0, T)$. Later, let for $x^0 \in E_n$, $R > 0$ and $k > 0$,

$$\varepsilon_k(x^{0\cdot}) = \left\{ x : \sum_{i=1}^{n} \left(\left(x_i - x_i^0 \right)^2 / R^{ni} \right) < (kR)^2 \right\}$$

be an ellipsoid.

Let D be an bounded domain in E_n, with the boundary of domain ∂D, and $0 \in \partial D$. ε is a such king of ellipsoid that $\bar{D} \subset \varepsilon$. $B(\varepsilon)$ is a set of all functions, satisfying in $\bar{\varepsilon}$ the uniform Lipchitz condition and having zero near the $\partial\varepsilon$.
Denote by (α) the vector $(\alpha) = (a_{1...,}a_n)$ and by $W^1_{2,n}(D)$ the Banach space of the functions $u(x)$ given on D with the finite norm

$$\|u\|_{W_{1,2,\alpha}} = \left(\int_D \left(u^2 + \sum_{i=1}^{n} \lambda_i(x) u_i^2 \right) dx \right)^{1/2} \tag{1}$$

there, $u_i = \dfrac{\partial u}{\partial x_i}$, $i = \overline{1, n}$

$$\lambda_i(x) = (|x|_\lambda)^{\alpha_i}, \ |x|_\alpha = \sum_{i=1}^{n} |x_i|^{\frac{2}{2+\alpha_i}} \tag{2}$$

$$0 \le \alpha_i < \frac{2}{n-1}$$

Let $\overset{0}{W}{}^1_{2,\alpha}(D)$ the Banach space of the functions from $C_0^\infty(D)$ closed by the norm of the space $W^1_{2,\alpha}(D)$.

Denote by $M(D)$ the set of all bounded functions in D.

Let $E \subset D$ be some compact. Denote by $A_\varepsilon(D)$ of all functions $u(x) \in C^\infty\overline{(D)}$ of which there exists some neighborhood of the compact E in which $u(x) = 0$.

The compact E is called the removable relative to the first boundary value problem for the operator L in the space $M(D)$ if all generalized solution of the equation $Lu = 0$ in $D \setminus E$ formed in zero on ∂D and belonging to the space $M(D)$ identically equal to zero. We will say that the function $n(x) \in \overset{\circ}{W}{}^1_{2,\varepsilon}(\varepsilon)$ if there exists the sequence of the functions $\{u_{(m)}(x)\}$, $m = 1, 2...$, such that $u_m(x) \in B(\varepsilon)$, $u_m(x) \ge 0$ for $x \in H$ and $\lim\limits_{m\to\infty} \|u_{(m)} - u\|_{W^1_{2,\alpha}(\varepsilon)} = 0$.

The function $u(x) \in W^1_{2,\alpha}(D)$ is nonnegative on ∂D in sense $W^1_{2,\alpha}(D)$ if there exists the sequence of the functions $\{u_{(m)}(x)\}$, $m = 1, 2...$ such that $u_{(m)}(x) \in C^{1,0}$, $u_{(m)}(x) \geq 0$ for $x \in \partial D$ and $\lim\limits_{m\to\infty} \|u_{(m)} - u\|_{W^1_{2,\alpha}(D)} = 0$. It is easy to determine the inequalities $u(x) \geq const$, $u(x) \geq v(x)$, $u(x) \leq 0$, and also equality $u(x) = 1$ on the set H in the sense $\overset{\circ}{W}{}^1_{2,\varepsilon}(\varepsilon)$.

Let $\omega(x)$ be measurable function in D, finite and positive for a.e. $x \in D$. Denote by $L_{p,\omega}(D)$ the Banach space of the functions given on D, with the norm

$$\|u\|_{L\,p,\omega(D)} = \left(\int \left(\omega(x)^{p/2}\,|u|^p\,dx\right)^{1/p}\right), \quad 1 < p < \infty \tag{3}$$

Let $W^1_{p,\alpha}(D)$ be a Banach space of the function given on $u(x)$, with the finite norm D.

$$\|u\|_{W^1_{p,\alpha}(D)} = \left(\int_D \left(|u|^P + \sum_{i=1}^n (\lambda_i(x))^{p/2}\,|u_{x_i}|^p\right) dx\right)^{1/p}, \; 1 < p < \infty \tag{4}$$

Analogously to $\overline{W}{}^{0,1}_{p,\alpha}(D)$, it is introduced the subspace $\overset{0}{W}{}^1_{p,\alpha}(D)$ for $1 < p < \infty$. The space conjugated to $\overset{0}{W}{}^1_{p,\alpha}(D)$, we will denote by $\overline{\overset{\circ}{W}}{}^1_{p,\alpha}(D)$.

The questions of compact removability for Laplace equation is studied by Carleson [1]. The compact removability for elliptic and parabolic equations of nonduvergent structure is considererd by Landis [2], Gadjiev, Mamedova [1]. The removability condition of compact in the space of continuous functions in the papers Harvey and Polking [4], Kilpelainen and Zhong [5] is considered. The different questions of qualitative properties of solutions of uniformly degenerated elliptic equations is studied by Chanillo and Weeden [6]. In paper [7] the second order uniform divergent elliptic operator is considered.

We will consider the elliptic operator in the bounded domain $D \subset E_n$

$$L = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial}{\partial x_j}\right) \tag{5}$$

In assumption that $\|a_{ij}(x)\|$ is a real symmetric matrix with measurable in D elements, more over, for all $\xi \in E_n$ and a.e. $x \in D$, the condition

$$\alpha \sum_{i=1}^n \lambda_i(x)\xi_i^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \alpha^{-1}\sum_{i=1}^n \lambda_i(x)\xi_1^2 \tag{6}$$

Here $\alpha \in (0, 1]$ is a const.

The function $u(x) \in W^1_{2,\alpha}(D)$ is called the generalized solution of the equation $Lu = f(x)$ in D, if for any function $\eta(x) \in \overset{0}{W}{}^1_{2,\alpha}(D)$ the integral identity

$$\int_D \sum_{i,j=1} a_{ij}(x)u_{x_i}\eta_{x_{ij}}dx = \int_D f\eta dx \tag{7}$$

is fulfilled.

Here $f(x)$ is a given function from $L_2(D)$.

Let $E \subset D$ be some compact. The function $n(x) \in W^1_{2,\alpha}(D)$ is called generalized solution of the equation $L_u = f(x)$ in $D \setminus E$ vanishing on ∂D if integral identity (7) is fulfilled for any function $\eta(x) \in A_E(D)$.

We will assume that the coefficients of the operator Z continued in $E_n \setminus D$ with saving condition (2),(6). For is, it is sufficient, for example, to assume that $a_{ij}(x) = \delta_{ij}\lambda_i(x)$ for $x \in E_n \setminus D$, $i,j = 1...n$ where δ_{ij} is a Croneker symbol.

Let $h(x) \in W^1_{2,\alpha}(D)$, $f^{(0)}(x) \in L_2(D)$, $f^i(x) \in L_{\alpha,\lambda^{-1}}(D)$, $i = 1,2...n$, be a given functions. Let us consider the first

$$Lu = f^{(0)}_{(x)} + \sum_{i=1}^{n} \frac{\partial f^i(x)}{\partial x_i}, \quad x \in D \tag{8}$$

$$\{u(x) - h\,(x)\} \in \overset{0}{W}{}^1_{2,\alpha}(D)$$

The function $u(x) \in W^1_{2,\alpha}(D)$ we will call generalized solution of problem (8) if for any function $\eta(x) \in \overset{0}{W}{}^1_{2,\alpha}(D)$, the integral identity

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} \eta_{x_j} dx = \int_D (-f^0\eta + \sum_{i=1}^{n} f^i \eta_{x_i}) dx \tag{9}$$

is fulfilled.

Our aim to get the necessary and sufficient condition of compact removability E in the class bounded functions.

Z-capacity potential $u(x)$ is weak solution of the equation $Lu = -\mu$ equaling to zero $\partial\varepsilon$ and can be represented in the following form

$$u(x) = \int_\varepsilon g(x,z) d\mu(z)$$

where μ measure on H.

On the other side, there exists the sequence of the functions $\{\eta^{(m)}\,(x)\}$, $m = 1,2...$, such that $\eta^{(m)}(x) \in W^1_{2,\alpha}(\varepsilon)$, $\eta^{(m)}(x) = 1$ for $x \in H$ and $\lim\limits_{m\to\infty} \left\|\eta^{(m)} - u\right\|_{W^1_{2,\alpha}(\varepsilon)} = 0$.

We conclude that it first is equal to $\mu(H)$ at any natural m, while the left part tends to $cap_L(H)$ as $m \to \infty$. Thus

$$cap_L(H) = \mu(H)$$

Theorem. *Let relative to the coefficients of the operator L condition (2)-(6) be fulfilled. Then, for removability of the compact $E \subset D$ relative to the first boundary value problem for the operator L in the space $\mu(D)$, it is necessary and sufficient that*

$$cap_L(E) = 0. \tag{10}$$

For proof we used property of capacity $cap_L(E)$ and some auxiliary proposition.

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