Rasoul HASHEMI

ON A CONJECTURE OF ZIMMERMAN ABOUT AUTOMORPHISMS OF S_n

Abstract

All groups considered are finte. For a group G and $\alpha \in Aut(G)$, we set

$$S(G,\alpha) = \{g \in G | g\alpha = g^2\}$$

We let $s(G,\alpha) = \dfrac{|S(G,\alpha)|}{|G|}$ and define the function $s(G)$ by $s(G) = \max\limits_{\alpha \in Aut(G)} s(G,\alpha)$ we calculate for the permutation group S_n.

1. Introduction. The function $s(G)$ was investigated by Zimmerman in [8], who proved the following result (his notation was slightly different from ours):

Notation: we will write simply $S(\alpha)$ for $S(G,\alpha)$ and $s(a)$ for $s(G,\alpha)$.

Theorem A. *Suppose $\alpha \in Aut(G)$ with $s(\alpha) \geq \frac{5}{12}$, then $s(a) = s(G)$ and one of the following holds:*

I. G is abelian of odd order and $s(G) = 1$.

II. G splits over an abelian, odd-order subgroup of index 2, coinciding with $S(\alpha)$ and $s(G) = \frac{1}{2}$.

III. $Z(G)$ is abelian of odd order, $G/Z(G) \cong A4$ and $G \cap Z(G) = \{1\}$. In this case, $s(G) = \frac{5}{12}$.

Zimmerman conjectured that if $s(G) > \frac{1}{3}$, then G is soluble. It turns out that the conjecture is true, but that one can prove much stronger results. Peter V. Hegarty proved two theorems in [2].

One of these is the complete classification of those groups G of even order such that $s(G) > \frac{1}{6}$, plus a partial classification of those for which $s(G) = \frac{1}{6}$. The number $\frac{1}{6}$ arises naturally, as follows. Clearly, an even order groups satisfies $s(G) = 1$ if G has an odd order direct factor O of index with $s(O) = 21$. In [3], Liebeck proved that if G is non-abelian of odd order, then $s(G) \leq \frac{1}{3}$, and he classified all odd order groups where equality holds. The following theorems can be seen in [2].

Theorem B (i). *Let G be a group of even order such that $(G) \geq \frac{1}{6}$. Let $\alpha \in Aut(G)$ be such that $(\alpha) \geq \frac{1}{6}$. If $s(a) = s(G)$ then of the following holds:*

I. $s(G) = \frac{1}{2}$, G is of type II in theorem A.

II. $s(G) = \frac{5}{12}$, G is of type III in theorem A.

III.$s(G) = \frac{1}{4}$, G has a normal, odd order subgroup of index 4 which coincides with $s(a)$.

IV. $s(G) = \dfrac{5}{24}$, *G is a split extension of a group of type II by* Z_2.

V. $s(G) = \dfrac{9}{48}$, *G is a central extension of an odd order subgroup* $Z \subset S(\alpha)$ *by a group H of order 48. H has a homocyclic, normal Sylow-2 subgroup, which is acted upon fixed-point freely by the elements of H of order 3.*

VI. $s(G) = \dfrac{25}{144}$, *G is a central extension of an odd order subgroup* $Z \subset S(\alpha)$ *by* $A_4 \times A_4$.

VII. $s(G) = \dfrac{1}{6}$, *G has an order subgroup of index 2, which is isomorphic to one of the groups of theorems 4.5 and 4.10 of [3].*

If $s(\alpha) \neq s(G)$ *then G is of type I above, has order divisible by 3, and* $s(\alpha) = \dfrac{1}{6}$.

(ii) If G is one of the groups of types I through VI above, then G does indeed possess an automorphism α *such that* $s(\alpha) = s(G) > \dfrac{1}{6}$.

Theorem C. *If* $s(G) > \dfrac{7}{60}$, *then G is soluble.*

For more information, one can be seen in [1,2,3,4,5].

In this paper, we consider $G = S_n$, then we have, every element of $Aut(S_n)$ is an inner automorphism [7]. So for a given $\sigma \in Aut(S_n)$ there exists an element $g \in S_n$ such that $x\sigma = x^g (x \in S_n)$. If σ is " squaring elements " automorphism ,i.e., $x\sigma = x^2$ then x is an even permutation, for, otherwise x^g will be an odd permutation and x^2 is even, a contradiction. Moreover, for such an automorphism, g is of even order, because $x^g = x^2$ yields, $x^{g^k} = x^{2^k}$ for every positive integer k, if $k = |g|$ (order of g) then k is the least integer that satisfies $x^{2^{k-1}} = 1$, so $k = 2$. This yields in turn $x^3 = 1$, i. e., every element of the set

$$S(G, \sigma) = \{x \in S_n | x\sigma = x^2\}, \ (G = S_n)$$

Is of the form $x = c_1c_2...c_m$, where $m \geq 1$, $c_1c_2...c_m$ are disjoint 3-cycles. Also we conclude that g is a product of disjoint transposition.

Following Hegarty [2] and let $s(G, \sigma) = \dfrac{|S(G, \sigma)|}{n!}$ and $s(G) = \max\limits_{\sigma \in Aut(G)} s(G, \sigma)$. The lower bounds of this number is of interest in classification of groups (one may see Zimmerman [8] and Hegarty [2]). In this paper we give explicit values for $s(G)$ $(G = S_n)$ for every $n \geq 2$ our calculation gives a generating set for maximal abelian subgroup of S_n of order as well. (we have used $[m]$, for the integer part of m).

By the above considerations if $x\sigma = x^2$ then $x\sigma = x^g$ where g is the form $g = (1, 2)(3, 4)(5, 6)...(2m - 1, 2m)$ where $m \geq 2$ and $(i, i + 1)$ is a transposition. It is easy to see that if $g_1 = (a_1, b_1)(a_2, b_2)...(a_m, b_m)$ (disjoint transpositions), and $x\sigma_1 = x^{g_1}$, where $x\sigma_1 = x^2$, then $|S(G, \sigma)| = |S(G, \sigma_1)|, (G = S_n)$. So to calculate $s(G)$ we only need to calculate $|S(G, \sigma)|$ for all values of σ where $x\sigma = x^g$ and g one of the value:

$$\begin{aligned}
g_1 &= (1,2) \\
g_2 &= (1,2)(3,4) \\
g_3 &= (1,2)(3,4)(5,6) \\
&\quad . \\
&\quad . \\
&\quad . \\
g_k &= (1,2)(3,4)...(2k-1,\ 2k),\ k=[\frac{n}{2}].
\end{aligned}$$

Notation. For simplicity let $t_i = |S(G,\sigma_i)|$ where $x\sigma_i = x^{g_i}$, $x\sigma_i = x^2$ for $i = 1, 2...k$. Then our main results are as follows :

Proposition 1.1. Let $G = S_n$ then

(i) For every $n \geq 2$, $t_1 = |S(G,\sigma_1| = 2n-3$

(ii) For every $n \geq 4$, $t_2 = 4n^2 - 32n + 65$

(iii) For every $n \geq 6, t_3 = 24n^3 - 492n^2 + 3354n - 7587$

Proposition 1.2. Let $G = S_n$, $n \geq 8$ and $k = \left[\frac{n}{2}\right]$. Then, for every $m\,(4 \leq m \leq k)$:

$$t_m = 1 + 2m\,(n-2m) + 8\,\tbinom{m}{3} + 4\,\tbinom{m}{3}\sum_{i=2}^{m-2}\left(\prod_{j=2m}^{2m+i-2}(n-j)\,2^i\,(i-1)!\,\tbinom{n-3}{i-1} + \right.$$

$$\left. +2\,\tbinom{m}{2}\sum_{i=1}^{m-1}2^i\prod_{j=2m}^{2m+i}(n-j)\right).$$

Proposition 1.3. Let $G = S_n$, $n \geq 2$ and $k = \left[\frac{n}{2}\right]$ then, $s\,(G) = \frac{1}{n!}\max\{t_1, t_2, ..., t_k\}$.
Proofs based on the classification of the elements $x = c_1c_2\ldots c_p$ (decomposed into disjoint 3–cycles) and enumerating their numbers satisfying $x\sigma = x^2$ for every $g_1, g_2, \ldots, g_k$. First we note that if $x^{g_i} = x^2 (1 \leq i \leq k)$, then every component c_i of x should have common elements with g_i. Indeed, if $c_j = (a,b,c)$ then $c_j^2 = (a,c,b)$ and $c_j^{g_i} = (ag_i, bg_i, cg_i) = (a,c,b)$ yields $bg_i = c$, $cg_i = b$, $ag_i = a$ .So $g_i = (b,c)\ g\prime_i$ where $g_i' \in st(a)$ (stabilizer of the element a). By these preliminaries and letting $A = \{1, 2, \ldots., n\}$ we give the proofs.

Proofs of 1.1. (i): Let $g_1 = (1,2)$, then every $x = (1,2,c)$ where $c \in A - \{1,2\}$ satisfies $x^{g_1} = x^2$ and there are exactly $n-2$ values x. For each x of his kind x^{-1} and (1) the identity permutation are also satisfying $x^{g_1} = x^2$. So there are $2(n-2)+1 = 2n-3$ solutions for $x^{g_1} = x^2$. No more x satisfies $x^{g_1} = x^2$, for, if $x = c_1c_2\ldots c_p$, $p > 1$, then c_i 's could not be disjoint, which is a contradiction, this proves (i).

(ii): Let $g_2 = (1,2)(3,4)$.There are four different classes of the elements $x \in A_n$ satisfying $x^{g_2} = x^2$, as follows :

(I) $(1,2,c),(2,1,c), \quad c\in A-\{1,2,3,4\}$
(II) $(3,4,c'),(4,3,c'), \quad c'\in A-\{1,2,3,4\}$
(III) $(5,6,c),(6,5,c'),(1,2,c),(4,3,c')c, \quad c'\in A-\{1,2,3,4\} \quad c\neq c'$
(IV) $(2,1,c),(3,4,c'),(2,1,c),(4,3,c')c, \quad c'\in A-\{1,2,3,4\} \quad c\neq c'$

Then, there are $4(n-4)+4(n-4)(n-5)+1$ different values for x. Any other 3-cycle different from above appearing in the disjoint the decomposition $x=c_1c_2\dots c_p$ should fix 1,2,3 and 4, the doesn't satisfy $x^{g_2}=x^2$ The prove (ii), that $t_2=4(n-4)(1+n-5)+1=4n^2-32n+65$.

(iii): Let $g_3=(1,2)(3,4)(5,6)$. All of the classes of values for $x=c_1c_2c_3\dots c_p$ satisfying $x^{g_3}=x$ are as follows:
(I) $(1,2,c),(2,1,c), \quad c\in A-\{1,2,3,4,5,6\}$
(II) $(3,4,c),(4,3,c), \quad c\in A-\{1,2,3,4,5,6\}$
(III) $(5,6,c),(6,5,c)c\in A-\{1,2,3,4,5,6\}$

These classes contain $6(n-6)$ elements all together.

(IV) $x=c_1c_2$ where c_1 and c_2 are disjoint 3–cycles of the classes I,II and III indeed this class contains :

$$(2,1,c),(3,4,c'),(2,1,c),(3,4,c'),\quad (3,4,c),(5,6,c')$$

$$(2,1,c),(5,6,c'),(2,1,c),(5,6,c'),\quad (3,4,c),(6,5,c')$$

$$(1,2,c),(4,3,c'),\quad (2,1,c),(4,3,c'),\quad (4,3,c),(5,6,c')$$

$$(1,2,c),(6,5,c'),\quad (2,1,c),(6,5,c'),\quad (4,3,c),(6,5,c')$$

where $c\neq c'$ and $c,c'\in A-\{1,2,3,4,5,6\}$. This class contains $12(n-6)(n-7)$ elements.

(V). $x=c_1c_2c_3$ where 3-cycles are disjoint elements of the first three classes. To enumerate the elements of this class we note that each element this class maybe made from the elements of the class (IV) , by multiplying a new 3–cycle, indeed this class consists of the following case of elements:

$$\begin{gathered}(a,b,c)(3,4,c')(5,6,c")\\ (a,b,c)(3,4,c')(6,5,c")\\ (a,b,c)(4,3,c')(5,6,c")\\ (a,b,c)(4,3,c')(6,5,c")\end{gathered} \tag{1}$$

where $(a=1,b=2)$ or $(a=2,b=1)$; $c,c',c''\in A-\{1,2,3,4,5,6\}$ and $c\neq c'\neq c''$. Also

$$\begin{gathered}(a,b,c)(1,2,c\prime)(5,6,c")\\ (a,b,c)(1,2,c\prime)(6,5,c")\\ (a,b,c)(2,1,c\prime)(5,6,c")\\ (a,b,c)(2,1,c\prime)(6,5,c")\end{gathered} \tag{2}$$

where $(a = 3, b = 4)$ or $(a = 4, b = 3)$; $c, c', c'' \in A - \{1, 2, 3, 4, 5, 6\}$ and $c \neq c' \neq c''$.

$$\begin{aligned}&(a, b, c)(1, 2, c\prime)(3, 4, c")\\&(a, b, c)(1, 2, c\prime)(4, 3, c")\\&(a, b, c)(2, 1, c\prime)(3, 4, c")\\&(a, b, c)(2, 1, c\prime)(4, 3, c")\end{aligned}\tag{3}$$

where $(a = 5, b = 6)$ or $(a = 6, b = 5)$; $c, c', c'' \in A - \{1, 2, 3, 4, 5, 6\}$ and $c \neq c' \neq c''$. And

$$\begin{aligned}&(1, 3, 5)(2, 6, 4), (1, 3, 6)(2, 5, 4)\\&(1, 4, 5)(2, 6, 3), (1, 4, 6)(2, 5, 3)\\&(2, 3, 5)(1, 6, 4), (2, 3, 6)(1, 5, 4)\\&(2, 4, 5)(1, 6, 3), (2, 4, 6)(1, 5, 3)\end{aligned}\tag{4}$$

Obviously, $8 + 24(n - 6)(n - 7)(n - 8)$ element obtained and then $t_3 = |S(G, \sigma_3)| = 1+6(n-6)+12(n-6)(n-7)+24(n-6)(n-7)(n-8)+8 = 24n^3-492n^2+3354n-7587$. This compelets the proof.

Proof of 1.2: Let $g_m = (1, 2)(3, 4) \ldots ..(2m - 1, 2m)$ wher $4 \leq m \leq k$. Every value of x satisfying $x^{g_m} = x^2$ should be in the form $x = c_1c_2 \ldots c_t$ where c_i' s are disjoint 3–cycles and $1 \leq t \leq m$.

Generalizing the method of classification of 3–cycles in 1.1 give us:

$$t_1 = 1 + 2\binom{1}{1}(n - 2).$$

$$t_2 = 1 + 2\binom{2}{1}(n - 4) + 2\binom{2}{1}(n - 4)(n - 5).$$

$$t_3 = 1 + 2\binom{3}{1}(n - 6) + [8\binom{3}{2} + 4\binom{3}{2}(n - 6)(n - 7)] + 8\binom{3}{2}(n - 6)(n - 7)(n - 8)$$

$$t_4 = 1+2\binom{4}{1}(n-8)+[8\binom{4}{3}+4\binom{4}{2}(n-8)(n-9)]+[16\binom{4}{3}(n-8)+8\binom{4}{2}(n-8)\times$$
$$\times(n - 9)(n - 10)] + [0 + 16\binom{4}{2}(n - 8)(n - 9)(n - 10)(n - 11)].$$

$$t_5 = 1+2\binom{5}{1}(n-10)+[8\binom{5}{3}+4\binom{5}{2}(n-10)(n-11)]+[32\binom{5}{3}(n-10)+8\binom{5}{2}(n-10)\times$$
$$\times(n-11)(n-12)+[64\binom{5}{3}(n-10)(n-11)+16\binom{5}{2}(n-10)(n-11)(n-12)(n-13)]+$$
$$+[0 + 32\binom{5}{2}(n - 10)(n - 11)(n - 12)(n - 13)(n - 14)].$$

And by an induction method on constructing $c_1c_2\dots c_m$ from $c_1c_2\dots c_{m-1}$, we get:

$$t_m = 1+2\binom{m}{1}(n-2m)+[8\binom{m}{3}+4\binom{m}{2}(n-2m)(n-2m-1)]+[8\binom{m}{3}(2m-6)(n-2m)+$$

$$+2\times 4\binom{m}{2}(n-2m)(n-2m-1)(n-2m-3)]+[8\binom{m}{3}(2m-6)(2m-4)(n-2m)(n-2m-1)$$

$$+2^2\times 4\binom{m}{2}(n-2m)(n-2m-1)(n-2m-3)(n-2m-5)]+...$$

$$....+[0+2^m\binom{m}{2}(n-2m)(n-2m-1)...(n-2m-m+1)].$$

The sequence $8, 8(2m-6), 8(2m-6)(2m-4), ..., 8(2m-6)(2m-4)...4\times 2, 0$ is indeed the defined sequence $\{a_i\}$ and $1, (n-2m), (n-2m)(n-2m-1), (n-2m)(n-2m-1)(n-2m-2), \dots, 0$ is the defined sequence $\{b_i\}$. This gives the result at once, because

$$t_m = 1+2\binom{m}{1}(n-2m)+8\binom{m}{3}+\binom{m}{3}\sum_{i=2}^{m-2}a_ib_i+4\binom{m}{2}\sum_{i=1}^{m-1}2^{i-1}b_{i+2},$$

where for a given $n\geq 8$, $k=\left[\frac{2}{n}\right]$ and $4\leq m\leq k$,

$$a_1=8,\quad a_i=2^{i+2}(i-1)!\binom{m-3}{i-1}, i\geq 2$$

$$b_1=1,\quad b_i=\prod_{j=2m}^{2m+i-2}(n-j),\ i\geq 2$$

whence $\binom{m-3}{i-1}$ is used for the binomial coefficient.

Corollary 1.4. *$H=<\sigma_1\sigma_{2,\dots,}\alpha_{\left[\frac{n}{2}\right]}$ is the maximal abelian 2-subgroup of $Aut(Sn)$ and $|H|=2^{\left[\frac{n}{2}\right]}$.*

Proof : For every i and j $1\leq i,j\leq\left[\frac{n}{2}\right]$, $\sigma_i\sigma_j=\sigma_j\sigma_i$, for by the definition we get $x\sigma_i\sigma_j=x\sigma_j\sigma_i$, for every x. So H is abelian .Every σ_j is of order 2 ,for, $x\sigma_i^2=(x\sigma_i)\,\sigma_i=x^{g_i}\sigma_i=x^{g_i^2}=x$, moreover , every element of order 2 is in H, for,

$$x\sigma_1\sigma_2=x^{(3,4)}=x\tau_1\implies x\tau_1\implies\tau_1\in H.$$

$$x\sigma_1\tau\sigma_3=x^{(5,6)}=x\tau_2\implies\tau_2\in H.$$

So, $x^{(i,i+1)}\in H$ for every i. Consequently H is a maximal abelian 2-subgroup of $Aut(S_n)$. Obviously, $H\cong Z_2^{\left[\frac{n}{2}\right]}$ and $|H|=2^{\left[\frac{n}{2}\right]}$.

As a result our calculation we give the following table for $G = S_n$, the same notation g_1, g_2 and the corresponding $\sigma_1, \sigma_2, \ldots$, and $s(\sigma) = |S(G, \sigma)|$.

n	$s(\sigma_1)$	$s(\sigma_2)$	$s(\sigma_3)$	$s(\sigma_4)$	$s(\sigma_5)$	$s(\sigma_6)$	...	$s(G)$
4	5	1	–	–	–	–		$\frac{5}{4!}$
5	7	5	–	–	–	–		$\frac{7}{5!}$
6	9	17	9	–	–	–		$\frac{17}{6!}$
7	11	37	15	–	–	–		$\frac{37}{7!}$
8	13	65	45	33	–	–		$\frac{65}{8!}$
9	15	101	147	105	–	–		$\frac{147}{9!}$
10	17	145	369	225	81	–		$\frac{369}{10!}$

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