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A HARNACK INEQUALITY FOR THE SOLUTION OF KOLMOGOROV EQUATION

Abstract

In the paper, an inequality of Harnack type is obtained for non-negative solutions of Kolmogorov equations in one-dimensional case under the condition of Cordes type.

In the paper we consider an operator of the form

$$L = a\,(x,y,z)\,\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial y} - \frac{\partial}{\partial t}, \quad (x,y,t) \in R^3, \tag{1}$$

where the coefficient $a\,(x,y,t)$ is measurable, bounded and satisfies the ellipticity condition

$$0 < C_1 \le a\,(x,y,t) \le C_2. \tag{2}$$

The equation $Lu = 0$ is called the Kolmogorov equation. This equation was first introduced and investigated in A.N. Kolmogorov's classic paper [1].

The paper is devoted to investigation of internal properties of the solutions of the equation under consideration, i.e. to the proof of a Harnack type inequality under the Cordes type condition. Note that subject to the Cordes type condition for parabolic equations with discontinuous coefficients, the similar results were proved in the works of Yu. Mozer [2], E.M. Landis [3] and R.Ya. Glagoleva [4]. In the papers of N.V. Krylov and M.V. Safonov [5], Harnack inequality without restriction of Cordes type condition was proved for solving parabolic equations with measurable coefficients.

Recently, there is great interest to investigation of quality properties of Kolmogorov equations. So, for instance, in the papers [8,9]. Harnack type inequality for non-negative values of this equation is obtained. [8] considers the Kolmogorov equation in divergent form, where the Holder continuity condition with the exponent $\alpha\,(0 < \alpha < 1)$ is required from the coefficients of the equation and their derivatives. Their methods are based on the mean-value theorem for solving the equation. In the paper [9] the Kolmogorov equation in non-divergent form is considered and the Holder continuity condition with the exponent $\alpha\,(0 < \alpha \le 1)$ is required from the coefficients.

In the present paper, such restrictions are not imposed: we consider the Kolmogorov equation in non-divergent form with a bounded and measurable coefficient but subject to the Cordes type condition. The method used in the paper is very close to Landis method, i.e. it is based on the lemma on increase of the solution of Kolmogorov equation proved by the author in [7].

1^0. Denote by $C_{x_0,R}^{t_1,t_2;y_1,y_2}$ a cylinder defined by the inequalities:

$$t_1 < t < t_2, \quad y_1 < y < y_2, \quad |x - x_0| < R.$$

Give some definitions and statements without proof from [6].

Let $D \in R^3$ be a bounded domain. The aggregate of points (x_0, y_0, t_0) for each of which there will be found such $h > 0$ that even if one of the cylinders

$$C_{x_0,h}^{t_0-h,t_0;y_0-h+(x_0-h)(t-t_0),y_0+(x_0-h)(t-t_0)}$$

or

$$C_{x_0,h}^{t_0-h,t_0;y_0+(x_0+h)(t-t_0),y_0+(x_0+h)(t-t_0)+h}$$

belongs to the domain D, will be denoted by M.

We'll call the set $\Gamma(D) = \overline{\partial D \backslash M}$ an eigen boundary of the domain D and the set $\gamma(D) = \partial D \backslash \Gamma(D)$ an upper cover of D.

Theorem 1. (Maximum principle). *Let D be a bounded domain in R^3, $\gamma(D)$ be its upper cover, $\Gamma(D)$ its eigen boundary.*

Let in $D \cup \gamma(D)$ operator (1) be defined, and $u(x, y, t)$ be a sub solution (super solution) for operator (1) in $D \cup \gamma(D)$ (see [6]). Then

$$\sup_D u = \overline{\lim_{\substack{(x,y,t)\to\Gamma(D)\\(x,y,t)\in D}}} u(x,y,t), \quad \left(\inf_D u = \underline{\lim_{\substack{(x,y,t)\to\Gamma(D)\\(x,y,t)\in D}}} u(x,y,t) \right).$$

2^0. We'll consider Cordes type equations, i.e such equations

$$Lu = 0, \tag{3}$$

for which the constrants C_1 and C_2 of inequality (2) satisfy the condition

$$\frac{2C_2}{C_1} < 3.$$

For negative solution of equation (3) prove a Harnack type inequality.

As a preliminary we prove an auxiliary lemma being a corollary of the increase theorem (see [7]).

Assume

$$b = \min\left(\frac{1}{10C_2}, 1\right).$$

Lemma 1. *Let in the cylinder $C_{\eta,\eta}^{0,b\eta^2;0,2b\eta^3}$ the domain D intersecting the cylinder $C_{\eta,\frac{\eta}{2}}^{\frac{1}{2}b\eta^2;b\eta^2;b\eta^3,2b\eta^3}$ and having parallel points on the eigen boundary of the cylinder $C_{\eta,\eta}^{0,b\eta^2;0,2b\eta^3}$ be situated. Denote by Γ the part of the boundary D situated sbrongly interior to the cylinder $C_{\eta,\eta}^{0,b\eta^2;0,2b\eta^3}$. Let in D, the solution of equation (3) that is continuous in $\overline{D}$, positive in D and vanishing on Γ be defined.*

Then for any $K > 0$ there will be found $\delta > 0$ dependent on K, C_1 and C_2 such that from the inequality

$$mesD < \delta\eta^6 \tag{4}$$

it follows the inequality

$$\frac{\sup\limits_{\overline{D}} u}{\sup\limits_{D\cap C^{\frac{1}{2}b\eta^2,b\eta^2;b\eta^3,2b\eta^3}_{\eta,\frac{\eta}{2}}} u} > K. \tag{5}$$

Proof. Take $\beta = C_1$ and $S = \frac{2C_2}{C_1}$. Then $b_0 = \frac{1}{10C_2}$ and by the condition $b = \min\left(\frac{1}{10C_2}, 1\right)$, i.e. the condition $b \le b_0$ is satisfied.

Let ξ be a constant of theorem 3 from [7]. Then in this case ξ depends only on C_1 and C_2. Let further m be such a least natural number that

$$\left(1+\frac{\xi}{2}\right)^m < K.$$

Assume

$$\delta = \frac{b^2}{4^9 m^6}.$$

Divide the difference

$$C^{0,b\eta^2;0,2b\eta^3}_{\eta,\eta} \backslash C^{\frac{1}{2}b\eta^2,b\eta^2;b\eta^3,2b\eta^3}_{\eta,\frac{\eta}{2}} \tag{6}$$

into m parts by the eigen boundaries Γ_i, of the cylinders

$$C^{(i)} = C^{\frac{1}{2}b\eta^2\left(1-\frac{i}{m}\right),b\eta^2;b\eta^3\left(1-\frac{i}{m}\right),2b\eta^3}_{\eta,\frac{\eta}{2}\left(1+\frac{i}{m}\right)}, \quad i = 0,1,...,m-1.$$

Γ_0 coincides with the eigen boundary of the cylinder $C^{\frac{1}{2}b\eta^2,b\eta^2;b\eta^3,2b\eta^3}_{\eta,\frac{\eta}{2}}$.

Assume

$$M_i = \max_{D\cap\Gamma_i} u, \quad i = 0,1,...,m-1.$$

Let M_i be attained at the point $\left(x^i, y^i, t^i\right) \in \Gamma_i$. Consider the cylinder

$$C_1^{(i)} = C^{t^i-b\left(\frac{\eta}{2m}\right)^2,t^i;y^i+\left(x^i-\frac{\eta}{2m}\right)\left(t-t^i\right)-2b\left(\frac{\eta}{2m}\right)^3,y^i+\left(x^i-\frac{\eta}{2m}\right)\left(t-t^i\right)}_{x^i,\frac{\eta}{2m}}.$$

Show that

$$C_1^{(i)} \subset C^{(i+1)}, \quad i = 0,1,...,m-1.$$

Let (x,y,t) be any point from $C_1^{(i)}$. From $\left(x^i, y^i, t^i\right) \in \Gamma_i$ we have

$$\left.\begin{array}{c} \left|x^i-\eta\right| \le \frac{\eta}{2}\left(1+\frac{i}{m}\right) \\ \frac{1}{2}b\eta^2\left(1-\frac{i}{m}\right) \le t^i \le b\eta^2 \\ b\eta^3\left(1-\frac{i}{m}\right) \le y^i \le 2b\eta^3 \end{array}\right\}. \tag{7}$$

It is easy to show that

$$|x-\eta| < \frac{\eta}{2}\left(1+\frac{i+1}{m}\right) \quad \textit{and} \quad \frac{1}{2}b\eta^2\left(1-\frac{i+1}{m}\right) < t < b\eta^2. \tag{8}$$

It remains to show

$$b\eta^3\left(1-\frac{i+1}{m}\right) < y < 2b\eta^3. \tag{9}$$

The right side of inequality (9) is obvious, prove the left side. From $(x,y,t)\in C_1^{(i)}$ and (8) we get

$$y>y^i-\frac{2b\eta^3}{8m^3}+\left(x^i-\frac{\eta}{2m}\right)\left(t-t^i\right)\geq$$

$$\geq b\eta^3\left(1-\frac{i}{m}\right)-\frac{b\eta^3}{4m^3}+\left(\eta+\frac{\eta}{2}\left(1+\frac{i}{m}\right)-\frac{\eta}{2m}\right)\left(t^i-\frac{b\eta^2}{4m^2}-t^i\right)=$$

$$=b\eta^3\left(1-\frac{i}{m}-\frac{3}{8m^2}-\frac{i+1}{8m^3}\right).$$

Now it suffices to show

$$b\eta^3\left(1-\frac{i}{m}-\frac{3}{8m^2}-\frac{i+1}{8m^3}\right)\geq b\eta^3\left(1-\frac{i+1}{m}\right).$$

If we simplify the last inequality, we get

$$8m^2-4m\geq 0,$$

and this is true for any $m\in N$, i.e. (9) is proved. Consequently, we proved that

$$C_1^{(i)}\subset C^{(i+1)},\quad i=0,1,...,m-1.$$

In the cylinder $C_1^{(i)}$ we consider the cylinders

$$C_2^{(i)}=C_{x^i,\frac{\eta}{32m}}^{t^i-\frac{b}{2}\left(\frac{\eta}{8m}\right)^2,t^i;y^i+\left(x^i-\frac{\eta}{2m}\right)\left(t-t^i\right)-\frac{b}{2}\left(\frac{\eta}{4m}\right)^3,y^i+\left(x^i-\frac{\eta}{2m}\right)\left(t-t^i\right)}$$

and

$$C_3^{(i)}=C_{x^i,\frac{\eta}{32m}}^{t^i-b\left(\frac{\eta}{2m}\right)^2,t^i-15b\left(\frac{\eta}{8m}\right)^2;\ y^i+\left(x^i-\frac{\eta}{2m}\right)\left(t-t^i\right)-\frac{17b}{2}\left(\frac{\eta}{4m}\right)^3,\ y^i+\left(x^i-\frac{\eta}{2m}\right)\left(t-t^i\right)-\frac{15b}{2}\left(\frac{\eta}{4m}\right)^3}.$$

It is clear that

$$mesC_3^{(i)}=\frac{b^2\eta^6}{2\cdot 4^8m^6}. \tag{10}$$

Denote the set $D\cap C_1^{(i)}$ by D'. If now we apply theorem 5 from [7] to the cylinders $C_1^{(i)}$, $C_2^{(i)}$, $C_3^{(i)}$ and domain D'; we get

$$\sup_{D'}u\geq\left[1+\xi\frac{mesE}{mesC_3^{(i)}}\right]\sup_{D'\cap C_2^{(i)}}u. \tag{11}$$

On the other hand

$$\sup_{D'}u=\sup_{D'\cap C_1^{(i)}}u\leq\sup_{D\cap C^{(i+1)}}u=u\left(x^{i+1},y^{i+1},t^{i+1}\right)=M_{i+1}. \tag{12}$$

In addition, the point $\left(x^i,y^i,t^i\right)\in D'\cap C_2^{(i)}$, therefore

$$\sup_{D'\cap C_2^{(i)}}u\geq u\left(x^i,y^i,t^i\right)=M_i. \tag{13}$$

Then from (11), (12) and (13) we get

$$M_{i+1} \geq \left[1 + \xi \frac{mesE}{mesC_3^{(i)}}\right] M_i. \tag{14}$$

Now, taking into account (10), we get

$$mesE = mes\left(C_3^{(i)} \backslash D'\right) > mes\left(C_3^{(i)}\right) - mesD >$$

$$> \frac{b^2\eta^6}{2 \cdot 4^8 m^6} - \frac{1}{2} \cdot \frac{b^2\eta^6}{2 \cdot 4^8 m^6} = \frac{1}{2} \cdot \frac{b^2\eta^6}{2 \cdot 4^8 m^6} = \frac{1}{2} mes\left(C_3^{(i)}\right).$$

Thus, from (14) we deduce

$$M_{i+1} \geq \left(1 + \frac{\xi}{2}\right) M_i.$$

If we repeat all these operations for $i = 0, 1, ..., m-1$, we get

$$M_m > \left(1 + \frac{\xi}{2}\right)^m M_0$$

and so, by the maximum principle (theorem 1)

$$\sup_D u > K \sup_{D \cap C_{\eta, \frac{\eta}{2}}^{\frac{1}{2}b\eta^2, b\eta^2; b\eta^3, 2b\eta^3}} u.$$

The lemma is proved.

Consider the following transformation:

$$x' = x + x_0 - \eta, \quad t' = t + t_0 - b\eta^2, \quad y' = y + (x_0 - \eta)\left(t - b\eta^2\right) + y_0 - 2b\eta^3. \tag{15}$$

If we apply transformation (15) to the cylinders $C_{\eta,\eta}^{0, b\eta^2; 0, 2b\eta^3}$ and $C_{\eta, \frac{\eta}{2}}^{\frac{1}{2}b\eta^2, b\eta^2; b\eta^3, 2b\eta^3}$ then these cylinders will pass to the following ones

$$\left.\begin{array}{l} C_1 = C_{x_0,\eta}^{t_0 - b\eta^2, t_0; y_0 + (x_0-\eta)(t-t_0) - 2b\eta^3, y_0 + (x_0-\eta)(t-t_0)} \\ C_2 = C_{x_0, \frac{\eta}{2}}^{t_0 - \frac{1}{2}b\eta^2, t_0; y_0 + (x_0-\eta)(t-t_0) - b\eta^3, y_0 + (x_0-\eta)(t-t_0)} \end{array}\right\}. \tag{16}$$

Then we can get the statement, similar to lemma 1 in cylinders (16).

Lemma 2. *Let all the conditions of lemma 1 be satisfied in cylinders (16). Then for any $K > 0$ there will be found $\delta > 0$ dependent on K, C_1 and C_2, such that from the inequality $mesD < \delta\eta^6$ it follows the inequality*

$$\frac{\sup_{\overline{D}} u}{\sup_{D \cap C_2} u} > K. \tag{17}$$

Now we can formulate a Harnack type inequality for non-negative solution of equation (3).

Theorem 2. (Harnack inequality). *Let in the cylinder* $C_{R,R}^{0,bR^2;0,2bR^3}$ *the non-negative solution* $u(x,y,t)$ *of equation (3) be defined.*

Then,

$$\sup_{C_{R,\frac{1}{32}R}^{\frac{1}{32}bR^2,\frac{2}{32}bR^2;bR^3,\frac{17}{16}bR^3}} u(x,y,t) \Big/ \inf_{C_{R,\frac{1}{32}R}^{\frac{31}{32}bR^2,bR^2;\frac{31}{16}bR^3,2bR^3}} u(x,y,t) < C_3, \tag{18}$$

where $C_3 > 0$ *is a constant dependent on* C_1 *and* C_2.

Proof. It is clear that it suffices to prove the theorem for the case $R = 1$. For convenience denote

$$C_1 = C_{1,1}^{0,b;0,2b}, \quad C_2 = C_{1,\frac{1}{32}}^{\frac{31}{32}b,b;\frac{31}{16}b,2b}, \quad C_3 = C_{1,\frac{1}{32}}^{\frac{1}{32}b,\frac{2}{32}b;b,\frac{17}{16}b}.$$

Then being proved inequality (18) will take the form

$$\sup_{C_3} u / \inf_{C_2} u < C.$$

The theorem will be proved if from the assumption

$$\sup_{C_3} u = 2$$

it will follow that

$$\inf_{C_2} u > \nu,$$

where $\nu > 0$ is a constant dependent on C_1 and C_2.

Indeed, we can take the function $V = \frac{2u}{\sup\limits_{C_3} u}$ and prove inequality (18) for V, and this proves inequality (18) for the function u. Therefore, we suppose that $\sup\limits_{C_3} u = 2$.

Assume

$$\widetilde{C}_3 = C_{1,\frac{1}{16}}^{0,\frac{2}{32}b;\frac{15}{16}b,\frac{17}{16}b}.$$

Denote by G_1 the point set $(x,y,t) \in \widetilde{C}_3$, where $u(x,y,t) > 1$.

Assume in lemma 2 $K = 2^7$ and find the appropriate δ. Further we assume

$$\varepsilon_0 = \left(\frac{1}{4096}\right)^6 \delta. \tag{19}$$

Consider two separate cases

$$mesG_1 \geq \varepsilon_0$$

and

$$mesG_1 < \varepsilon_0.$$

Case 1. $mesG_1 \geq \varepsilon_0$.

Let $S = \frac{2C_2}{C_1}$, $\beta = C_1$. Equation (3) is a Cordes type equation, therefore $S < 3$. Besides, $G_1 \subset C_{1,1}^{0,b;0,2b} \subset C_{1,1}^{0,1;0,2}$. Then by the property of (S,β) capacity $\gamma_{s,\beta}(G_1) \geq CmesG_1 \geq C\varepsilon_0$ (see. [6]).

Let μ be an admissible measure on G_1 such that

$$\mu G_1 > \frac{1}{2}\gamma_{s,\beta}(G_1) > \frac{C}{2} mesG_1.$$

Assume

$$V(x,y,t) = \int\limits_{G_1} g_{s,\beta}(x,\xi;y,\varsigma;t-\tau)\, d\mu(\xi,\varsigma,\tau) - b^{-s}\exp\left[-\frac{1}{5\beta b}\right]\mu(G_1)\,(see.[7]).$$

Then, taking into account inequality (20) (see [7]), we find that on the eigen boundary of C_1 (outside of $\overline{G}_1$) it is fulfilled

$$V \le 0.$$

Further, μ is an admissible measure, therefore $V \le 1$ in $C_1 \backslash \overline{G}_1$. This means that on the eigen boundary of $C_1 \backslash \overline{G}_1$

$$u \ge V,$$

therefore, by the maximum principle

$$u|_{C_1 \backslash \overline{G}_1} \ge V|_{C_1 \backslash \overline{G}_1}.$$

By inequality (21) (see [7])

$$u|_{C_2} \ge V|_{C_2} \ge b^{-s}\exp\left[-\frac{1}{5,3\beta b}\right]\mu G_1 - b^{-s}\exp\left[-\frac{1}{5\beta b}\right]\mu G_1 >$$

$$> \frac{C\varepsilon_0 b^{-s}}{2}\left(\exp\left[-\frac{1}{5,3\beta b}\right] - \exp\left[-\frac{1}{5\beta b}\right]\right).$$

Thus, in case 1 for ν we can take

$$\frac{C\varepsilon_0 b^{-s}}{2}\left(\exp\left[-\frac{1}{5,3\beta b}\right] - \exp\left[-\frac{1}{5\beta b}\right]\right).$$

Case 2. $mesG_1 < \varepsilon_0$.

Assume

$$C^{(\rho)} = C^{\frac{b}{32}\left(1-\rho^2\right),\frac{2}{32}b;b\left(1-\rho^3\right),\frac{17}{16}b}_{1,\frac{1}{32}+\rho}.$$

It is clear that $C^{(0)} = C_3$ and $C^{\left(\frac{1}{32}\right)} \subset \widetilde{C}_3$.

Assume

$$G^{(1)}_\rho = G_1 \cap \left(C^{(\rho)} \backslash C^{(0)}\right).$$

From (19) we have

$$mesG^{(1)}_{\frac{1}{64}} < \left(\left(\frac{1}{64}\right)^2\right)^6 \delta.$$

On the other hand

$$mesG^{(1)}_\rho \ge O\left(\rho^6\right) \quad as \quad \rho \to 0.$$

Then

$$mesG_{\rho}^{(1)} \geq O\left(\rho^{12}\right) \quad as \quad \rho \to 0,$$

and therefore for rather small $\rho > 0$

$$mesG_{\rho}^{(1)} \geq \left(\rho^{2}\right)^{6} \delta.$$

Therefore there will be found ρ_1, $0 < \rho_1 < \frac{1}{64}$ such that

$$mesG_{\rho_1}^{(1)} = \left(\rho_1^{2}\right)^{6} \delta. \tag{20}$$

On the eigen boundary of the cylinder $C^{\left(\rho_1^2\right)}$ find the point $\left(x^1, y^1, t^1\right) \in G_1$ where $u\left(x^1, y^1, t^1\right) \geq 2$. By the maximum principle it is possible to find such a point because $\sup\limits_{C_3} u = 2$ and $C_3 \subset C^{\left(\rho_1^2\right)}$.

Take the cylinder

$$C_{(1)} = C_{x^1, \rho_1^2}^{t^1 - b\rho_1^4, t^1; y^1 - 2b\rho_1^6 + \left(x^1 - \rho_1^2\right)\left(t - t^1\right), y^1 + \left(x^1 - \rho_1^2\right)\left(t - t^1\right)}.$$

It is easy to prove that this sylinder is disposed in the margin between the cylinders $C^{(0)}$ and $C^{(\rho_1)}$.

Assume

$$V_1(x, y, t) = u(x, y, t) - 1.$$

We have $V_1\left(x^1, y^1, t^1\right) \geq 1$ and $V_1(x, y, t) > 0$ in G_1, $V_1(x, y, t) \leq 0$ outside of G_1.

Denote by $D_{(1)}$ that component of the set $G_1 \cap C_{(1)}$ that contains the point $\left(x^1, y^1, t^1\right)$. Applying lemma 2 to the cylinder $C_{(1)}$ and to the domain $D_{(1)}$ in it, that is possible by (20):

$$\sup_{D_{(1)}} u > \sup_{D_{(1)}} V_1 \geq 2 \cdot 2^6.$$

Denote by G_2 the points set $(x, y, t) \in \widetilde{C}_3$, where

$$u(x, y, t) > 2^6.$$

Consider $C^{(\rho_1 + \rho)}$, $0 < \rho < \frac{1}{32} - \rho_1$. Assume

$$G_{\rho}^{(2)} = G_2 \cap \left(C^{(\rho_1 + \rho)} \backslash C^{(\rho_1)}\right).$$

From relation $0 < \rho < \frac{1}{32} - \rho_1$ it follows that $\rho_1 < \rho + \rho_1 < \frac{1}{32}$, and therefore $C^{(\rho_1 + \rho)} \in \widetilde{C}_3$. Since $\rho_1 < \frac{1}{64}$, and $G_2 \subset G_1$, then from (19)

$$mesG_{\frac{1}{64}}^{(2)} < \left(\left(\frac{1}{64}\right)^2\right)^6 \delta.$$

We again get

$$mesG_{\rho}^{(2)} \geq O\left(\rho^{6}\right) \qquad as \qquad \rho \to 0,$$

consequently,

$$mesG_{\rho}^{(2)} \geq O\left(\rho^{12}\right) \qquad as \quad \rho \to 0.$$

Then for rather small positive ρ by the same reasonings that as before

$$mesG_{\rho}^{(2)} \geq \left(\rho^2\right)^6 \delta.$$

Therefore, there will be found such ρ_2 from the interval $\left(0, \frac{1}{64}\right)$ that

$$mesG_{\rho_2}^{(2)} = \left(\rho_2^2\right)^6 \delta. \tag{21}$$

Find on the eigen boundary $C^{\left(\rho_1+\rho_2^2\right)}$ the point $\left(x^2, y^2, t^2\right)$, where $u\left(x^2, y^2, t^2\right) > 2 \cdot 2^6$. Take the cylinder

$$C_{(2)} = C_{x^2, \rho_2^2}^{t^2 - b\rho_2^4, t^2; y^2 - 2b\rho_2^6 + \left(x^2-\rho_2^2\right)\left(t-t^2\right), y^2 + \left(x^2-\rho_2^2\right)\left(t-t^2\right)}.$$

It is clear that this cylinder is disposed in the margin between the cylinders $C^{(\rho_1)}$ and $C^{\left(\rho_1+\rho_2^2\right)}$.

Assume

$$V_2\left(x, y, t\right) = u\left(x, y, t\right) - 2^6,$$

thus, $V_2\left(x^2, y^2, t^2\right) > 2^6$ and $V_2\left(x, y, t\right) > 0$ in G_2, $V_2\left(x, y, t\right) \leq 0$ outside of G_2. Denote by $D_{(2)}$ the component of the set $G_2 \cap C_{(2)}$ that contains the point $\left(x^2, y^2, t^2\right)$.

Applying lemma 2 to the cylinder $C_{(2)}$ and the domain $D_{(2)}$ in it, we find (from (21))

$$\sup_{D_{(2)}} u > \sup_{D_{(2)}} V_2 \geq 2^6 \cdot 2^7 = 2 \cdot 2^{2 \cdot 6}.$$

If $\rho_1 + \rho_2 < \dfrac{1}{64}$, we continue the process. Denote by G_3 the points set $(x, y, t) \in \widetilde{C}_3$, where $u > 2^{2 \cdot 6}$.

Consider $C^{(\rho_1+\rho_2+\rho)}$, $0 < \rho < \dfrac{1}{64} - \rho_1 - \rho_2$. It follows from this that $\rho_1 + \rho_2 < \rho + \rho_1 + \rho_2 < \frac{1}{64}$, and therefore $C^{(\rho_1+\rho_2+\rho)} \subset \widetilde{C}_3$.

Assume

$$G_{\rho}^{(3)} = G_3 \cap \left(C^{(\rho_1+\rho_2+\rho)} \backslash C^{(\rho_1+\rho_2)}\right)$$

and find such $\rho_3 < \dfrac{1}{64}$ that

$$mesG_{\rho_3}^{(3)} = \left(\rho_3^2\right)^6 \delta,$$

and on the eigen boundary of $C^{\left(\rho_1+\rho_2+\rho_3^2\right)}$ there is a point $\left(x^3, y^3, t^3\right)$, where $u\left(x^3, y^3, t^3\right) > 2 \cdot 2^{2 \cdot 6}$. Then in the margin between $C^{(\rho_1+\rho_2)}$ and $C^{(\rho_1+\rho_2+\rho_3)}$ we find the cylinder $C_{(3)}$ and the domain $D_{(3)}$ in it and etc.

We'll continue this process until for the first time there will be $\rho_1 + \rho_2 + ... + \rho_k \geq \dfrac{1}{64}$.

Note that therewith $\rho_k < \dfrac{1}{64}$, $\rho_1 + ... + \rho_k < \dfrac{1}{32}$. In the sequel, the moment when the sum $\rho_1 + ... + \rho_k$ will exceed $\frac{1}{64}$ will occur: otherwise we could continue the process infinitely, and since at its each step the value of u increases more than 2^6 times, the function u could be unbounded in $\widetilde{C}_3$.

So, let

$$\rho_1 + ... + \rho_{k-1} < \frac{1}{64}$$

and

$$\rho_1 + ... + \rho_k \geq \frac{1}{64}. \tag{22}$$

To each number i, $i = 1, 2, ..., k$ there corresponds the set $G^{(i)}_{\rho_i} \subset \widetilde{C}_3$ satisfying the equality

$$mesG^{(i)}_{\rho_i} = \left(\rho_i^2\right)^6 \delta \tag{23}$$

and

$$u\big|_{G^{(i)}_{\rho_i}} > 2^{6(i-1)}. \tag{24}$$

From (22) it follows that there will be found such i_0 that

$$\rho_{i_0} > \frac{1}{2^{\frac{i_0}{2}+8}}.$$

Otherwise,

$$\rho_1 + \rho_2 + ... + \rho_k \leq \frac{1}{2^{\frac{1}{2}+8}} + \frac{1}{2^{\frac{2}{2}+8}} + ... + \frac{1}{2^{\frac{k}{2}+8}} < \frac{1}{256} \cdot \frac{1}{2^{\frac{1}{2}} - 1} < \frac{1}{64},$$

but it is impossible by (22).

Then (23), (24) give us

$$mesG^{(i_0)}_{\rho_{i_0}} = \left(\rho_{i_0}^2\right)^6 \delta > \left(\frac{1}{2^{i_0+16}}\right)^6 \delta = 2^{-6(i_0+16)}\delta, \tag{25}$$

$$u\big|_{G^{(i_0)}_{\rho_{i_0}}} > 2^{(i_0-1)6}. \tag{26}$$

Let μ be an admissible measure defined on $G^{(i_0)}_{\rho_{i_0}}$ and such that $\mu\left(G^{(i_0)}_{\rho_{i_0}}\right) > \frac{\gamma_{s,\beta}\ G^{(i_0)}_{\rho_{i_0}}}{2}$, so that on account of the property of $(s.\beta)$-capacity (see [6])

$$\mu\left(G^{(i_0)}_{\rho_{i_0}}\right) > \frac{C}{2} mesG^{(i_0)}_{\rho_{i_0}},$$

i.e.

$$\mu\left(G^{(i_0)}_{\rho_{i_0}}\right) > \frac{C}{2} 2^{-6(i_0+16)}\delta. \tag{27}$$

Consider the function

$$V(x,y,t) = 2^{6(i_0-1)} \left[\int\limits_{G^{(i_0)}_{\rho_{i_0}}} g_{s,\beta}(x,\xi;y,\varsigma;t-\tau)\, d\mu(\xi,\varsigma,\tau) - b^{-s} \exp\left[-\frac{1}{5\beta b}\right] \mu\left(G^{(i_0)}_{\rho_{i_0}}\right) \right].$$

On the lower foot of the cylinder C_1 outside of $\overline{G}_{\rho_{i_0}}^{(i_0)}$ this function negative, on the lateral bounds that are eigen boundaries, it is negative on account of inequality (20) (saee [7]). Outside of the set $G_{\rho_{i_0}}^{(i_0)}$ C_1 in doesn't exceed $2^{6(i_0-1)}$. Therefore, by the maximum principle it doesn't exceed u everywhere in $C_1 \backslash \overline{G}_{\rho_{i_0}}^{(i_0)}$.

Applying inequality (21) (see [7]), we find

$$\inf_{C_2} u > \inf_{C_2} V \geq 2^{6(i_0-1)} b^{-s} \left(\exp\left[-\frac{1}{5,3\beta b} \right] - \exp\left[-\frac{1}{5\beta b} \right] \right) \mu \left(G_{\rho_{i_0}}^{(i_0)} \right) >$$

$$> 2^{6(i_0-1)} b^{-s} \left(\exp\left[-\frac{1}{5,3\beta b} \right] - \exp\left[-\frac{1}{5\beta b} \right] \right) \cdot \frac{C}{2} \cdot 2^{-6(i_0+16)} \delta =$$

$$= 2^{-103} C \delta b^{-s} \left(\exp\left[-\frac{1}{5,3\beta b} \right] - \exp\left[-\frac{1}{5\beta b} \right] \right).$$

Accept the number standing in the right side of the inequality for ν. It obviously depends on C_1 and C_2.

We proved the theorem for $R = 1$. If we make transformations $x' = \alpha x$, $t' = \alpha^2 t$, $y' = \alpha^3 y$ for $\alpha = R$ and take into account that operator (1) remains invariant in this transformation, we can confirm that the theorem is true for any $R > 0$.

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