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# ON DEFECT BASICITY OF THE SYSTEM OF EIGEN FUNCTIONS OF A SPECTRAL PARAMETER WITH A SPECTRAL PROBLEM IN THE BOUNDARY CONDITIONS

**Abstract**

*We consider the spectral problem*

$$-y''(x) + q(x)\,y(x) = \lambda y(x)\,, x \in (0,1)\,,$$

$$y'(0) = (a_0\lambda + b_0)\,y(0)\,,$$

$$y'(1) = (a_1\lambda + b_1)\,y(1)\,,$$

*where $\lambda$ is a spectral parameter, $q(x) \in C[0,1]$, $q(x) > 0$, $x \in [0,1]$, $a_i, b_i, i = 0,1$ are real constants, and $a_0 < 0$, $a_1 < 0$, $b_0 > 0$, $b_1 < 0$.*

*We study general characteristics of location of eigen values on a real axis, oscillation properties of eigenfunctions, basis properties in the space $L_p(0,1)$, $1 < p < \infty$ of the subsystems of eigenfunctions of this problem.*

Consider the spectral problem

$$-y''(x) + q(x)\,y(x) = \lambda y(x)\,, x \in (0,1)\,, \tag{1}$$

$$y'(0) = (a_0\lambda + b_0)\,y(0)\,, \tag{2}$$

$$y'(1) = (a_1\lambda + b_1)\,y(1)\,, \tag{3}$$

where $\lambda$ is a spectral parameter, $q(x) \in C[0,1]$, $q(x) > 0$, $x \in [0,1]$, $a_i, b_i, i = 0,1$ are real constants, and $a_0 \neq 0$, $a_1 \neq 0$, arising for example while solving by the method of separation of variables a problem on heat propagation in a bar at the ends of which the concentrated heat capacities are placed [1,2].

In the case $a_0 < 0$, $a_1 > 0$ problem (1)-(3) was considered in [3], where in particular it was proved that after removing two any functions of different parity ordinal numbers, the system of functions forms a basis in the space $L_p(0,1)$, $1 < p < \infty$ (i.e. this system forms a defect basis with defect number 2).

The basis properties in the space $L_p$, $1 < p < \infty$, of the system of eigenfunctions (root functions) of problem (1)-(3) in the case $q \equiv 0$, $a_0 < 0$, $a_1 > 0$, $b_0 = b_1 = 0$ was studied in [4], in the case $q \equiv 0$, $a_0 < 0$, $a_1 > 0$, $b_0 = b_1 = 0$ in [5]. Necessary and sufficient conditions of defect basicity (with defect number 2) in $L_p(0,1)$, $1 < p < \infty$ of the system of root functions of this problem was found in these papers.

The present paper is devoted to investigation of basis properties in the space $L_p(0,1)$, $1 < p < \infty$ of the system of eigenfunctions of problem (1)-(3) in the case $a_0 < 0$, $a_1 < 0$, $b_0 > 0$, $b_1 < 0$.

Everywhere in the sequel suppose that the following conditions are fulfilled

$$a_0 < 0, a_1 < 0, b_0 > 0, b_1 < 0.$$

It is known that [6] there exists a unique solution $y(x,\lambda)$ of equation (1) satisfying the initial conditions

$$y(0,\lambda)=1,\quad y'(0,\lambda)=(a_0\lambda+b_0)\,, \tag{4}$$

and the function $y(x,\lambda)$ for each fixed $x\in[0,1]$ is an entire function $\lambda$.

Along with problem (1.1)-(1.3) consider the following boundary value problems:

$$\left.\begin{array}{c} -y''(x)+q(x)\,y(x)=\lambda y(x)\,, x\in(0,1)\,, \\ y'(0)=(a_0\lambda+b_0)\,y(0)\,, y(1)=0; \end{array}\right\} \tag{5}$$

$$\left.\begin{array}{c} -y''(x)+q(x)\,y(x)=\lambda y(x)\,, x\in(0,1)\,, \\ y'(0)=(a_0\lambda+b_0)\,y(0)\,, y'(1)=0. \end{array}\right\} \tag{6}$$

The eigenvalues of problems (5) and (6) are real, simple and form the infinitely increasing sequences

$$\mu_1<\mu_2<...<\mu_k<... \quad and \quad \nu_1<\nu_2<...<\nu_k<...,$$

respectively; the eigenfunctions $\vartheta_k(x)$ and $w_k(x)$, $k\in N$, corresponding to the eigenvalues $\mu_k$ and $\nu_k$ have exactly $k-1$ simple zeros in the interval (0,1) [7].

Note that the eigenvalues $\mu_k$ and $\nu_k$, $k\in N$ of problem (5) and (6) are the zeros of entire functions $y(1,\lambda)=1$ and $y'(1,\lambda)$, respectively.

The function

$$F(\lambda)=y'(1,\lambda)\,/y(1,\lambda)$$

was determined for the values

$$\lambda\in D\equiv(\mathbf{C}\backslash\mathbf{R})\cup\bigcup_{k=1}^{\infty}(\mu_{k-1},\mu_k)$$

and is a meromorphic function of finite order, $\nu_k$ and $\mu_k$, $k\in N$ are zeros and poles of this function, respectively, where $\mu_0=-\infty$.

By lemma 1.3 and theorem 1.1 from [8] the following relations hold:

$$\frac{dF(\lambda)}{d\lambda}=-\frac{\int\limits_0^1 y^2(x,\lambda)\,dx-a_0}{y^2(1,\lambda)},\quad \lambda\in D. \tag{7}$$

$$\lim_{\lambda\to-\infty}F(\lambda)=+\infty. \tag{8}$$

Multiplying the both sides of equation (1) by $y(x)$, integrating the obtained equality from 0 to 1 using integration by parts and taking into account boundary conditions of problems (5), (6), we get:

$$\int\limits_0^1\left\{y'^2(x)+q(x)\,y^2(x)\right\}dx+b_0y^2(0)=\lambda\left[\int\limits_0^1 y^2(x)\,dx-a_0\right], \tag{9}$$

whence it follows that $\mu_k > 0$ and $\nu_k > 0$, $k \in \mathbf{N}$. From (7) we get that the function $F(\lambda)$ is strictly decreasing on the interval $(-\infty, \mu_1)$ and $F(\nu_1) = 0$, $\nu_1 \in (-\infty, \mu_1)$. Consequently, $F(0) > 0$.

Following the appropriate reasonings carried out by proving supposition 4 from [9], we see the validity of the following statement.

**Lemma1.** *It holds the representation*

$$F(\lambda) = \sum_{k=1}^{\infty} \frac{\lambda c_k}{\mu_k (\lambda - \mu_k)}, \tag{10}$$

*where* $c_k = \underset{\lambda = \mu_k}{res} F(\lambda > 0)$.

From formula (10) the validity of the following relations holds:

$$F'(\lambda) = -\sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \mu_k)^2}, \quad \lambda \in D, \tag{11}$$

$$F''(\lambda) = 2\sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \mu_k)^3}, \quad \lambda \in D. \tag{12}$$

By (11), we have $F'(\lambda) < 0$ for $\lambda \in \bigcup_{k=41}^{\infty} (\mu_{k-1}, \mu_k)$ (that also follows from formula (7)), and by (12), we have $F''(\lambda) < 0$ for $\lambda \in (-\infty, \mu_1)$ that means that the function $F(\lambda)$ is convex upwards on the interval $(-\infty, \mu_1)$.

It is obvious that the eigen values of problem (1)-(3) are the roots of the equation

$$F(\lambda) = (a_1 \lambda + b_1). \tag{13}$$

The following lemmas are valid

**Lemma 2.** *The eigenvalues of boundary value problem (1)-(3) are real, simple and form no more than countable set, not having finite limit point.*

The proof is carried out by the scheme of the proof of lemmas 1.1 and 1.2 from [6].

**Lemma 3.** *If equation (13) has a solution on the interval* $(\mu_{k-1}, \mu_k) \cap \mathbf{R}_{\mp}$, $k = 1, 2, 3, ...$, *this solution is unique, where* $\mathbf{R}_{\nu} = \{\mu \in \mathbf{R} : 0 < \nu\mu \leq \infty\}$, $\nu = \mp$.

**Proof.** Suppose that $\lambda^* \in (\mu_0, \mu_1) \cap R_{-} \equiv (-\infty, 0)$ is the solution of equation (13). Multiplying the both sides of equation by $y(x, \lambda^*)$, integrating the obtained equality from 0 to 1, and taking into account boundary conditions (2) and (3), we get

$$\int_0^1 \left\{ y'^2(x, \lambda^*) + q(x) y^2(x, \lambda^*) \right\} dx + b_0 - b_1 y^2(1, \lambda^*) =$$

$$= \lambda^* \left[ \int_0^1 y^2(x, \lambda^*) dx - a_0 + a_1 y^2(1, \lambda^*) \right]. \tag{14}$$

Since $q(x) > 0$, $x \in [0,1]$, $b_0 > 0$, $b_1 > 0$, $\lambda^* < 0$, from (14) it follows the validity of the inequality

$$\int_0^1 y^2(x,\lambda^*)\,dx - a_0 + a_1 y^2(1,\lambda^* x) < 0. \tag{15}$$

Taking into account (7), from (15) we get

$$\frac{d}{d\lambda}\left(F(\lambda) - (a_1\lambda + b_1)\right)|_{\lambda=\lambda^*} > 0,$$

that means that only strictly decreasing the function $F(\lambda) - (a_1\lambda + b_1)$ on the interval $(-\infty, 0)$ accepts the value 0. Consequently, equation (13) on the interval $(-\infty, 0)$ has a unique solution $\lambda^*$. The case $\lambda^* \in (\mu_{k-1}, \mu_k) \cap R_+$, $k = 1, 2, ...$ is considered similarly. The lemma is proved

Denote by $s(\lambda)$ the number of zeros of the function $y(x,\lambda)$ located in the interval $(0,l)$.

**Lemma 4 [8].** I*f* $\lambda \in (\mu_{k-1}, \mu_k]$, $k \in N$, *then* $s(\lambda) = k - 1$.

It holds the following oscillation

**Theorem 1.** *The eigen values of problem (1)-(3) form an infinitely increasing sequence* $\lambda_1 < \lambda_2 < ...\lambda_k < ...$, *moreover* $\lambda_1 < 0$ *and* $\lambda_k > 0$ *for* $k \geq 2$. *The eigenfunction* $y_1(x)$ *corresponding to the eigenvalue* $\lambda_1$ *has no zeros in the interval (0,1), the eigenfunction* $y_k(x)$, $k \geq 2$, *corresponding to the eigenvalue* $\lambda_k$ *has exactly* $k - 2$ *simple zeros in the interval* $(0,1)$.

**Proof.** By (8) and (10) we have

$$\lim_{\lambda \to \mu_{k-1}+0} F(\lambda) = +\infty, \qquad \lim_{\lambda \to \mu_k - 0} F(\lambda) = -\infty. \tag{16}$$

Taking into account the convexity of the function $F(\lambda)$ in the interval $(-\infty, 0)$ and relations $b_1 < 0$ and $F(0) > 0$, we get that equation (13) has the solutions $\lambda = \lambda_1 \in (-\infty, 0)$ and $\lambda = \lambda_2 \in (0, \mu_1)$. Based around lemma 4, the eigenfunctions $y_1(x) = y(x,\lambda_1)$ and $y_2(x) = y(x,\lambda_2)$ have no zeros in the interval $(0,1)$.

Let now $\lambda \in (\mu_{k-1}, \mu_k)$, $k = 2, 3, ....$ By (7), (16) and lemma 3, equation (13) has a unique solution $\lambda = \lambda_{k+1}$. Again, based on lemma 4, the eigenfunction $y(x) = y(x,\lambda_k)$, $k = 3, 4, ...$, has $k - 2$ simple zeros in the interval $(0,1)$. The theorem is proved.

Problem (1)-(3) is reduced to the eigenvalue problem for the operator $L$ in Hilbert space $H = L_2(0,1) \oplus C^2$ with the scalar product

$$(\widehat{y}, \widehat{u}) = (\{y(x), m, n\}, \{u(x), s, t\}) = (y,u)_{L_2} + |a_0|^{-1} m\bar{s} + |a_1|^{-1} n\bar{t}, \tag{17}$$

where $(\cdot,\cdot)_{L_2}$ is a scalar product in $L_2(0,1)$,

$$L\widehat{y} = L\{y,m,n\} = \left\{-y''(x) + q(x)\,y(x),\, y'(0) - b_0 y(0),\, y'(1) - b_1 y(1)\right\},$$

with domain of definition

$$D(L) = \{\{y,m,n\} \in H : y, y' \in AC[0,1], \quad -y'' + q(x)\,y(x) \in L_2(0,1),$$

$$m = a_0 y(0), \quad n = a_1 y(1)\}$$

everywhere dense in $H$ [10]. It is obvious that the operator $L$ was defined well in $H$.

Problem (1)-(3) takes the form

$$L\widehat{y} = \lambda \widehat{y}, \quad \widehat{y} \in D(L),$$

i.e. the eigenvalues $\lambda$ of problem (1)-(3) and operator $L$ coincide together with their multiplicities, and there is a correspondence between eigen and associated functions [10]

$$y_k(x) \longleftrightarrow \{y_k(x), m_k, n_k\}, \quad m_k = a_0 y_k(0), n_k = a_1 y_k(1).$$

Note that the operator $L$ is not self-adjoint in $H$. Define the operator $J : H \to H$ in the following way: $J\{y, m, n\} = \{y, m, -n\}$. The operator $J$ is unitary and symmetric in $H$ with a spectrum consisting of two eigen values: -1 with multiplicity 1 and +1 with infinite multiplicity. Consequently, this operator generates Pontryagin space $\Pi_1 = L_2(0,1) \oplus C^2$ with inner product ($J$-metrics) [11]

$$(\widehat{y}, \widehat{u})_{\Pi_1} = [\{y, m, n\}, \{u, s, t\}] = (y, u)_{L_2} - a_0^{-1} m\bar{s} + a_1^{-1} n\bar{t}. \tag{18}$$

**Lemma 5 [12].** *The operator $L$ is $J$-self adjoint in $\Pi_1$; if $L^*$ be the adjoint operator of the operator $L$ in $H$, then $L^* = JLJ$. The system of eigenvectors $\{y_k(x)\}_{k=1}^{\infty}$, $\widehat{y}_k = \{y_k(x), m_k, n_k\}$, of the operator $L$ forms the Riesz basis in $H$.*

Note that each element $\widehat{y}_k = \{y_k(x), m_k, n_k\}$, $k \in N$, where $m_k = a_0 y_k(0)$, $n_k = a_1 y_k(1)$ of the system of eigenvectors $\{\widehat{y}_k\}_{k=1}^{\infty}$ of the operator $L$ satisfies the relation

$$L\widehat{y}_k = \lambda_k \widehat{y}_k. \tag{19}$$

The element $\widehat{\vartheta}_k^* = \{\vartheta_k^*(x), s_k^*, t^*\}$ of the system of eigenvectors $\{\vartheta_k^*\}_{k=1}^{\infty}$ of the operator $L^*$ satisfies the equality

$$L^* \widehat{\vartheta}_k^* = \lambda_k \widehat{\vartheta}_k^*. \tag{20}$$

Based on lemma 5 and relations (19), (20), we have

$$\widehat{\upsilon}_k^* = J\widehat{y}_k, \quad k = 1, 2, .... \tag{21}$$

Denote

$$\delta_k = \|y\|_{L_2(0,1)}^2 - a_0^{-1} m_k^2 + a_1 n_k^2, \quad k = 1, 2, ... \tag{22}$$

where $\|\cdot\|_{L_2(0,1)}$ is the norm in the space $L_2(0,1)$.

From (6) and (7) we have

$$\delta_k \neq 0, k = 1, 2, .... \tag{23}$$

Indeed, by lemma 1, the eigenvalues of problem (1)-(3) (of the operator $L$) are simple. Consequently,

$$F'(\lambda) - a_1 \neq 0. \tag{24}$$

Taking into account relations (7), from (24) we get

$$\frac{\int_0^1 y^2(x,\lambda_k)\,dx - a_0}{y^2(1,\lambda_k)} + a_1 \neq 0,$$

whence it follows that

$$\int_0^1 y_k^2(x)\,dx - a_0 + a_1 y_k^2(1) \neq 0.$$

Inequality (23) directly follows from the last inequality.

Since the operator $L$ is $J$-self-adjoint in $\Pi_1$, then

$$\left(\widehat{y}_k, \widehat{\vartheta}_l\right) = (\widehat{y}_k, J\widehat{y}_l) = [\widehat{y}_k, \widehat{y}_l] = 0,\ k,l \in N, \quad k \neq l. \tag{25}$$

From (21) and (7) we have

$$\left(\widehat{y}_k, \widehat{\vartheta}_k\right) = (\widehat{y}_k, J\widehat{y}_k) = (\{y, m_k, n_k\}, \{y, m_k, -n_k\}) =$$

$$= \|y\|^2_{L_2(0,1)} - a_0^{-1} m_k^2 + a_1 n_k^2, \quad k \in N. \tag{26}$$

From (23), (25) and (26) we find

$$\left(\widehat{y}_k, \delta_k^{-1}\widehat{\vartheta}_l\right) = \delta_{k,l}, \tag{27}$$

where $\delta_{k,l}$ is Kronecker's symbol. Consequently, the element $\widehat{\vartheta}_k = \{\vartheta_k(x), s_k, t_k\}$ of the system $\left\{\widehat{\vartheta}_k\right\}_{k=1}^{\infty}$ adjoint to the system $\{\widehat{y}_k\}_{k=1}^{\infty}$, $\widehat{y}_k = \{y_k, m_k, n_k\}$ is defined by the equality

$$\widehat{\vartheta}_k = \delta_k^{-1}\vartheta_k^*, \quad k \in N. \tag{28}$$

Let $r, l, r \neq l$ be arbitrary fixed natural numbers. From theorem 4.2 and remark 4.4 [12], it follows that if

$$\Delta_{r,l} = \begin{vmatrix} s_r & s_l \\ t_r & t_l \end{vmatrix} \neq 0, \tag{29}$$

then the system $\{y_k(x)\}_{k=1, k\neq r,l}^{\infty}$ forms a basis in the space $L_p(0,1)$, $1 < p < \infty$ (for $p = 2$ the Riesz basis), if $\Delta_{r,l} = 0$, then this system is neither complete nor minimal in $L_p(0,1)$, $1 < p < \infty$.

Taking into account (4) and (28), from (29) we get

$$\Delta_{r,l} = \begin{vmatrix} s_r & s_l \\ t_r & t_l \end{vmatrix} = \begin{vmatrix} \delta_r^{-1} m_r & \delta_l^{-1} m_l \\ -\delta_r^{-1} n_r & -\delta_l^{-1} n_l \end{vmatrix} = -\delta_r^{-1}\delta_l^{-1} \begin{vmatrix} m_r & m_l \\ n_r & n_l \end{vmatrix} =$$

$$= -\delta_r^{-1}\delta_l^{-1} \begin{vmatrix} a_0 y_r(0) & a_0 y_l(0) \\ a_1 y_r(1) & a_1 y_l(1) \end{vmatrix} = -a_0 a_1 \delta_r^{-1}\delta_l^{-1} \begin{vmatrix} 1 & 1 \\ y_r(1) & y_l(1) \end{vmatrix}. \tag{30}$$

From (4) and theorem 1 we have

$$y_1(1) > 0, \quad (-1)^k y_k(1) > 0, \quad k = 2, 3, 4.... \tag{31}$$

Taking into account relation (31), from (30) we get that if $l$ is odd, then $\Delta_{1,l} \neq 0$; if $r, l \geq 2$ and have different parities, then $\Delta_{r,l} \neq 0$.

Thus, we proved the following theorem that gives sufficient conditions of defect basicity of the system of eigenfunctions of problem (1)-(3).

**Theorem 2.** *Let $r, l, r \neq l$ be arbitrary fixed natural numbers such that if $r = 1$, then $l$ is an odd number, and if $r, l \geq 2$, then they have different parities. Then the system of eigen functions $\{y_k(x)\}_{k=1, k\neq r,l}^{\infty}$ of problem (1)-(3) forms a basis in the space $L_p(0,1)$, $1 < p < \infty$, for $p = 2$ the Riesz basis.*

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Received February 15, 2012; Revised May 23, 2012