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STUDY OF ONE CLASS PROBLEMS MOVING SOURCES IN SYSTEMS OF OPTIMAL CONTROL BY WITH THE DISTRIBUTED PARAMETERS

Abstract

For the solution of a problems of optimal control of moving sources which condition is described by totality of parabolic type equation and systems of the ordinary differential equations, existence and uniqueness theorems are proved, sufficient conditions of differentiability of a target functional and an expression for its gradient are obtained, necessary conditions of optimality in the form of integral maximum principles are established.

Now, in view of complexity of the solution of a problem of optimum control of the moving sources which condition is described by the differential equations with partial derivatives and systems of the ordinary differential equations, are studied insufficiently [1,5]. For some classes of linear and nonlinear boundary value problems in which participates pulse functions, questions of existence and uniqueness of the generalized solution are investigated. In studied work the problem of optimal control by the moving sources, totality by the parabolic type equation and systems of the ordinary differential equations is considered under entry and boundary conditions. For this problem theorems of existence and uniqueness of the solution are proved, sufficient conditions of Frechet differentiability of a target functional and expression for its gradient are obtained, necessary conditions of optimality in the form of integral maximum principles are established.

1. Problem statement

Let $l > 0, T > 0$ are the given numbers $\Omega_t = (0,l) \times (0,t)$, $\Omega = \Omega_T$. The functional spaces $W_2^{1,0}(\Omega)$, $W_2^{1,1}(\Omega)$, $V_2(\Omega)$, $V_2^{1,0}(\Omega)$ used below, are introduced, for example in [4].

Let the condition of operated process is described by functions $u(x,t)$ and $s(t)$. Let's assume that in area Ω function $u(x,t)$ satisfies the following equation parabolic type

$$u_t = a^2 u_{xx} + \sum_{k=1}^{n} p_k(t)\delta(x - s_k(t)), \tag{1}$$

with initial and boundary conditions

$$u(x,o) = \varphi(x),\ 0 \le x \le l, \tag{2}$$

$$u_x\,|_{x=0} = 0,\ \ u_x\,|_{x=l} = 0,\ \ 0 < t \le T, \tag{3}$$

where $a > 0$ is the given number, $\varphi(x) \in L_2(0,l)$is the given function; $\delta(\cdot)$ is the Dirak's function; $p(t) = (p_1(t), p_2(t), ..., p_n(t)) \in L_2^n(0,T)$ is the control function.

Let's assume also that the functions $s_k(t) \in C[0,T], k = \overline{1,n}$, are the solution of the following problem of Cauchy:

$$\dot{s}_k(t) = f_k(s_k(t), \vartheta(t), t),\ 0 < t \leq T,\ s_k(o) = s_{k0}, k = \overline{1,n}, \tag{4}$$

where $s_{k0} \in R^n$; $\vartheta = \vartheta(t) = (\vartheta_1(t), \vartheta_2(t), ..., \vartheta_r(t)) \in L_2^r(0,T)$ are the control function; function $f(s,\vartheta,t) = (f_1(s,\vartheta,t), f_2(s,\vartheta,t), ..., f_n(s,\vartheta,t))$ is continuous, has continuous with respect to s, ϑ for $(s,\vartheta,t) \in E^n \times E^r \times [o,T]$.

Pair of functions $\bar{\vartheta} = (p(t), \vartheta(t))$ we will call controls. For brevity we denote by $H = L_2^n(0,T) \times L_2^r(0,T)$ a Hilbert space of pairs $\bar{\vartheta} = (p(t), \vartheta(t))$ with scalar product

$$< \bar{\vartheta}^1, \bar{\vartheta}^2 >_H = \int_0^T [p^1(t)p^2(t) + \vartheta^1(t)\vartheta^2(t)]dt$$

and the norm $\|\bar{\vartheta}\|_H = \sqrt{(< \bar{\vartheta}, \bar{\vartheta} >_H)} = \sqrt{(\|p\|_{L_2}^2 + \|\vartheta\|_{L_2}^2)}$, where $\bar{\vartheta}^k = (p^k, \vartheta^k), k = 1,2$.

Let's put

$$V = \left\{(p,\vartheta) \in H : 0 \leq p_i \leq A_i, 0 \leq |\vartheta_j| \leq B_j, i = \overline{1,n}, j = \overline{1,r}\right\}, \tag{5}$$

where $A_i > 0, i = \overline{1,n}$, $B_j > 0, j = \overline{1,r}$, are the given numbers and we will consider a functional

$$J(\tilde{\vartheta}) = \int_0^l \int_0^T [u(x,t) - \tilde{u}(x,t)]^2 dxdt + \alpha_1 \sum_{k=1}^n \int_0^T [p_k(t) - \tilde{p}_k(t)]^2 dt +$$

$$+ \alpha_2 \sum_{m=1}^r \int_0^T [\vartheta_m(t) - \tilde{\vartheta}_m(t)]^2 dt, \tag{6}$$

where $\bar{\vartheta} = (p(t), \vartheta(t)) \in H$ is the control fiunction; $\alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 > 0$ are the given parameters; $\tilde{u}(x,t)) \in L_2(\Omega)$, $\omega = (\tilde{p}(t), \tilde{\vartheta}(t)) \in H$, $\tilde{p}(t) = (\tilde{p}_1(t), \tilde{p}_2(t), ..., \tilde{p}_n(t)) \in L_2^n(0,T)$, $\tilde{\vartheta}(t) = (\tilde{\vartheta}_1(t), \tilde{\vartheta}_2(t), ..., \tilde{\vartheta}_r(t) \in L_2^r(0,T))$ are the given functions.

It is required to find such controls $\bar{\vartheta} = (p(t), \vartheta(t))$ from the set V and functions $(u(x,t), s(t))$ that the functional (6) accepted the smallest possible value at conditions (1)-(4).

2. Existence and uniqueness of the solution

Definition. *A problem on finding the function* $(u(x,t), s(t)) = (u(x,t;\bar{\vartheta}), s(t;\vartheta))$ *from conditions (1)-(4) for the given control* $\bar{\vartheta} = (p(t), \vartheta(t)) \in V$ *is said to be a reduced problem. Under the solution of the reduced problem (1)-(4), corresponding to the control* $\bar{\vartheta} = (p(t), \vartheta(t)) \in V$ *we understand the functions* $(u(x,t), s(t))$ *from* $(V_2^{1,0}(\Omega), C[0,T])$, *where the function* $u = u(x,t)$ *satisfies the integral identity*

$$\int_0^l \int_0^T [-u\eta_t + a^2 u_x \eta_x] dxdt = \int_0^l \varphi(x)\eta(x,0)dx + \sum_{k=1}^n \int_0^T p_k(t)\eta(s_k(t),t)dt, \tag{7}$$

for $\forall\eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ *and* $\eta(x,T) = 0$, *the function* $s_k(t) = s_k(t;\vartheta)$ *satisfies the integral equation*

$$s_k(t) = \int_0^t f_k(s_k(\tau), \vartheta(\tau), \tau)d\tau + s_{k0},\ 0 \le t \le T, k = \overline{1,n}. \tag{8}$$

It follows from results of papers [2],[4] follows that for each fixed $\overline{\vartheta} \in V$ the reduced problem (1) - (4) has a unique solution from $(V_2^{1,0}(\Omega), C\,[0,T])$. Let the conditions accepted at the statement of problem (1) - (6) be fulfilled. Then problem (1) - (6) has at least one solution. It should be noted that the problem (1) - (6) for $\alpha_j = 0, j = \overline{1,2}$ is incorrect in the classical sense [8]. However it holds

Theorem 1. *There exists a dense subset K of the space H such that for any $\omega \in K$ for $\alpha_i > 0, i = \overline{1,2}$, problem (1)-(6) has a unique solution.*

Proof. Prove continuity of the functional $J_0(\tilde{\vartheta}) = \|u(x,t) - \tilde{u}(x,t)\|^2_{L_2(\Omega)}$. Let $\Delta\overline{\vartheta} = (\Delta p, \Delta\vartheta) \in V$ be an increment of a control on the element $\overline{\vartheta} = (p, \vartheta) \in V$ such that $\overline{\vartheta} + \Delta\overline{\vartheta} \in V$. Denote $\Delta u \equiv \Delta u(x,t) = u(x,t;\overline{\vartheta} + \Delta\overline{\vartheta}) - u(x,t,\overline{\vartheta})$, $u \equiv u(x,t;\overline{\vartheta})$, $\Delta s_k \equiv \Delta s_k(t) = s_k(t;\overline{\vartheta} + \Delta\overline{\vartheta}) - s_k(t;\overline{\vartheta})$, $s_k \equiv s_k(t;\overline{\vartheta})$.

It follows from (1)-(4) that function $\Delta u$is generalized solution of the boundary value problem

$$\Delta u_t = a^2\Delta u_{xx} + \sum_{k=1}^{n}[(p_k + \Delta p_k)\delta(x - (s_k + \Delta s_k)) - p_k\delta(x - s_k)],\ (x,t) \in \Omega, \tag{9}$$

$$\Delta u_x|_{x=0} = \Delta u_x|_{x=l} = 0,\ t \in [0,T], \tag{10}$$

$$\Delta u|_{t=0} = 0, x \in [0,l], \tag{11}$$

and functions $\Delta s_k, k = \overline{1,n}$ is solutions of the Cauchy problem

$$\Delta\dot{s}_k(t) = \Delta f_k(s_k(t), \vartheta(t), t),\ \Delta s_k(0) = 0, k = \overline{1,n}, \tag{12}$$

where $\Delta f_k(s_k(t), \vartheta(t), t) = f_k(s_k + \Delta s, \vartheta + \Delta\vartheta, t) - f_k(s_k, \vartheta, t)$.

Prove that for the function $\Delta u(x,t)$ it holds the estimation

$$\|\Delta u\|_{V_2^{1,0}(\Omega)} \le c_1 \left\|\Delta\overline{\vartheta}\right\|_{L_2(0,T)}, \tag{13}$$

where $c_1 > 0$ is a constant.

Multiplying both members of equation (9) on $\eta = \eta(x,t)$ and integrating in parts received equality, we receive estimation:

$$\int_0^l \int_0^T [\Delta u_t\eta + a^2\Delta u_x\eta_x]dxdt = \sum_{k=1}^{n}\int_0^T [(p_k + \Delta p_k)\eta(s_k + \Delta s_k, t) - p_k\eta(s_k,t)]\,dt. \tag{14}$$

Let $t_1, t_2 \in [0,T]$ such that $t_1 \le t_2$. In identity (14) we will put

$$\eta(x,t) = \begin{cases} \Delta u(x,t)\ \ , \ t \in (t_1,t_2], \\ 0, t \in [0,t_1] \cup (t_2,T], \end{cases}$$

and applying the formula of finite increments for the function $\Delta u(s_k(t)+\Delta s_k,t)$ in the form

$$\Delta u(s_k+\Delta s_k,t)=\Delta u(s_k,t)+\Delta u_x(\bar{s}_k,t)\Delta s_k,\ \bar{s}_k=s_k+\theta\Delta s_k,\theta\in[0,1],$$

we receive energetic balance equation for problem (9) - (12):

$$\frac{1}{2}\left\|\Delta u(x,t)\right\|^2_{L_2(0,l)}\Big|^{t=t_2}_{t=t_1}+a^2\left\|\Delta u_x(x,t)\right\|^2_{L_2(\Omega_t)}\Big|^{t=t_2}_{t=t_1}=$$

$$=\sum_{k=1}^{n}\int_{t_1}^{t_2}[(p_k+\Delta p_k)\Delta s_k\Delta u_x(\bar{s}_k,t)+\Delta p_k\Delta u(s_k,t)]dt, \tag{15}$$

where $\bar{s}_k=s_k+\theta\Delta s_k,\theta\in[0,1]$.

Applying the Cauchy-Bunyakovskii inequality to the right member of equation (15), we have

$$\frac{1}{2}\left\|\Delta u(x,t)\right\|^2_{L_2(0,l)}\Big|^{t=t_2}_{t=t_1}+a^2\left\|\Delta u_x(x,t)\right\|^2_{L_2(\Omega_t)}\Big|^{t=t_2}_{t=t_1}\le$$

$$\sum_{k=1}^{n}\left(\left\|p_k\right\|_{L_2(t_1,t_2)}+\left\|\Delta p_k\right\|_{L_2(t_1,t_2)}\right)\max_{t_1\le t\le t_2}\left|\Delta s_k(t)\right|\left\|\Delta u_x(\bar{s}_k,t)\right\|_{L_2(t_1,t_2)}+$$

$$\left.+\left\|\Delta p_k\right\|_{L_2(t_1,t_2)}\left\|\Delta u(s_k,t)\right\|_{L_2(t_1,t_2)}\right]. \tag{16}$$

As function $\Delta s(t)$ is the solution of a problem of Cauchy (12), from properties of function $f(s,\vartheta,t)$ as rather small $\varepsilon=t_2-t_1$ we have

$$\max_{t_1\le t\le t_2}\left|\Delta s_k(t)\right|\le C_2\left\|\Delta\vartheta\right\|_{L_2(t_1,t_2)},\forall k,1\le k\le n,$$

and besides, it is simple to show that inequalities are true:

$$\left\|\Delta u(s_k,t)\right\|_{L_2(t_1,t_2)}\le c_3\left\|\Delta u\right\|_{V_2^{1,0}(\Omega)},$$

$$\left\|\Delta u_x(\bar{s}_k,t)\right\|_{L_2(t_1,t_2)}\le c_4\left\|\Delta u\right\|_{V_2^{1,0}(\Omega)},$$

where $c_3>0,c_4>0$ are some constants.

But, then we majorize the right hand side inequality (16) from above as follows

$$\frac{1}{2}\left\|\Delta u(x,t)\right\|^2_{L_2(0,l)}\Big|^{t_2}_{t_1}+a^2\left\|\Delta u_x(x,t)\right\|^2_{L_2(\Omega_T)}\Big|^{t_2}_{t_1}\le c_5\left\|\Delta\overline{\vartheta}\right\|_H\left\|\Delta u\right\|_{V_2^{1,0}(\Omega)}, \tag{17}$$

as $\left\|\Delta\overline{\vartheta}\right\|_{L_2(t_1,t_2)}\to 0$,where $c_5>0$ is some constant. As well as in work [2, pp. 166-168], for the any $t\in[0,T]$ we will break a segment $[0,t]$ into final number sub-segments, on each of which the inequality (17) is carried out. Then, having combined the received inequalities for everyone sub-segments, we have estimation

$$\frac{1}{2}\left\|\Delta u(x,t)\right\|^2_{L_2(0,l)}+a^2\left\|\Delta u_x(x,t)\right\|^2_{L_2(\Omega_T)}\le C_5\left\|\Delta\overline{\vartheta}\right\|_H\left\|\Delta u\right\|_{V_2^{1,0}(\Omega)},$$

from where the inequality (13) follows. Then $\|\Delta u\|_{V_2^{1,0}(\Omega)} \to 0$ as $\left\|\Delta\bar{\vartheta}\right\|_H \to 0$. Hence and from the trace theorem [8,p.161] we get $\|\Delta u(x,t)\|_{L_2(\Omega)} \to 0$ as $\left\|\Delta\bar{\vartheta}\right\|_H \to 0$.

Increment of a functional $J_0(\bar{\vartheta})$ representable in a look

$$J_0(\bar{\vartheta}+\Delta\bar{\vartheta}) - J_0(\bar{\vartheta}) = 2\int\limits_0^l\int\limits_0^T [u(x,t)-\tilde{u}(x,t)]\Delta u(x,t)dxdt + \|\Delta u(x,t)\|^2_{L_2(\Omega)}.$$

Hence and the fact $\|\Delta u(x,t)\|_{L_2(\Omega)} \to 0$ as $\left\|\Delta\bar{\vartheta}\right\|_H \to 0$, it follows continuity of the functional $J_0(\bar{\vartheta})$.

The functional $J_0(\bar{\vartheta})$ from below is bounded and owing to proved is continuous in V. Besides, H- evenly convex and is reflective Banakh's space [7]. Then from Bidou's theorem provided in work [9], existence of a dense subset K of space H such follows that for any $\omega = (\tilde{p}(t), \tilde{\vartheta}(t)) \in H$ as $\alpha_i > 0, i = \overline{1,2}$ a problem (1) - (6) has the unique solution. The theorem is proved.

3. Necessary conditions of optimality

Let $\psi = \psi(x,t)$is a solution from $V_2^{1,0}(\Omega)$ of the conjugated to (1)-(3) problem

$$\psi_t + a^2\psi_{xx} = -2[u(x,t)-\tilde{u}(x,t)],\ (x,t)\in\Omega, \tag{18}$$

$$\psi_x\,|_{x=0} = 0,\ \psi_x\,|_{x=l} = 0,\ \ 0\le t<T, \tag{19}$$

$$\psi(x,T) = 0,\ 0\le x\le l, \tag{20}$$

and $q_k(t)-$is a solution from $C\,[0,T]$ of the conjugated to (4) problem

$$\dot{q}_k(t) = -\frac{\partial f_k}{\partial s_k}q_k(t) + p_k(t)\psi_x(s_k(t),t),\ \ 0\le t<T, q_k(T)=0, k=\overline{1,n}. \tag{21}$$

Integrating in parts identity

$$\int\limits_{\Omega}\left(\psi_t + a^2\psi_{xx} + 2[u(x,t)-\tilde{u}(x,t)]\right)\eta_1(x,t)d\Omega = 0$$

the function $\psi = \psi(x,t)$ satisfies the integral identity

$$\int\limits_0^l\int\limits_0^T[\psi\eta_{1t} + a^2\psi_x\eta_{1x}]dxdt = 2\int\limits_0^l\int\limits_0^T[u(x,t)-\tilde{u}(x,t)]\eta_1(x,t)dxdt, \tag{22}$$

for $\forall\eta_1 = \eta_1(x,t)\in W_2^{1,1}(\Omega)$ and $\eta_1(x,0)=0$, function $q_k(t)$ satisfies the integral equation

$$q_k(t) = \int\limits_t^T\left[\frac{\partial f_k}{\partial s_k}q_k(\tau) - p_k(\tau)\psi_x(s_k(\tau),\tau)\right]d\tau,\ 0\le t\le T, k=\overline{1,n}. \tag{23}$$

The conjugated problem (9) - (12) is the mixed problem for the linear parabolic equation. If in relations (9) - (12), instead of a variable t we take a new independent variable $\tau = T - t$, we get a boundary value problem of the same type as (1) - (4). Therefore, it follows from the facts established for problem (1) - (4) that for each given $\overline{\vartheta} = (p(t), \vartheta(t)) \in V$ problem (9) - (12) has a unique solution from $(V_2^{1,0}(\Omega), C[0,T])$.

The function

$$H(t,s,\psi,q,\overline{\vartheta}) = -\left\{\sum_{k=1}^{n}[-f_k(s_k(t),\vartheta(t),t)q_k(t) + \psi(s_k(t),t)p_k(t)+\right.$$

$$\left.\alpha_1\left(p_k(t)-\tilde{p}_k(t)\right)^2\Big] + \alpha_2\sum_{m=1}^{r}\left(\vartheta_m(t)-\tilde{\vartheta}_m(t)\right)^2\right\}, \tag{24}$$

is said to be Hamilton-Pontryagin function of problem (1) - (6).

Theorem 2. *Let:*

1) functions $f_k(s_k,\vartheta,t), k=\overline{1,n}$, be continuous in totality of all its arguments together with all its partial derivatives wit respect to variables s and ϑ as $(s,\vartheta,t) \in R^n \times R^r \times [0,T]$;

2) functions $f_k(s_k,\vartheta,t)$, $f_{ks} = \dfrac{\partial f_k(s_k,\vartheta,t)}{\partial s}$, $f_{k\vartheta} = \dfrac{\partial f_k(s_k,\vartheta,t)}{\partial \vartheta}$, $k=\overline{1,n}$, satisfy to Lipshits's condition on s and ϑ, i.e.

$$\begin{aligned}
|f_k(s_k+\Delta s_k,\vartheta+\Delta\vartheta,t) - f_k(s_k,\vartheta,t)| &\le L(|\Delta s_k| + |\Delta\vartheta|),\\
|f_{ks}(s_k+\Delta s_k,\vartheta+\Delta\vartheta,t) - f_{ks}(s_k,\vartheta,t)| &\le L(|\Delta s_k| + |\Delta\vartheta|),\\
|f_{k\vartheta}(s_k+\Delta s_k,\vartheta+\Delta\vartheta,t) - f_{k\vartheta}(s_k,\vartheta,t)| &\le L(|\Delta s_k| + |\Delta\vartheta|),
\end{aligned}$$

Be fulfilled for all $(s_k+\Delta s_k,\vartheta+\Delta\vartheta,t)$, $(s_k,\vartheta,t) \in E^n \times E^r \times [0,T]$, where $L = const \ge 0$.

Then, if $(\psi(x,t),q(t))$- the solution of the conjugated problem (9) - (12), the functional (6) is Frechet differentiable and the expression

$$J'(\overline{\vartheta}) = \left(\frac{\partial J(\overline{\vartheta})}{\partial p}, \frac{\partial J(\overline{\vartheta})}{\partial \vartheta}\right) = \left(-\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial \vartheta}\right). \tag{25}$$

is valid for its gradient.

Proof. Consider the increment of the functional

$$\Delta J \equiv J(\bar{\vartheta}+\Delta\bar{\vartheta}) - J(\bar{\vartheta}) = 2\int_0^l\int_0^T[u(x,t)-\tilde{u}(x,t)]\Delta u(x,t)dxdt + \int_0^l\int_0^T|\Delta u(x,t)|^2dxdt+$$

$$+\sum_{k=1}^{n}\left\{2\alpha_1\int_0^T[p_k(t)-\tilde{p}_k(t)]\,\Delta p_k(t)dt + \alpha_1\int_0^T|\Delta p_k(t)|^2dt\right\}+$$

$$+\sum_{m=1}^{r}\left\{2\alpha_2\int_0^T\left[\vartheta_m(t)-\tilde{\vartheta}_m(t)\right]\cdot\Delta\vartheta_m(t)dt + \alpha_2\int_0^T|\Delta\vartheta_m(t)|^2dt\right\} \tag{26}$$

where $\bar{\vartheta} = (p, \vartheta) \in V, \bar{\vartheta} + \Delta\bar{\vartheta} \in V$, $\Delta u(x,t) \equiv u(x,t;\bar{\vartheta} + \Delta\bar{\vartheta}) - u(x,t;\bar{\vartheta})$, $u \equiv u(x,t;\bar{\vartheta})$.

If in (22) we set $\eta_1 = \Delta u(x,t)$, in (14) $\eta = \psi(x,t)$ and we will subtract the obtained relations, we have

$$2\int_0^l\int_0^T [u(x,t)-\tilde{u}(x,t)]\Delta u(x,t)dxdt = \sum_{k=1}^{n}\int_0^T [(p_k+\Delta p_k)\psi(s_k+\Delta s_k,t)-p_k\psi(s_k,t)]dt. \tag{27}$$

It follows from (12) that the function $\Delta s_k(t)$ satisfies the integral identity:

$$\int_0^T \left[\Delta s_k(t)\dot{\theta}_k(t) + \Delta f_k(s_k(t),\vartheta(t),t)\theta_k(t)\right] dt = 0, \tag{28}$$

for $\forall\theta_k(t) \in C\,[0,T]\,, \theta_k(T) = 0, k = \overline{1,n}\,.$

It follows from (21) that the function $q_k(t)$ satisfies the integral identity:

$$\int_0^T \left[q_k(t)\dot{\theta}_{1k}(t) - \left(\frac{\partial f_k}{\partial s_k}q_k(t) - p_k(t)\psi_x(s_k(t),t)\right)\theta_{1k}(t)\right] dt = 0, \tag{29}$$

for $\forall\theta_{1k}(t) \in C\,[0,T]\,, \theta_{1k}(0) = 0, k = \overline{1,n}$.

In the same way, if in (29) we set $\theta_{1k}(t) = \Delta s_k(t)$, in (28) $\theta_k(t) = q_k(t)$ and summing the obtained relations, we have:

$$[\Delta s_k(t)q_k(t)]\,\big|_0^T = \int_0^T \left[\left(\frac{\partial f_k}{\partial s_k}q_k(t) - p_k(t)\psi_x(s_k(t),t)\right)\Delta s_k(t) - \Delta f_k q_k(t)\right] dt.$$

Considering the theorem's condition, we can represent the function $\Delta f_k = \Delta f_k(s_k(t),\vartheta(t),t)$ in the form

$$\Delta f_k = \frac{\partial f_k}{\partial s_k}\Delta s_k + \sum_{m=1}^{r}\frac{\partial f_k}{\partial \vartheta_m}\Delta\vartheta_m + R_1,$$

where $R_1 = o\left(\sqrt{\|\Delta s\|^2_{L_2(0,T)} + \|\Delta\vartheta\|^2_{L_2(0,T)}}\right)$. Then from the last equality we have:

$$[\Delta s_k(t)q_k(t)]\,\big|_0^T = \int_0^T \left(\frac{\partial f_k}{\partial s_k}q_k(t) - p_k(t)\psi_x(s_k(t),t))\right)\Delta s_k(t) -$$
$$\left.\sum_{m=1}^{r}\frac{\partial f_k}{\partial \vartheta_m}\Delta\vartheta_m(t)q_k(t) - \frac{\partial f_k}{\partial s_k}\Delta s_k(t)q_k(t)\right] dt + R_1$$

on considering (12) and (21) we get

$$\int_0^T p_k(t)\psi_x(s_k(t),t)\Delta s_k(t)dt = -\sum_{m=1}^{r}\int_0^T \frac{\partial f_k}{\partial \vartheta_m}\Delta\vartheta_m(t)q_k(t)dt + R_1. \tag{30}$$

It is clear that under the assumptions made above, on Taylor's formula fairly decomposition:

$$\psi(s_k+\Delta s_k,t)=\psi(s_k,t)+\psi_x(s_k,t)\Delta s_k+o(\|\Delta s_k\|).$$

Considering this formula in (27), we get

$$2\int_0^l\int_0^T[u(x,t)-\tilde{u}(x,t)]\Delta u(x,t)dxdt=\sum_{k=1}^n\int_0^T[(p_k(t)\psi_x(s_k(t),t)\Delta s_k(t)+$$

$$+\psi(s_k(t),t)\Delta p_k(t)+\psi_x(s_k(t),t)\Delta p_k(t)\Delta s_k(t)+o(\|\Delta s_k\|)]dt.$$

As relation (30) is fulfilled, from the last equality we have

$$\int_0^\ell\int_0^T[u(x,t)-\widetilde{u}(x,t)]\Delta u(x,t)dxdt=\sum_{k=1}^n\int_0^T\left[-\sum_{m=1}^r\frac{\partial f_k}{\partial\vartheta_m}q_k(t)\Delta\vartheta_m(t)+\right.$$

$$\left.+\psi(s_k,t)\Delta p_k\right]dt+R_2, \tag{31}$$

where $R_2=\sum\limits_{k=1}^n\int\limits_0^T\left[\psi_x(s_k(t),t)\Delta p_k(t)\Delta s_k(t)+o(\|\Delta s_k\|)\right]dt+R_1$.

According to the usual scheme (see, for example, [2]) it is possible to prove justice of an assessment

$$\|\Delta s\|_{C[0,T]}\le c_5\|\Delta\vartheta\|_{L_2(0,T)}, \tag{32}$$

where $c_5>0$ is some constant.

From here we get $R_2=o(\|\Delta\bar{\vartheta}\|_H)$. On the other hand, it follows from estimation (13) that

$$\|\Delta u(x,t)\|_{L_2(\Omega)}=O\left(\|\Delta\bar{\vartheta}\|_H\right).$$

Considering these estimations in expressions (26) and (31), we have:

$$\Delta J=\sum_{k=1}^n\left(J_1(k)+\sum_{m=1}^r J_2(k,m)\right)+o(\|\Delta\bar{\vartheta}\|_H),$$

where

$$J_1(k)=\int_0^T\left[\psi(s_k(t),t)+2\alpha_1\left(p_k(t)-\tilde{p}_k(t)\right)\right]\Delta p_k(t)dt,$$

$$J_2(k,m)=\int_0^T\left[-\frac{\partial f_k(s_k(t),\vartheta(t),t)}{\partial\vartheta_m}q_k(t)+2\alpha_2\left(\vartheta_m(t)-\tilde{\vartheta}_m(t)\right)\right]\Delta\vartheta_m(t)dt.$$

Hence, allowing for expression of Hamilton-Pontryagin function, we get

$$\Delta J=\left(-\frac{\partial H}{\partial\bar{\vartheta}},\Delta\bar{\vartheta}\right)_H+o(\|\Delta\bar{\vartheta}\|_H),$$

that shows Frechet differentiability of functional (1) and validity of formula (25). The theorem is proved.

Theorem 3. *Let all conditions of theorem 2 be fulfilled and* $(u^*(x,t), s^*(t))$, $(\psi^*(x,t), q^*(t))$ *be solutions of problems (1) - (4) and (18) - (21), re-spectively for* $\overline{\vartheta} = \overline{\vartheta}^* \in V$*. Then for optimality of the control* $\overline{\vartheta}^* = (p^*(t),\vartheta^*(t))$*the condition*

$$H(t, s^*, \psi^*, q^*, \overline{\vartheta}^*) = \max_{\overline{\vartheta} \in V} H(t, s^*, \psi^*, q^*, \overline{\vartheta}), \forall (x,t) \in \Omega. \tag{33}$$

should be fulfilled.

Proof. Assume that $\overline{\vartheta}^* = (p^*(t),\vartheta^*(t))$ is an optimal control. Assume the contrary, i.e. there will be found such control $\tilde{\vartheta} = \bar{\vartheta}^* + h\Delta\bar{\vartheta} \in V$ and the number $\beta > 0$, for which

$$H(t, s^*, \psi^*, q^*, \tilde{\vartheta}) - H(t, s^*, \psi^*, q^*, \bar{\vartheta}^*) \geq \beta > 0, \tag{34}$$

where $h > 0$ is some number, $\tilde{\vartheta} \equiv (p^* + h\Delta p, \vartheta^* + h\Delta\vartheta), \Delta\overline{\vartheta} = (\Delta p, \Delta\vartheta)$.

If in (34) we take into account formula (24), we get

$$h\left(J'(\breve{\vartheta}), \Delta\bar{\vartheta}\right)_H \leq -\beta < 0,$$

where $\breve{\vartheta} = h\theta_1\Delta\bar{\vartheta} \equiv h\theta_1(\Delta p, \Delta\vartheta) \in V, \theta_1 \in (0,1)$ are some numbers. Hence and from the finite increment formula we have:

$$J(\tilde{\vartheta}) - J(\bar{\vartheta}^*) = h\left(J'(\widehat{\vartheta}), \Delta\bar{\vartheta}\right)_H = h\left(J'(\breve{\vartheta}), \Delta\bar{\vartheta}\right)_H + h\left(J'(\widehat{\vartheta}) - J'(\breve{\vartheta}), \Delta\bar{\vartheta}\right)_H \leq$$

$$\leq -\beta + h0(\left\|\Delta\bar{\vartheta}\right\|_H), \tag{35}$$

where $\widehat{\vartheta} = h\theta_2\Delta\bar{\vartheta} \equiv h\theta_2(\Delta p, \Delta\vartheta) \in V, \theta_2 \in (0,1)$are some numbers.

Let $0 < h_1 < h$ be such a number that $-\beta + h_1 0(\left\|\Delta\bar{\vartheta}\right\|_H) < 0$. Assume $\overset{\approx}{\vartheta} = \overline{\vartheta}^* + h_1\Delta\overline{\vartheta}$. Reasoning as in the proof of inequality (35), we get

$$J(\overset{\approx}{\vartheta}) - J(\bar{\vartheta}^*) \leq -\beta + h_1 0(\left\|\Delta\bar{\vartheta}\right\|_H) < 0$$

This contradicts optimality of the control $\bar{\vartheta}^*$. Hence we get validity of relation (33). The theorem is proved.

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