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ASYMPTOTICS OF EIGENVALUE DISTRIBUTION AND TRACE FORMULA OF ONE SINGULAR

Abstract

The spectrum and eigenvalue aysmptotics of boundary problem value for differential operator equation on semiaxis is studied. Trace formula for operator associated with this problem is established.

Let H be separable Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Consider in space $L_2\left((0,\infty),H\right)$ the problem

$$l[y] \equiv -y''(t) + ty(t) + Ay(t) + q(t)y(t) = \lambda y(t), \tag{1}$$

$$y'(0) + \lambda y(0) = 0, \tag{2}$$

where $A = A^*,\ A > E,\ E$ an identity operator in $H,\ A^{-1} \in \sigma_\infty$. Denote eigenvalues and orthonormal eigen-vectors of operator A by $\gamma_1 \le \gamma_2 \le ...$ and $\varphi_1, \varphi_2,...$ respectively.

Suppose $q(t)$ is weakly measurable, $\|q(t)\| < const,\ q^*(t) = q(t),\ \forall t \in [0,\infty)$ and the following conditions are held:

1. $\sum\limits_{k=1}^{\infty}\int\limits_0^\infty |(q(t)\varphi_k,\varphi_k)|\,dt < const,\ \forall t \in [0,\infty)$;
2. $\dfrac{q_k(t)}{t} \equiv \dfrac{(q(t)\varphi_k,\varphi_k)}{t}$ is summable on $(0,\infty)$, $\int\limits_0^\infty \dfrac{q_k(t)}{t}dt = 0,\ \forall k = \overline{1,\infty}$;
3. $\int\limits_0^1 \dfrac{q_k(t)}{t^5} < \infty,\ \ \forall k = \overline{1,\infty}.$

Introduce the space $\mathbb{L}_2 = L_2((0,\infty),H) \oplus H$ with scalar product for elements

$$Y = (y(t), y_0) \in \mathbb{L}_2,\quad Z = (z(t), z_0) \in \mathbb{L}_2 : (Y,Z)_{\mathbb{L}_2} = \int\limits_0^\infty (y(t), z(t))dt + (y_0, z_0).$$

Define in $\mathbb{L}_2$ operator L_0 for case $q(t) \equiv 0$

$$D(L_0) = \{Y \in \mathbb{L}_2 / l[y] \in L_2(0,\infty), H),\ y_0 = y(0)\},$$

$$L_0 Y = \left\{l(y), -y'(0)\right\}.$$

One could show, that L_0 is selfadjoint operator in $\mathbb{L}_2$.

Denote operator, corresponding to case $q(t) \neq 0$, by $L : L = L_0 + Q$ where $Q : QY = \{q(t)y(t), 0\}$ is bounded selfadjoint operator in $\mathbb{L}_2$.

In the paper the eigenvalue asymptotics of problem (1), (2) is studied. Also trace formula for operator L is established.

The asymptotics of eigenvalue distributions of problems for differential operator equations with parameter dependent boundary conditions are investigated in [1], [2], [3] and others, in [3], [4] also trace formulas for appropriate operators are established.

1. The asymptotics of eigenvalues. Begin with studding of the spectrum operator L_0.

Condition $A > E$ yields positive-definiteness of L_0 in $\mathbb{L}_2$. Let $y_k(t) = (y(t), \varphi_k)$. Since the system of vectors $\{\varphi_k\}$ are basis in H, then $(y(t), y(t)) = \sum_{k=1}^{\infty} |y_k(t)|^2$,

$$((tE + A)y(t), y(t)) = \sum_{k=1}^{\infty} (t + \gamma_k) |y_k(t)|^2 . \tag{1.1}$$

Theorem 1.1. *If A^{-1} is compact operator in H, then spectrum of L_0 is discrete.*

Proof. Since L_0 is positive-definite, by Rellich's (see [5, p.386) it is enough to show precompactness of the set of vectors

$$V = \{Y \in D(L_0)/(L_0Y, Y) =$$

$$= \left. \int_0^{\infty} \left[\left\| y'(t) \right\|^2 + ((tE + A)y(t), y(t)) \right] dt \leq 1 \right\} \tag{1.2}$$

in $\mathbb{L}_2$.

To proof this theorem the following two lemma are usefull.

Lemma 1.1. *For any given number $\varepsilon > 0$ we can find $N = N(\varepsilon)$ such that all $Y \in V$ satisfy*

$$\int_N^{\infty} (y(t), y(t))dt < \infty. \tag{1.3}$$

Proof. Consider partition of semiaxis (N, ∞) into subintervals Ω_k of the same length $\frac{\varepsilon}{4}$. The mean value theorem yields, there is a point t_k in each subinterval Ω_k, such that

$$(y(t_k), y(t_k)) \leq \frac{\int_{\Omega_k} ((tE + A)y(t), y(t)dt}{\int_{\Omega_k} tdt}. \tag{1.4}$$

Chose a number $N = N(\varepsilon)$ as large so that for all $\Omega_k \subset (N, \infty)$ to hold the inequality $\int_{\Omega_k} tdt > 1$.

Hence,

$$\left| \|y(t)\|^2 - \|y(t_k)\|^2 \right| \leq 2 \int_{\Omega_k} \left[(y'(t), y'(t)) + ((tE + A)y(t), y(t)) \right] dt. \tag{1.5}$$

(1.4) and (1.5) yield

$$\int_{\Omega_k} \|y(t)\|^2 dt < \frac{\varepsilon}{4} \int_{\Omega_k} ((tE + A)y(t), y(t))dt +$$

$$+\frac{\varepsilon}{2}\int\limits_{\Omega_k}\left[(y'(t),y'(t))+((tE+A)y(t),y(t))\right]dt. \qquad (1.6)$$

Summing (1.6) over all k and using $Y\in V$ we get $\int\limits_N^\infty (y(t),ty(y))dt<\varepsilon$, which proves the lemma.

Lemma 1.2. *If given any $\varepsilon>0$ there is $R=R(\varepsilon)$, such that*

$$\int\limits_0^N\sum_{k=R+1}^{\infty}|y_k(t)|^2\,dt+\sum_{k=R+1}^{\infty}|y_k(0)|^2<\varepsilon. \qquad (1.7)$$

Proof. For $Y\in V$,

$$\int\limits_0^N\sum_{k=R+1}^{\infty}|y_k(t)|^2\,dt\le\frac{1}{\gamma_R}\int\limits_0^N(Ay(t),y(t))dt\le\frac{1}{\sqrt{\gamma_R}}. \qquad (1.8)$$

On the other hand

$$\sum_{k=R+1}^{\infty}|y_k(0)|^2=\sum_{k=R+1}^{\infty}\int\limits_0^\infty(y_k^2(t))'dt\le$$

$$\le 2\left(\sum_{k=R+1}^{\infty}\int\limits_0^\infty(y_k'(t))^2dt\right)^{\frac{1}{2}}2\left(\sum_{k=R+1}^{\infty}\int\limits_0^\infty(y_k(t))^2dt\right)^{\frac{1}{2}}\le\frac{2}{\sqrt{\gamma_R}}. \qquad (1.9)$$

Since $\gamma R\to\infty$, whenever $R\to\infty$, then $\forall\varepsilon>o,\ \exists R,\ \dfrac{2}{\sqrt{\gamma_R}}<\varepsilon$.

The last relation with (1.8) and (1.9) proves the assertion of lemma.

Denote by $E_R(N)$ the set of all vector functions $\widetilde{Y}=\{\widetilde{y}_1,...,\widetilde{y}_R\}$, where $\widetilde{y}_k=\{y_k(t),y_k(0)\}$ $(t\le N)$. Define the functions $y_k(t)$ as $y_k(t)=0$ for $t>N$. The lemma 1.1. and 1.2. yield, that the set $E_R(N)$ is $\varepsilon-$net in $\mathbb{L}_2$ for V. Since $|y_k(0)|\le const,\ (k=\overline{1,R})$ and one could apply criteria of precompactness to $y_k(t)$ in $L_2(0,N)$ ([6], p. 291), then $E_R(N)$ is precompact in $\mathbb{L}_2$. That is why, V also precompact, which completes the proof of discreteness of spectrum of L_0.

From the following relations for resolvents of operators L_0 and L

$$R_\lambda(L)=R_\lambda(L_0)-R_\lambda(L_0)QR_\lambda(L),$$

where Q is bounded operator, we get discreteness of psectrum of L. Denote eigenvalues of operators L_0 and L by $\lambda_1\le\lambda_2\le...$ and $\mu_1\le\mu_2\le....$, respectively.

Now turn to study of eigenvalue asymptotics of operator L_0.

Suppose $\gamma_k\sim ak^\alpha,\ k\to\infty,\ a>0,\ \alpha>0$. By virtue of spectral expansion of A we get the following problem for coefficients $y_k(t)=(y(t),\varphi_k)$

$$-y_k''(t)+ty_k(t)+\gamma_ky_k(t)=\lambda y_k(t)\qquad t\in(0,\infty) \qquad (1.10)$$

$$y_k'(0)+\lambda y_k(0)=0. \qquad (1.11)$$

The solution of problem (1.10), (1.11) from $L_2(0,\infty)$ in case $t+\gamma_k>\lambda$ is

$$\psi(t,\lambda)=\sqrt{t+\gamma_k-\lambda}K_{\frac{1}{3}}\left\{\frac{2}{3}(t+\gamma_k-\lambda)^{\frac{3}{2}}\right\}.$$

But in case $t+\gamma_k<\lambda$ one could write it as function of real argument like

$$\psi(t,\lambda)=\frac{\pi}{\sqrt{3}}\sqrt{\lambda-t-\gamma_k}\times$$

$$\times\left\{J_{\frac{1}{3}}\left(\frac{2}{3}(\lambda-\gamma_k-t\right)^{\frac{3}{2}}\right\}+J_{-\frac{1}{3}}\left\{\left(\frac{2}{3}(\lambda-\gamma_k-t\right)^{\frac{3}{2}}\right\}. \tag{1.13}$$

To satisfy (1.11) it is necessary and sufficient to hold

$$(\lambda-\gamma_k)\left\{J_{\frac{2}{3}}\left(\frac{2}{3}(\lambda-\gamma_k\right)^{\frac{3}{2}}-J_{-\frac{2}{3}}\left(\frac{2}{3}(\lambda-\gamma_k\right)^{\frac{3}{2}}\right\}+$$

$$+\lambda\sqrt{\lambda-\gamma_k}\left\{J_{\frac{1}{3}}\left(\frac{2}{3}(\lambda-\gamma_k\right)^{\frac{3}{2}}-J_{-\frac{1}{3}}\left(\frac{2}{3}(\lambda-\gamma_k\right)^{\frac{3}{2}}\right\}=0 \tag{1.14}$$

at last for one $\gamma_k(\lambda\neq\gamma_k)$. Therefore, the spectrum of L_0 consists of those real $\lambda\neq\gamma_k$, which satisfy equation (1.14) at last for one k.

Denote $z=\sqrt{\lambda-\gamma_k}$. Then equation on becomes like

$$z\left\{J_{\frac{2}{3}}\left(\frac{2}{3}z^3\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}z^3\right)\right\}+$$

$$+(z^2+\gamma_k)\left\{J_{\frac{1}{3}}\left(\frac{2}{3}z^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}z^3\right)\right\}=0. \tag{1.15}$$

Find the asymptotics of those solutions of equation (1.14) which greater than γ_k, other words real roots of (1.15).

In virtue of ([7], p. 973) the asymptotics

$$J_v(z)=\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{v\pi}{2}-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{z}\right)\right),$$

for large $|z|$, we get the following equation equivalent to (1.15)

$$\cos\left(\frac{\frac{4}{3}z^3-\frac{\pi}{2}}{2}\right)+O\left(\frac{1}{z}\right)=0,$$

from which

$$z=\sqrt[3]{\frac{9\pi}{8}+\frac{3\pi m}{2}+O\left(\frac{1}{m}\right)}=\left(\frac{3\pi}{2}m\right)^{\frac{1}{3}}+\frac{1}{4}\frac{1}{m^{\frac{2}{3}}}+O\left(\frac{1}{m^{\frac{5}{3}}}\right). \tag{1.16}$$

Find the asymptotics of eigenvalues of L_0, which are less that γ_k. These eigenvalues corresponds to imaginary roots of (1.15). Taking $\sqrt{\lambda-\gamma_k}=iy(y>0)$ we get

$$iy\left\{J_{\frac{2}{3}}\left(-\frac{2}{3}iy^3\right)-J_{-\frac{2}{3}}\left(-\frac{2}{3}iy^3\right)\right\}+$$

$$+\left(\gamma_k - y^2\right)\left\{J_{\frac{1}{3}}\left(-\frac{2}{3}iy^3\right) + J_{-\frac{1}{3}}\left(-\frac{2}{3}iy^3\right)\right\} = 0. \qquad (1.17)$$

By using relations ([7], p. 981)

$$zJ_v'(z)vJ_v(z) = zJ_{v-1}(z), \qquad (1.18)$$

$$zJ_v'(t) - vJ_v(z) = -zJ_{v+1}(z), \qquad (1.19)$$

and

$$J_{\frac{2}{3}}\left(-\frac{2}{3}iy^3\right) - J_{-\frac{2}{3}}\left(-\frac{2}{3}iy^3\right) \sim e^{-\frac{2}{3}iy^3}\frac{\sqrt{3}}{\pi}\left(1 - \frac{1}{2y^3}\right) \qquad (1.20)$$

in (1.17), we get the following equivalent equation

$$y^2 + y - \gamma_k + O\left(\frac{1}{y^3}\right) = 0,$$

and

$$y \sim \sqrt{\gamma_k + \frac{1}{4}} - \frac{1}{2}.$$

Thus,

$$\lambda = \gamma_k + (iy)^2 \sim \sqrt{\gamma_k + \frac{1}{4}} - \frac{1}{2}. \qquad (1.21)$$

We come to the following assertion.

Lemma 1.3. *Eigenvalues of L_0 form two sequences*

$$\lambda_k \sim \sqrt{\gamma_k + \frac{1}{4}} - \frac{1}{2};\ \lambda_{m,k} = \gamma_k + z_m^2,$$

where $z_m = c_1 m^{\frac{1}{3}} + O\left(\frac{1}{m^{\frac{2}{3}}}\right)$.

Obviously, beginning with some large K (1.15) has one imaginary root, and for $k \le K$ the number of imaginary roots (if they exist) is finite.

Denote the imaginary roots of (1.15) by $x_{m,k}$, where $m = \overline{M_k, \infty}$, and the real roots $x_{m,k}$, where $m = \overline{M_k, \infty}$ (after some sufficiently large k $M_k = 1$).

It is easy to prove the following two lemmas, which we will use later.

Lemma 1.4. *Equation (1.15) has no complex roots with exception imaginary roots.*

For large $|z|$ consider the rectangular contour l with vectors at points $\pm A_N \pm iB$, where $B > x_{m,k}$, $(m = \overline{0,\ M_k - 1})$. For every fixed k take $A_N = \sqrt[3]{\frac{3\pi N}{2} + \frac{15\pi}{8}}$. According to (1.16) $A_{N-1} < x_{N,k} < A_N$ for great N.

Lemma 1.5.

Denote by $N(\lambda, L_0)$ the distribution function of L_0

$$N(\lambda, L_0) = \sum_{\lambda_k(L_0)<\lambda} = N_1(\lambda) + N_2(\lambda),$$

where $N_1(\lambda) = \sum_{\lambda_k<\lambda} 1$, $N_2(\lambda) = \sum_{\lambda_{m,k}<\lambda} 1$. *Since* $\gamma_k \sim ak^\alpha$, $a > 0$, $\alpha > 0$, *then* $\lambda_k \sim \sqrt{\gamma_k} \sim const k^{\frac{\alpha}{2}}$. *So,* $N_1(\lambda) \sim c_2\lambda^{\frac{2}{\alpha}}$.

$N_2(\lambda)$ is the number of pairs of positive whole number (m,k) for which holds $x^2_{m,k}+\gamma_k \le \lambda$.

In virtue of lemma 1.3. for large m

$$(c-\varepsilon)m^{\frac{2}{3}} \le x^2_{m,k} \le (c-\varepsilon)m^{\frac{2}{3}}.$$

Also from asymptotics γ_k we have $(a-\varepsilon)k^{\alpha} < \gamma_k < (a+\varepsilon)k^{\alpha}$ ($\varepsilon < 0$ is quite small). So, according to lemmas 1.4. and 1.5.

$$N_2^{''}(\lambda) - O(1) < N_2(\lambda) < N_2'(\lambda) - O(1),$$

where $N_2'(\lambda)$ is the number of positive whole pairs (m,k) satisfying

$$(c-\varepsilon)m^{\frac{2}{3}} + (a-\varepsilon)k^{\alpha} < \lambda,$$

and $N_2''(\lambda)$ the number of pairs (m,k) satisfying∼

$$(a-\varepsilon)k^{\alpha} + (c-\varepsilon)m^{\frac{2}{3}} < \lambda.$$

Then, by the same way as in ([8], lemma 2) we get the following lemma.

Lemma 1.6. *If $\gamma_k \sim ak^{\alpha}$,$(a>0,\alpha>0)$ then $\mu_n \sim \lambda_n \sim dn^{\beta}$ where*

$$\beta = \begin{cases} \dfrac{2\alpha}{2+3\alpha}, & \alpha \in \left(0, \dfrac{2}{3}\right) \\ \dfrac{\alpha}{2}, & \alpha > \dfrac{2}{3} \\ \dfrac{1}{3}, & \alpha = \dfrac{2}{3} \end{cases} \tag{1.22}$$

2. Trace formula.

The following lemma is valid.

Lemma 2.1. *Let the conditions of lemma 1.6. be hold. Then for $\alpha > \dfrac{2}{3}$, there is subsequence $\{n_m\}$ of natural numbers, such that*

$$\lambda_k - \lambda_{n_m} \ge \frac{d}{2}\left(k^{\frac{\alpha}{2}} - n_m^{\frac{\alpha}{2}}\right), \quad k = n_m, \quad n_m+1, ...$$

Proof. According to lemma 1.6. for $\alpha > \dfrac{2}{3}$

$$\lim_{n\to\infty} \frac{\lambda_n}{n^{\frac{\alpha}{2}}} = d,$$

which yields, that $\lim\limits_{n\to\infty}\left(\mu_n - \dfrac{d}{2}n^{\frac{\alpha}{2}}\right) = \infty$. That is why one could chose such a subsequences that $n_1 < n_2 < ... < n_m ...$, for all $k \ge n_m$

$$\mu_n - \frac{d}{2}n^{\frac{\alpha}{2}} \ge \mu_{n_m} - \frac{d}{2}n_m^{\frac{\alpha}{2}},$$

or

$$\mu_k - \mu_{n_m} \geq \frac{d}{2}\left(k^{\frac{\alpha}{2}} - n_m^{\frac{\alpha}{2}}\right).$$

The lemma is proved.

Call $\lim\limits_{n\to\infty}\sum\limits_{n=1}^{n_m}(\mu_n - \lambda_n)$ regularized trace of operator L. As it will be shown below this limit independent of choice sequence $\{n_m\}$ satisfying the hypothesis of lemma 2.1.

Let $R_0(\lambda)$ and $R(\lambda)$ be resolvents of operators L_0 and L. From (1.22) it is clear that, they are trace class operators for $\alpha > 2$.

In virtue of lemma 2.1. for $\alpha > 2$ the following assertion is valid.

Lemma 2.2. *If $\|q(t)\| < const$ on interval $[0,\infty)$, and the conditions of lemma 1.6 hold. Then for $\alpha > 2$ the following relation is true*

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\mu_n - \lambda_n - (Q\psi_n, \psi_n)) = 0.$$

The proof of this lemma is analogous to proofs of lemma 2 and theorem 2 form [9], so we don't derive it here.

The eigen-vectors of L_0 in $\mathbb{L}_2$ are

$$\psi_{m,k} = \left\{\psi(t, x_{m,k}^2)\varphi_k, \psi(0, x_{m,k}^2)\varphi_k\right\}.$$

Calculate their norms. We have

$$\|\psi_{m,k}\| = \int_0^\infty \psi^2(t, x_{m,k}^2)dt + \psi^2(0, x_{m,k}^2).$$

Let $z^2 = \alpha^2$ and $z^2 = \beta^2$ in equation $-y_k''(t) + ty_k(t) = z^2 y_k(t)$. So, appropriate solutions are $\psi(t,\alpha^2), \psi(t,\beta^2)$. Multiplying the first of considered equations by $\psi(t,\beta^2)$ and the second one by $\psi(t,\alpha^2)$ and subtracting the second one form the first one, we get

$$\int_0^\infty \psi(t,\alpha^2), \psi(t,\beta^2)dt =$$

$$= \frac{\pi^2}{3}\alpha\beta\left[\frac{\alpha\left\{J_{\frac{2}{3}}\left(\frac{2}{3}\alpha^3\right) - J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha^3\right)\right\}\left\{J_{\frac{1}{3}}\left(\frac{2}{3}\beta^3\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}\beta^3\right)\right\}}{\alpha^2 - \beta^2} - \right.$$

$$\left. - \frac{\beta\left\{J_{\frac{1}{3}}\left(\frac{2}{3}\alpha^3\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}\alpha^3\right)\right\}\left\{J_{\frac{2}{3}}\left(\frac{2}{3}\beta^3\right) - J_{-\frac{2}{3}}\left(\frac{2}{3}\beta^3\right)\right\}}{\alpha^2 - \beta^2}\right]. \quad (2.1)$$

Going to limit as $\alpha \to \beta$ we obtain

$$\int_0^\infty \psi^2(t,\beta^2)dt = \frac{\pi^2\beta^4}{3}\left(J_{\frac{1}{3}}\left(\frac{2}{3}\beta^3\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}\beta^3\right)\right)^2\left(1 + \frac{\left(\beta^2 + \gamma_k\right)^2}{\beta^2}\right). \quad (2.2)$$

So

$$\|\psi_{m,k}\|^2_{\mathbb{L}_2} = \frac{\pi^2 x^2_{m,k}}{3}\left(J_{\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)\right)^2 \times$$

$$\times\left(x^2_{m,k}+1+\left(x^2_{m,k}+\gamma_k\right)^2\right). \qquad (2.2')$$

So orthonormal eigen-vectors of operator L_0 are

$$\psi_{m,k} = \frac{\sqrt{3}(\psi(x^2_{m,k},t)\varphi_k, \psi(x^2_{m,k},0)\varphi_k)}{\pi x_{m,k}\left(J_{\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)\right)\sqrt{x^2_{m,k}+1+\left(x^2_{m,k}+\gamma_k\right)^2}}.$$

Lemma 2.3. *If operator-valued function $q(t)$ satisfies condition 1 and $\alpha > 2$, then*

$$\frac{3}{\pi^2}\sum_{k=1}^{\infty}\sum_{m=M_k}^{\infty}\int_0^{\infty}\times$$

$$\times\left|\frac{q_k(t)\psi^2(x^2_{m,k},t)}{x^2_{m,k}\left(J_{\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)\right)\left(x^2_{m,k}+1+\left(x^2_{m,k}+\gamma_k\right)^2\right)}\right|dt+$$

$$+\frac{3}{\pi^2}\sum_{k=K}^{\infty}\int_0^{\infty}\times$$

$$\times\left|\frac{q_k(t)\psi^2(x^2_{0,k},t)dt}{x^2_{0,k}\left(J_{\frac{1}{3}}\left(\frac{2}{3}x^3_{0,k}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x^3_{0,k}\right)\right)\left(x^2_{0,k}+1+\left(x^2_{0,k}+\gamma_k\right)^2\right)}\right|dt<\infty,$$

where $x_{0,k}$ are imaginary, $x_{m,k}, k=\overline{1,\infty}$, $m=\overline{M_k,\infty}$ are real roots of equation (1.15).

Proof. Consider the first series in (2.3). Let $\varepsilon > 0$ is sufficiently small number. Take $t \in (0, x^2_{m,k} - x^{\varepsilon}_{m,k})$, then $z = x^2_{m,k} - t \in (x^2_{m,k}, x^{\varepsilon}_{m,k})$. At $t \in (x^2_{m,k} - x^{\varepsilon}_{m,k} \cdot x^2_{m,k} + x^{\varepsilon}_{m,k})$, we have $z \in \left(-x^{\varepsilon}_{m,k}, 0\right] \cup (0, x^{\varepsilon}_{m,k})$ and finally, for $t \in (x^2_{m,k} + x^{\varepsilon}_{m,k} + \infty)$ we will have $z \in (-\infty, -x^{\varepsilon}_{m,k})$. Therefore, since for $z \in (x^2_{m,k}, x^{\varepsilon}_{m,k})$

$$\psi(x^2_{m,k},t)=\sqrt{x^2_{m,k}-t}\left(J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt{x^2_{m,k}-t}^3\right)\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}\sqrt{x^2_{m,k}-t}^3\right)\sim\frac{e^{-i\sqrt{z}^3}}{z},$$

and for

$$z \in (-\infty, -x^{\varepsilon}_{m,k})\psi(x^2_{m,k},t)=\sqrt{t-x^2_{m,k}}K_{\frac{1}{3}}\left(\frac{2}{3}(t-x^2_{m,k})^{\frac{3}{2}}\right)\sim\frac{e^{-\sqrt{-z}^3}}{-z},$$

then

$$\left|\int_0^{\infty}q_k(t)\psi(x^2_{m,k},t)dt\right|\sim\left|\int_{x^2_{m,k}}^{x^{\varepsilon}_{m,k}}\frac{e^{-2i\sqrt{-z}^3}}{z^2}q_k(x^2_{m,k}-z)dz+\right.$$

$$+\int\limits_{x^{\varepsilon}_{m,k}}^{-x^{\varepsilon}_{m,k}} q_k(x^2_{m,k}-z)\psi^2(z)dz+\int\limits_{-\infty}^{-x^{\varepsilon}_{m,k}} q_k(x^2_{m,k}-z)\frac{e^{-2\sqrt{-z}^3}}{z^2}dz\Bigg|<$$

$$<\int\limits_0^{\infty}|q_k(z)|\,dz+\int\limits_{x^{\varepsilon}_{m,k}}^{-x^{\varepsilon}_{m,k}}\left|q_k(x^2_{m,k}-z)\psi^2(z)\right|dz+\int\limits_0^{\infty}|q_k(z)|\,dz. \tag{2.4}$$

For $\varepsilon\to 0$, we get

$$\lim_{\varepsilon\to 0}\left|\int\limits_{x^{\varepsilon}_{m,k}}^{-x^{\varepsilon}_{m,k}}\left|q_k(x^2_{m,k}-z)\psi^2(z)\right|dz\right|=\left|\int\limits_{-1}^{1}\left|q_k(x^2_{m,k}-z)\psi^2(z)\right|dz\right|<$$

$$<c\int\limits_{-1}^{1}|q_k(z)|\,dz<c\int\limits_{-1}^{1}|q_k(z)|\,dz. \tag{2.5}$$

In virtue of asymptotics $x_{m,k}\sim cm^{\frac{1}{3}}$ with (2.4) and (2.5) and condition 1) we obtain

$$\sum_{k=1}^{\infty}\sum_{m=M_k}^{\infty}\frac{3}{\pi}\int\limits_0^{\infty}\left|\frac{q_k(t)\psi^2(x^2_{m,k},t)dt}{x^2_{m,k}\left(J_{\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)\right)\left(x^2_{m,k}+1+\left(x^2_{m,k}+\gamma_k\right)^2\right)}\right|<$$

$$<\sum_{k=1}^{\infty}\int\limits_0^{\infty}|q_k(t)|\,dt\sum_{m=M_k}^{\infty}\frac{1}{m^{\frac{4}{3}}}<\infty.$$

Convergence of the second series in (2.3) follows from the following asymptotics at large k $(x_{0,k}=iy, y>0)$

$$\frac{\psi^2(x^2_{0,k}t)}{\left(J_{\frac{1}{3}}\left(\frac{2}{3}x^3_{0,k}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x^3_{0,k}\right)\right)^2}\sim\frac{e^{-2\sqrt{-y^2+t}^3}}{e^{-2y^3(y^2+t)}},y\sim\sqrt{\gamma_k}\sim ak^{\frac{\alpha}{2}},\alpha>2,$$

and from condition 1.

The lemma is proved.

Prove the following theorem by usin lemma 2.3.

Theorem 2.1. *Let the conditions of lemma 1.6. be held. If operator valued function $q(t)$ satisfies conditions 1-3, then*

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\lambda_n-\mu_n)=0.$$

Proof. According to lemma 2.1

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\lambda_n-\mu_n)=$$

$$= \sum_{k=K}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}\frac{3}{\pi}\frac{q_k(t)\psi^2(x_{m,k}^2,t)dt}{x_{m,k}^2\left(J_{\frac{1}{3}}\left(\frac{2}{3}x_{m,k}^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x_{m,k}^3\right)\right)\left(x_{m,k}^2+1+\left(x_{m,k}^2+\gamma_k\right)^2\right)}+$$

$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{\infty}\frac{3}{\pi}\frac{q_k(t)\psi^2(x_{m,k}^2,t)dt}{x_{m,k}^2\left(J_{\frac{1}{3}}\left(\frac{2}{3}x_{m,k}^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x_{m,k}^3\right)\right)\left(x_{m,k}^2+1+\left(x_{m,k}^2+\gamma_k\right)^2\right)}. \tag{2.6}$$

Calculate the first series in (2.6).

Denote

$$T_N(t)=\sum_{m=0}^{N}\frac{3}{\pi^2}\frac{\psi^2(x_{m,k}^2,t)}{x_{m,k}^2\left(J_{\frac{1}{3}}\left(\frac{2}{3}x_{m,k}^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x_{m,k}^3\right)\right)\left(x_{m,k}^2+1+\left(x_{m,k}^2+\gamma_k\right)^2\right)}.$$

Show the m-th term of the sum $T_N(t)$ as a residue at point $x_{m,k}$ of some function of complex variable z having poles at points $x_{0,k},...,x_{N,k}$.

Consider the following function

$$G(z)=\frac{6\psi(z^2,t)^2}{f(z)}, \tag{2.7}$$

where

$$f(z)=\pi^2 z\left[z\left(J_{\frac{2}{3}}\left(\frac{2}{3}z^2\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}z^2\right)\right)+\right.$$

$$\left.+(z^2+\gamma_k)\left(J_{\frac{1}{3}}\left(\frac{2}{3}z^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}z^3\right)\right)\right]\left(J_{\frac{1}{3}}\left(\frac{2}{3}z^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}z^3\right)\right). \tag{2.8}$$

Denote the factor in brackets in (2.8) by $F(z)$. The function $G(z)$ has simple poles at points $x_{m,k}$ and at zeros of function $J_{\frac{1}{3}}\left(\frac{2}{3}z^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}z^3\right)$. Zeros of latter one denote by α_m.

Find the residue at $x_{m,k}$. By virtue of recurrent formulas (1.18) and (1.19) we have

$$J'_{\frac{2}{3}}\left(\frac{2}{3}z^3\right)=2z^2\left[-\frac{1}{z^3}J_{\frac{2}{3}}\left(\frac{2}{3}z^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}z^3\right)\right],$$

$$J'_{-\frac{2}{3}}\left(\frac{2}{3}z^3\right)=2z^2\left[-\frac{1}{z^3}J_{-\frac{2}{3}}\left(\frac{2}{3}z^3\right)-J_{\frac{1}{3}}\left(\frac{2}{3}z^3\right)\right],$$

$$J'_{\frac{1}{3}}\left(\frac{2}{3}z^3\right)=2z^2\left[J_{-\frac{2}{3}}\left(\frac{2}{3}z^3\right)-\frac{1}{2z^3}J_{\frac{1}{3}}\left(\frac{2}{3}z^3\right)\right],$$

$$J'_{-\frac{1}{3}}\left(\frac{2}{3}z^3\right)=2z^2\left[-J_{\frac{2}{3}}\left(\frac{2}{3}z^3\right)-\frac{1}{2z^3}J_{-\frac{1}{3}}\left(\frac{2}{3}z^3\right)\right].$$

So, from the latter relations by virtue of (1.15) we get

$$F'(x_{m,k})=2x_{m,k}(x_{m,k}^2+1+(x_{m,k}^2+\gamma_k))\left(J_{\frac{1}{3}}\left(\frac{2}{3}x_{m,k}^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x_{m,k}^3\right)\right).$$

According to (2,2') $F'(x_{m,k}) \neq 0$ since $\psi_{m,k}$ is nontrivial solution of problem at $q(t) \equiv 0$, i.e. $x_{m,k}$ is a simple pole.

Hence,

$$\underset{z=x_{m,k}}{res} G(z) = \frac{3\psi^2(x^2_{m,k},t)}{\pi^2 x^2_{m,k}\left[J_{\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)\right]^2 (x^2_{m,k}+(x^2_{m,k}+\gamma_k)^2+1)}.$$

Now calculate residue at α_m. It is easy to show that, zeros of function $J_{\frac{1}{3}}\left(\frac{2}{3}x^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x^3\right)$ are real and simple.

Since

$$\left(J_{\frac{1}{3}}\left(\frac{2}{3}z^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}z^3\right)\right)'\Bigg| = 2\alpha^2_m\left(J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha^3_m\right)-J_{\frac{2}{3}}\left(\frac{2}{3}\alpha^3_m\right)\right)-$$

$$-\frac{1}{2\alpha^3_m}\left(J_{-\frac{1}{3}}\left(\frac{2}{3}\alpha^3_m\right)+J_{\frac{1}{3}}\left(\frac{2}{3}\alpha^3_m\right)\right) = -2\alpha^2_m\left(J_{\frac{2}{3}}\left(\frac{2}{3}\alpha^3_m\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha^3_m\right)\right),$$

then

$$\underset{z=\alpha_m}{res} G(z) = -\frac{3\psi^2(\alpha^2_m,t)}{\pi^2\alpha^4_m\left(J_{\frac{2}{3}}\left(\frac{2}{3}\alpha^3_m\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha^3_m\right)\right)^2}.$$

Thus, we get that

$$\sum_{m=0}^{N}\frac{3\psi^2(\alpha^2_m,t)}{\pi^2\alpha^4_m\left(J_{\frac{2}{3}}\left(\frac{2}{3}\alpha^3_m\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha^3_m\right)\right)^2}-$$

$$-\sum_{m=0}^{N}\frac{3\psi^2(\alpha^2_m,t)}{\pi^2 x^2_{m,k}\left[J_{\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)-J_{-\frac{1}{3}}\left(\frac{2}{3}x^3_{m,k}\right)\right]^2 (x^2_{m,k}+(x^2_{m,k}+\gamma_k)^2+1)}=$$

$$=\frac{1}{2\pi i}\int\limits_{|z|=r,0<\varphi<\pi} G(z)dz+\frac{1}{2\pi i}\int\limits_{l} G(z)dz, \tag{2.9}$$

where l is a rectangular contour with vertices at $\pm A_N + iB, \pm A_N$ which by passes point $x_{m,k}$ and α_m along semicircles below real axis, and $x_{m,k}$ and $-\alpha_m$ above it. For sufficiently large N

$$x_{N-1,k} < A_N < x_{N,k},\ \alpha_N < A_N < \alpha_{N+1}.$$

Since $G(z)$ is an odd function, then integral along lower part of l equals to zero. As $r \to 0$ $G(z)$ is equivalent to

$$\frac{zt\left(J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt{-t}^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}\sqrt{-t}^3\right)\right)^2}{[z^4-1+(z^2+1)(z^2+\gamma_k)]\,(z^2+1)}.$$

Since

$$J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt{-t}^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}\sqrt{-t}^3\right)\sim^{-\sqrt{t}^3}, t\in(0,\infty),$$

then

$$\lim_{r\to 0}\int\limits_0^\infty q_k(t)\int\limits_{|z|=r,0<\varphi<\pi} G(z)dzdt=0.$$

At $B=CA_N, C>0$ and great N on segment with vertices at A_N and A_N+iB by virtue of asymptotics $J_{-\frac{1}{3}}(z)+J_{\frac{1}{3}}(z)\sim e^{-iz}$, $\left(\sqrt{z^2-t}\right)^3\sim z^3-\dfrac{3}{2}$ for $t>0$ and $N\to\infty$, we have

$$\int\limits_{A_N}^{A_N+iB} G(z)dz\sim i\int\limits_0^B \frac{e^{3A_N^2v-v^3-\frac{3}{2}tv}dv}{e^{3A_N^2v-v^3\sqrt{v^2+A_N^2}}}=O\left(\frac{e^{-\frac{3}{2}A_N}}{A_Nt}\right)\to 0.$$

At $t=0,\ N\to\infty$

$$\int\limits_{A_N}^{A_N+iB} G(z)dz\sim\int\limits_0^{A_N}\frac{dv}{\sqrt{A_N^2+v^2}}=In\frac{A_N+\sqrt{2}A_N}{A_N}=const.$$

So, $\int\limits_0^\infty q_{k(t)}\int\limits_{A_N}^{A_N+iB} G(z)dzdt\to 0$ when $N\to\infty$. On the upper side of rectangle (on the segment with vertices $\pm A_N+iB$) for $t>0$

$$\int\limits_{A_N+iB}^{-A_N+iB} G(z)dz\sim\int\limits_{A_N}^{-A_N}\frac{e^{3u^2B-B^3-\frac{3}{2}tB}dv}{e^{3u^2B-B^2\sqrt{B^2+A_N^2}}}du=O\frac{e^{-\frac{3t}{2}CA_N}}{\sqrt{2}A_N}2A_N\to 0$$

when $N\to\infty$. Obviously, at $t=0$ the considered integral is bounded. So,

$$\int\limits_0^\infty q_k(t)dt\sim\int\limits_{A_N+iB}^{-A_N+iB} G(z)dzdt\to 0$$

when $N\to\infty$.

On the left side of rectangle in virtue of condition 2, we have

$$\int\limits_0^\infty q_k(t)\int\limits_{-A_N+iB}^{-A_N} G(z)dzdt\sim$$

$$\sim i\int\limits_0^\infty q_k(t)\int\limits_0^{A_N}\frac{e^{-\frac{3}{2}tv}}{\sqrt{A_N^2+v^2}}dvdt=\int\limits_0^\infty\frac{2q_k(t)}{3t}dtO\left(\frac{1}{A_N}\right)=0.$$

Therefore

$$\lim_{N\to\infty}\int_0^\infty q_k(t)\sum_{m=1}^N \frac{3\psi^2(\alpha_m^2,t)dt}{\pi^2\alpha_m^4\left(J_{\frac{2}{3}}\left(\frac{2}{3}\alpha_m^3\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha_m^2\right)\right)^2}=$$

$$=\lim_{N\to\infty}\int_0^\infty q_k(t)\sum_{m=1}^N \frac{3}{\pi^2}\times$$

$$\times\frac{\psi^2(x_{m,k}^2,t)}{x_{m,k}^2(x_{m,k}^2+1+(x_{m,k}^2+\gamma_k)^2\left[J_{\frac{1}{3}}\left(\frac{2}{3}x_{m,k}^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}x_{m,k}^3\right)\right]^2}. \tag{2.10}$$

Find

$$\sum_{m=1}^\infty \frac{\psi^2(\alpha_m^2,t)}{\alpha_m^4\left(J_{\frac{2}{3}}\left(\frac{2}{3}\alpha_m^3\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha_m^3\right)\right)^2}. \tag{2.11}$$

For this deal chose a function of complex variable with poles at α_m, so that the resides at poles are equal to the terms of this series. By taking x instead of zero in (2.6) one could show, that

$$\int_x^\infty \psi^2(t,\beta^2)dt=\frac{\pi^2}{3}(\beta^2-x)^2\left[\left\{J_{\frac{1}{3}}\left(\frac{2}{3}(\beta^2-x)^{\frac{3}{2}}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}(\beta^2-x)^{\frac{3}{2}}\right)\right\}^2+\right.$$

$$\left.+\left\{J_{\frac{2}{3}}\left(\frac{2}{3}(\beta^2-x)^{\frac{3}{2}}\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}(\beta^2-x)^{\frac{3}{2}}\right)\right\}^2\right]. \tag{2.12}$$

Denote $\beta^2-x=f(x,\beta)$ and the right hand side of (2.12) by $F(f(x,\beta))$. We have $F'_x=-F'_f, F_\beta=F'_f\cdot 2\beta=-F'_x\cdot 2\beta$. $F'_x=-\psi^2(x,\beta^2)$.

$$F'_\beta=2\beta\psi^2(x,\beta^2). \tag{2.13}$$

Using these relations we get that the function

$$g(z)=\frac{F(f(x,z)}{zv^2(z)},\quad \left(v(z)=J_{\frac{1}{3}}\left(\frac{2}{3}z^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}z^3\right)\right)$$

have poles of second order at α_m which equal to terms of series (2.11).

So,

$$\underset{z=\alpha_m}{res}g(z)=\lim_{z\to\alpha_m}\left((z-\alpha_m)^2\frac{F(f(x,z))}{zv^2(z)}\right)'=$$

$$=\lim_{z\to\alpha_m}\left((z-\alpha_m)^2\frac{F(f(x,z))}{\alpha_m v'(\alpha_m)^2(z-\alpha_m)^2+c_m(z-\alpha_m)^4}\right)=$$

$$=\lim_{z\to\alpha_m}\left((z-\alpha_m)^2\frac{F(f(x,z))}{\alpha_m v'(\alpha_m)^2+c_m(z-\alpha_m)^2}\right)=\frac{F'_{z=\alpha_m}}{\alpha_m v'(\alpha_m)^2}=$$

$$= \frac{2\alpha_m \psi^2(x,\alpha_m^2)}{4\alpha_m^5\left(J_{\frac{2}{3}}\left(\frac{2}{3}\alpha_m^3\right) - J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha_m^3\right)\right)^2} =$$

$$= \frac{\psi^2(x,\alpha_m^2)}{2\alpha_m^4\left(J_{\frac{2}{3}}\left(\frac{2}{3}\alpha_m^3\right) - J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha_m^3\right)\right)^2}.$$

Take as contour of integration the considered above contour l, which by passes point α_m below it and points α_m, 0 above them.

First consider the part of contour with vertices at A_N and $A_N + iB$:

$$\int_0^\infty q_k(t) \int_{A_n}^{A_N+iB} g(z)dzdt \sim \int_0^\infty q_k(t) \int_0^{A_N} e^{-\frac{3}{2}iv} A_N^3 dvdt =$$

$$= \int_0^\infty q_k(t) \left[A_N^3 \frac{e^{-\frac{3}{2}tA_N}}{-\frac{3}{2}t} + \frac{2A_N^3}{3t} \right] dt. \tag{2.14}$$

From condition 2

$$\int_0^\infty \frac{q_k(t)}{t} A_N^3 dt = 0.$$

According to conditions 3

$$\int_0^\infty \left| \frac{q_k(t)}{t} \right| A_N^3 e^{-\frac{3}{2}tA_N} dt =$$

$$= \int_0^\infty \left| \frac{q_k(t)}{t} \right| \frac{A_N^3}{1 + \frac{3}{2}tA_N \frac{\left(\frac{3}{2}tA_N\right)^2}{2!} + \frac{\left(\frac{3}{2}tA_N\right)^3}{2!} + \frac{\left(\frac{3}{2}tA_N\right)^4}{2!} + \ldots} dt <$$

$$< \int_0^\infty \left| \frac{q_k(t)}{t} \right| \frac{1}{\frac{\left(\frac{3}{2}tA_N\right)^4}{4!}} dt = \frac{const}{A_N} \int_0^\infty \left| \frac{q_k(t)^5}{t} \right| dt \to 0. \tag{2.15}$$

On the side with vertices $\pm A_N + iB$ of contour we have

$$\int_0^\infty q_k(t) \int_{-A_N+iB}^{A_N+iB} g(z)dzdt \sim \int_0^\infty q_k(t) \int_{-A_N}^{A_N} e^{-\frac{3}{2}tA_N} A_N^3 dudt =$$

$$= \int_0^\infty 2q_k(t) A_N^4 e^{-\frac{3}{2}tA_N} dt, \tag{2.16}$$

$$\left| \int_0^\infty 2q_k(t) A_N^4 e^{-\frac{3}{2}tA_N} dt \right| < \frac{const}{A_N} \int_0^\infty \left| \frac{q_k(t)}{t^5} \right| dt \to 0. \tag{2.17}$$

By condition 3 $\lim\limits_{t\to 0} q_k(t) = 0$. So, in the similar way as in (2.15), (2.16) we obtain convergence to zero of the integral along the left side of contour when $N \to \infty$. Consequently,

$$\lim_{N\to\infty} \int g(z)dz = 0, \tag{2.18}$$

$$\sum_{m=1}^{\infty} \int_0^{\infty} \frac{\psi^2(\alpha_m^2, t) q_k(t) dt}{\alpha_m^4 \left(J_{\frac{2}{3}} \left(\frac{2}{3} \alpha_m^3 \right) - J_{-\frac{2}{3}} \left(\frac{2}{3} \alpha_m^3 \right) \right)} = \lim_{N\to\infty} \int g(z)dz = 0. \tag{2.19}$$

From (2.10), (2.19) we obtain that the sum of the first series in (2.6) equals zero. It is possible to show analogously that the sum of the second series is also zero.

The theorem is proved.

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