Bala A. ISKENDEROV, Gulnar Kh. AKHUNDOVA, Seymur E. SULEYMANOV

ASYMPTOTIC EXPANSION OF THE CAUCHY PROBLEM SOLUTION AT LARGE VALUES OF TIME FOR BARENBLATT-JELTOV-KOCHINA EQUATION

Abstract

Asymptotic expansion of the solution of the Cauchy problem for the Barenblatt-Jeltov-Kochina equation was obtained at large time.

While studying fluid filtration in fissured porous rocks, G.I. Barenblatt, Yu. P. Jeltov and I.N. Kochina obtained in [1] an equation unsolved with respect to a time derivative of the form

$$(I-\eta\Delta_3)\,D_t u\,(t,x)=\chi\Delta_3 u\,(t,x)+f\,(t,x)\,,t>0, \qquad (0.1)$$

where Δ_3 is a Laplace operator in R_3 - threedimensional Euclidean space, I is a unit operator, η is a coefficient of permeability, λ is a piezoconductivity factor. In [1] different boundary value problems for equation (01) were stated and expressions for pressure difference at both sides of the discontinuity surface were obtained. Therewith, it would be interesting to obtain a pressure expression, i.e. an expression for the solution of boundary value problems in the explicit form and to study their quality properties for the equations more general than equation (0.1). The solutions of different boundary value problems and their asymptotic properties at large values of time for equations of type (0.1) in many-dimensional domains were studied in the papers [2]-[3]. For general systems of equations, unsolved with respect to time derivative, the Cauchy problem was studied in the papers [4], [5].

In the present paper, we obtain explicit form of the solution and its asymptotic expansion as $t\to+\infty$ of the following Cauchy problem:

$$(I-\eta\Delta_{n+m})\,D_t u\,(t,x,y)=\chi\Delta_n u\,(t,x,y)+f\,(t,x,y)\,, \qquad (1)$$

$$u\,(0,x,y)=\varphi\,(x,y)\,, \qquad (2)$$

where $x\in R_n$, $y\in R_m$, Δ_{n+m} is a Laplace operator with respect to (x,y), Δ_n with respect to x, the conditions on the functions $\varphi\,(x,y)\,,\ \ f\,(t,x,y)$ are given below.

Let $W_1^{(\nu)}\,(G)$ be S.L. Sobolev space, where $G\subseteq R_{n+m}$ is some domain. Define the space of functions $W_{1,p}^{(\nu)}\,(G):\ \varphi\,(z)\in W_{1,p}^{(\nu)}\,(G)$ if

$$\left\|\varphi,W_{1,p}^{(\nu)}\,(G)\right\|=\sum_{|\alpha|\le\nu}\left\|(1+|z|^p)\,D_z^{(\alpha)}\varphi\,(z)\,,L_1\,(G)\right\|<+\infty, z=(x,y)\,.$$

We'll study the classic solution of problem (1)-(2). For that it suffices the functions $\varphi\,(x,y),\ f\,(t,x,y)$ belong to some S.L. Sobolev space that will be determined below. The solution of problem (1)-(2) is determined as the Fourier inverse transformation from the solution $\widetilde{u}(t,\sigma)$, $\sigma \in R_{n+m}$ of Cauchy's duality problem corresponding to problem (1)-(2) with respect to Fourier transformation, σ is a variable dual to (x,y)

$$\left[1+\eta\left(\sigma_1^2+...+\sigma_n^2+\sigma_{n+1}^2+...+\sigma_{n+m}^2\right)\right]\frac{dV(t,\sigma)}{dt}=$$

$$=-\chi\left(\sigma_1^2+\sigma_2^2+...+\sigma_n^2\right)V(t,\sigma)+\widetilde{f}(t,\sigma), \tag{3}$$

$$V(0,\sigma)=\widetilde{\varphi}(\sigma), \tag{4}$$

where $V(t,\sigma)=\widetilde{u}(t,x,y)$, the sign $\sim$ over the function means Fourier transformation with respect to (x,y). For the solution of Cauchy problem (3)-(4) we have

$$V(t,\sigma)=Q(t,\sigma)\widetilde{\varphi}(\sigma)+\int_0^t\frac{Q(t-\tau,\sigma)}{1+\left(\sigma_1^2+...+\delta_{n+m}^2\right)}\widetilde{f}(t,\sigma)\,d\sigma,$$

where

$$Q(t,\sigma)=e^{-\frac{\chi\left(\sigma_1^2+\sigma_2^2+...+\sigma_n^2\right)t}{1+\eta\left(\sigma_1^2+\sigma_2^2+...+\sigma_{n+m}^2\right)}}. \tag{5}$$

Introduce the spherical coordinates

$$\begin{array}{l}\sigma_1=\rho\cos\varphi_1,\\ \sigma_2=\rho\sin\varphi_1\cos\varphi_2,\\ \sigma_3=\rho\sin\varphi_1\sin\varphi_2\cos\varphi_3,\\ \cdots\cdots\cdots\cdots\end{array} \tag{6}$$

$$\sigma_{n+m-1}=\rho\sin\varphi_1\sin\varphi_2...\sin\varphi_{n+m-2}\cos\varphi_{n+m-1},$$

$$\sigma_{n+m}=\rho\sin\varphi_1\sin\varphi_2...\sin\varphi_{n+m-2}\sin\varphi_{n+m-1},$$

here $\rho=|\sigma|=\left(\sigma_1^2+\sigma_2^2+...+\sigma_{n+m}^2\right)^{\frac{1}{2}}$, where $0\le\varphi_j\le\pi$, $j=1,2,...,n+m-2$, $0\le\varphi_{n+m-1}\le 2\pi$. We'll denote $\overline{\varphi}=(\varphi_1,\varphi_2,...,\varphi_n)$, $\overline{\overline{\varphi}}=\left(\varphi_{n+1},\varphi_{n+2},...,\varphi_{n+m-1}\right)$, $\overline{\sigma}=(\sigma_1,\sigma_2,...,\sigma_n)$, $\overline{\overline{\sigma}}=(\sigma_{n+1},\sigma_{n+2},...,\sigma_{n+m})$. From (6) it follows

$$|\overline{\sigma}|^2=\sigma_1^2+\sigma_2^2+...+\sigma_n^2=\rho^2\left(1-\sin^2\varphi_1...\sin^2\varphi_n\right)\equiv\rho^2T(\overline{\varphi})$$

Taking into account these denotation and performing Fourier inverse transformation over $V(t,\sigma)$, for solving the Cauchy problem (1), (2) we get

$$u(t,x,y)=\frac{1}{(2\pi)^{n+m}}\left\{\int_{R_{n+m}}\int Q(t,\sigma)\widetilde{\varphi}(\sigma)e^{i(z,\sigma)}d\sigma+\right.$$

$$\left.+\int_0^t\left[\int_{R_{n+m}}\int\frac{Q(t-\tau,\sigma)}{1+\eta|\sigma|^2},\widetilde{f}(\tau,\sigma)e^{i(z,\sigma)}d\sigma\right]d\tau\right\}\equiv$$

$$\equiv u_\varphi(t,x,y)+u_f(t,x,y). \tag{7}$$

For a sufficiently smooth function $\varphi(z)$, decreasing at infinity rather rapidly, we have

$$\widetilde{\varphi}(\sigma) = (-1)^{\mu}\left(1+|\sigma|^2\right)^{-\mu}\widetilde{(1-\Delta_{n+m})^{\mu}\varphi(z)},$$

where μ is a natural number that will be chosen below. Taking into account the Fourier transformation formula of convolution, from (7) we get

$$u_{\varphi}(t,x,y) = \frac{(-1)^{\mu}}{(2\pi)^{n+m}}\int\limits_{R_{n+m}} e^{-\frac{\chi|\overline{\sigma}|^2 t}{1+\eta|\sigma|^2}}\widetilde{\varphi}(\sigma)\,e^{-(z,\sigma)}d\sigma =$$

$$= \frac{(-1)^{\mu}}{(2\pi)^{n+m}}\int\limits_{R_{n+m}}\left(1+|\overline{\sigma}|^2\right)^{-\mu}\widetilde{(1-\Delta_{n+m})^{\mu}\varphi(z)}e^{-\frac{\chi|\overline{\sigma}|^2 t}{1+\eta|\sigma|^2}}e^{i(z,\sigma)}d\sigma =$$

$$= (-1)^{\mu}(2\pi)^{n+m}\int\limits_{R_{n+m}} G(z-\xi,t)\,(1-\Delta_{n+m})^{\mu}\varphi(\xi)\,d\xi, \tag{8}$$

where

$$G(t,z) = \frac{1}{(2\pi)^{n+m}}\int\limits_{R_{n+m}}\left(1+\sigma^2\right)^{-\mu}e^{-\frac{\chi|\overline{\sigma}|^2 t}{1+\eta|\sigma|^2}t}e^{i(z,\sigma)}d\sigma,$$

we choose μ so that the integral in (8) converges absolutely. For that we set $\mu \geq \left[\frac{n+m}{2}\right]+1$. Formula (8) gives representation of the classic solution of Cauchy problem (1)-(2). Now study the asymptotic expansion $G(t,z)$ as $t\to+\infty$. To this end in the expression $G(t,z)$ pass to spherical coordinates (6). Then we get

$$G(t,z) = \frac{1}{(2\pi)^{n+m}}\int\limits_0^{\pi}...\int\limits_0^{\pi}\int\limits_0^{2\pi}\sin^{m-2}\varphi_{n+1}...\sin\varphi_{n+m-2}\left\{\int\limits_0^{\infty}\frac{\rho^{n+m-1}}{(1+\rho^2)^{\mu}}\times\right.$$

$$\left.\times\left[\int\limits_0^{\pi}...\int\limits_0^{\pi}\sin^{n+m-2}\varphi_1...\sin^{m-1}\varphi_n\,Q(t,\rho,\overline{\varphi})\,e^{i(z,\rho T_1(\varphi))}\right.\right\}d\overline{\overline{\varphi}}, \tag{9}$$

where the vector-function $T_1(\varphi) = \left(\cos\varphi_1, \sin\varphi_1\cos\varphi_2, ..., \sin\varphi_1\sin\varphi_2...\sin\varphi_{n+m-2}\right)$. Denote

$$G^*\left(t,z,\overline{\overline{\varphi}}\right) = \frac{1}{(2\pi)^{n+m}}\int\limits_0^{\infty}\int\limits_0^{\pi}...\int\limits_0^{\pi}\sin^{m-2}\varphi_1...\sin^{m-1}\varphi_n\frac{\rho^{n+m-1}}{(1+\rho^2)^{\mu}}\times$$

$$\times e^{-\frac{\chi\rho^2 T(\overline{\varphi})}{1+\eta\rho^2}t}e^{i(z,\rho T_1(\varphi))}d\overline{\varphi}d\rho. \tag{10}$$

The function $Q(t,\rho,\overline{\varphi})$ accepts the greatest value at the point $(\rho,\overline{\varphi}_0) = \left(0,\frac{\pi}{2},...,\frac{\pi}{2}\right)$. This points is a non-degenerate boundary point of maximum for the function $T_2(\psi^*) = -\frac{\chi\rho^2 T(\overline{\varphi})}{1+\eta\rho^2}$, $\psi^* = (\rho,\overline{\varphi})$ in the domain $\Omega = [0,\delta]\times D_n$, $D_n = \left\{\varphi_j : 0\leq\varphi_j\leq\pi, j=1,2,...,n\right\}$, since

$$\det\left\|\frac{\partial^2 T_2(\psi^*)}{\partial\varphi_k\partial\varphi_j}\right\|_{\psi^*=\psi_0^*} = 2^{n+1},\quad k,j=0,1,...,n;\quad \varphi_0=\rho,$$

$\psi_0^* = (0, \overline{\varphi}_0)$, $\overline{\varphi}_0 = \left(\frac{\pi}{2}, \frac{\pi}{2}, ..., \frac{\pi}{2}\right)$, $\delta > 0$ is a sufficiently small number. Represent the integral in (10) in the form

$$G^*\left(t, z, \overline{\overline{\varphi}}\right) = \frac{1}{(2\pi)^{n+m}} \left[\int_0^\delta + \int_\delta^\infty\right] \int_0^\pi ... \int_0^\pi \sin^{n+m-2}\varphi_1 ... \sin^{m-1}\varphi_n \frac{\rho^{n+m-1}}{(1+\rho^2)^\mu} \times$$

$$\times e^{-\frac{\chi\rho^2 T(\overline{\varphi})}{1+\eta\rho^2}t} e^{i(z,\rho T_1(\varphi))} d\overline{\varphi} d\rho \equiv G_1^*\left(t, z, \overline{\overline{\varphi}}\right) + G_2^*\left(t, z, \overline{\overline{\varphi}}\right). \tag{11}$$

At first study $G_1^*\left(t, z, \overline{\overline{\varphi}}\right)$ as $t \to +\infty$. To this end, we write expansion of a unit for domain D_n

$$1 \equiv \psi_1(\overline{\varphi}) + \psi_2(\overline{\varphi}) \tag{12}$$

where $\psi_1(\overline{\varphi})$ and $\psi_2(\overline{\varphi})$ are infinitely differentiable functions, $\psi_1(\overline{\varphi}) \equiv 1$ for $|\overline{\varphi} - \overline{\varphi}_0| \leq \varepsilon$, $\psi_1(\overline{\varphi}) \equiv 0$ for $|\overline{\varphi} - \overline{\varphi}_0| \geq 2\varepsilon$ and $0 \leq \psi_1(\overline{\varphi}) \leq 1$ for $\varepsilon \leq |\overline{\varphi} - \overline{\varphi}_0| \leq 2\varepsilon$; $\psi_2(\overline{\varphi}) \equiv 0$ for $|\overline{\varphi} - \overline{\varphi}_0| \leq \varepsilon$, $\psi_2(\overline{\varphi}) \equiv 1$ for $|\overline{\varphi} - \overline{\varphi}_0| \geq 2\varepsilon$, $0 \leq \psi_2(\overline{\varphi}) \leq 1$ for $\varepsilon \leq |\overline{\varphi} - \overline{\varphi}_0| \leq 2\varepsilon$, where $\varepsilon > 0$ is a sufficiently small number. Using expansion (12), represent $G_1^*\left(t, z, \overline{\overline{\varphi}}\right)$ in the form

$$G_1^*\left(t, z, \overline{\overline{\varphi}}\right) = \frac{1}{(2\pi)^{n+m}} \int_0^\delta d\rho \left[\int \cdots \int_{D_n} \psi_1(\overline{\varphi}) + \int \cdots \int_{D_n} \psi_2(\overline{\varphi})\right] \sin^{n+m-2}\varphi_1 ... \sin^{m-1}\varphi_n \times$$

$$\times \frac{\rho^{n+m-1}}{(1+\rho^2)^\mu} e^{-\frac{\chi\rho^2 T(\overline{\varphi})}{1+\eta\rho^2}t} e^{i(z,\rho T_1(\varphi))} d\overline{\varphi} \equiv G_{1,1}^*\left(t, z, \overline{\overline{\varphi}}\right) + G_{1,2}^*\left(t, z, \overline{\overline{\varphi}}\right). \tag{13}$$

Consider each summand in (13) as $t \to +\infty$ separately. Since the point $\psi^* = \psi_0^*$ is a non-degenerate maximum point for the function $T(\psi^*)$, then the statement of the item 4.1 of the book [6] (p. 132-133) is applicable to $G_{1,1}^*\left(t, z, \overline{\overline{\varphi}}\right)$ as $t \to +\infty$. Then we get

$$G_{1,1}^*\left(t, z, \overline{\overline{\varphi}}\right) = 2^{-\frac{n+3}{2}} (2\pi)^{-\frac{n+m}{2}} t^{-\frac{n+m}{2}} \left[a_{n+m} + t^{-\frac{1}{2}} O\left(|z|^{n+m}\right)\right.,$$

where

$$a_{n+m} = \frac{1}{(n+m)!2^{n+m}} L_{T_2}^{n+m}\left(\frac{\partial}{\partial\psi^*}\right) \left[\psi_1(\overline{\varphi}) \sin^{n+m-2}\varphi_1 ... \sin^{m-1}\varphi_n \frac{\rho^{n+m-1}}{(1+\rho^2)^\mu} \times\right.$$

$$\left.\times e^{tT_2(\psi^*, \psi_0^*)} e^{i(z,\rho T_1(\varphi))}\right]_{\psi^* = \psi_0^*},$$

the differential operator L_{T_2} is determined in the following way:

$$L_{T_2}\left(\frac{\partial}{\partial\psi^*}\right) = \left\langle T_2''(\psi_0^*)^{-1} \nabla_{\psi^*}, \nabla_{\psi^*}\right\rangle, \nabla_{\psi^*} = \left(\frac{\partial}{\partial\varphi_0}, \frac{\partial}{\partial\varphi_1}, ..., \frac{\partial}{\partial\varphi_n}\right),$$

$$T_2(\psi^*, \psi_0^*) = T_2(\psi^*) - \frac{1}{2}\left\langle T_2''(\psi_0^*)(\psi^* - \psi_0^*), (\psi^* - \psi_0^*)\right\rangle, \tag{14}$$

since $T_2(\psi_0^*) = 0$. Since

$$T_2''(\psi^*) = \left(\frac{\partial T_2(\psi^*)}{\partial\varphi_k \partial\varphi_j}\right), \quad k, j = 0, 1, ..., n,$$

$$T_2^{''}\left(\psi_0^*\right)=\begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ & \dots\dots\dots & & \\ 0 & 0 & \dots & 2 \end{pmatrix}, \quad \det T^{''}\left(\psi_0^*\right)=2^{n+1},$$

hence

$$\left(T_2^{''}\left(\psi_0^*\right)\right)^{-1}=\begin{pmatrix} \frac{1}{2} & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \dots & 0 \\ & \dots\dots\dots & & \\ 0 & 0 & \dots & \frac{1}{2} \end{pmatrix}.$$

Then

$$\frac{1}{2}\left\langle T_2^{''}\left(\psi_0^*\right)\left(\psi^*-\psi_0^*\right),\left(\psi^*-\psi_0^*\right)\right\rangle=\sum_{j=0}^{n}\left(\psi_j^*-\psi_{0j}^*\right)^2,$$

$$L_{T_2}\left(\frac{\partial}{\partial\psi^*}\right)=\frac{1}{2}\sum_{j=0}^{n}\frac{\partial^2}{\partial\varphi_j^2}. \tag{15}$$

From (14) and (15), as $t\to+\infty$ we get

$$G_{1,1}^*\left(t,z,\overline{\overline{\varphi}}\right)=C_0\left(n,m\right)t^{-\frac{n+m}{2}}\left[1+t^{-\frac{1}{2}}O\left(|z|^{n+m}\right)\right. \tag{16}$$

uniformly with respect to $\overline{\overline{\varphi}}$.

Now consider $G_{1,2}^*\left(t,z,\overline{\overline{\varphi}}\right)$ as $t\to+\infty$. Since in $\sup\psi_2\left(\overline{\varphi}\right)$ $|\overline{\varphi}-\overline{\varphi}_0|\geq\varepsilon$, then in the integrand expression $G_{1,2}^*\left(t,z,\overline{\overline{\varphi}}\right)$ the function $T_2\left(\psi^*\right)$ with respect to ρ has a non-degenerate maximum point $\rho=0$. For applying as $t\to+\infty$ the Laplace method [6] (ch. II. p. 54-151) to $G_{1,2}^*\left(t,z,\overline{\overline{\varphi}}\right)$, change the integration order in it, that is valid by absolute integrability of the integrand function

$$G_{1,2}^*\left(t,z,\overline{\overline{\varphi}}\right)=\frac{1}{(2\pi)^{n+m}}\int\cdots\int_{D_n}\psi_2\left(\overline{\varphi}\right)\sin^{n+m-2}\varphi_1\dots\sin^{m-1}\varphi_n\times$$

$$\times\left[\int_0^{\delta}\frac{\rho^{n+m-1}}{\left(1+\rho^2\right)^{\mu}}e^{-\frac{\chi\rho^2T(\overline{\varphi})}{1+\eta\rho^2}t}e^{i(z,\rho T_1(\varphi))}d\rho\right]d\overline{\varphi}. \tag{17}$$

Denote by $\Gamma_0\left(t,z,\varphi\right)$ the inner integral in (17)

$$\Gamma_0\left(t,z,\varphi\right)=\int_0^{\delta}\frac{\rho^{n+m-1}}{\left(1+\rho^2\right)^{\mu}}e^{-\frac{\chi\rho^2T(\overline{\varphi})}{1+\eta\rho^2}t}e^{i(z,\rho T_1(\varphi))}d\rho. \tag{18}$$

As $t\to+\infty$, to (18) apply theorem 1.6 (item 2^0) from [6] (p. 75). Then we get

$$\Gamma_0\left(t,z,\varphi\right)=t^{-\frac{1}{2}}\sum_{k=0}^{N}a_kt^{-\frac{k}{2}}+t^{-\frac{N}{2}-1}O\left(|z|^{n+m}\right) \tag{19}$$

uniformly with respect to φ, where

$$a_k\left(z,\varphi\right)=\frac{(-1)^{k+1}2^k}{k!}\Gamma\left(\frac{k+1}{2}\right)\left(h\left(\rho,0\right)\frac{d}{d\rho}\right)^k\left[\frac{\rho^{n+m-1}}{\left(1+\rho^2\right)^{\mu}}e^{i(z,\rho T_1(\varphi))}\right]_{\rho=0},$$

where

$$h(\rho,0)=\frac{\left(1+\eta\rho^{2}\right)^{\frac{3}{2}}}{2\left(-\chi T\left(\overline{\varphi}\right)\right)^{\frac{1}{2}}}.$$

For $k=0,1,...,n+m-2$ all $a_k=0$. For $k=n+m-1$ we have

$$a_{n+m-1}\left(z,\varphi\right)=\frac{\left(-1\right)^{\frac{n+m+1}{2}}}{\left[\chi T\left(\overline{\varphi}\right)\right]^{\frac{n+m-1}{2}}}\Gamma\left(\frac{n+m}{2}\right). \tag{20}$$

Substituting the expression $a_{n+m-1}\left(z,\varphi\right)$ from (20) into (19), for $\Gamma_0\left(t,z,\varphi\right)$ as $t\to+\infty$ we get

$$\Gamma_0\left(t,z,\varphi\right)=\frac{\left(-1\right)^{\frac{n+m+1}{2}}}{\left(\chi T\left(\overline{\varphi}\right)\right)^{\frac{n+m-1}{2}}}\Gamma\left(\frac{n+m}{2}\right)t^{-\frac{n+m}{2}}\left(1+t^{-\frac{1}{2}}O\left(|z|^{n+m}\right)\right), \tag{21}$$

uniformly with respect to $\overline{\varphi}$. Substituting the expression $\Gamma_0\left(t,z,\varphi\right)$ from (21) into (17) as $t\to+\infty$, we get

$$G^{*}_{1,2}\left(t,z,\overline{\overline{\varphi}}\right)=-\frac{\left(-\chi^{-1}\right)^{\frac{n+m-1}{2}}}{\left(2\pi\right)^{n+m}}\Gamma\left(\frac{n+m}{2}\right)t^{-\frac{n+m}{2}}\int\cdots\int_{D_n}\psi_2\left(\overline{\varphi}\right)\sin^{n+m-2}\varphi_1...\sin^{m-1}\varphi_n\times$$

$$\times\frac{d\overline{\varphi}}{\left(T\left(\overline{\varphi}\right)\right)^{\frac{n+m-1}{2}}}+t^{-\frac{n+m}{2}}O\left(|z|^{n+m}\right). \tag{22}$$

Since $\varphi_2\left(\overline{\varphi}\right)$ lies outside of $|\overline{\varphi}-\overline{\varphi}_0|\le\varepsilon$, then the integrand function in (22) has no singularities, and therefore the integral in (22) converges. From (13), (16), (22) as $t\to+\infty$ for $G^{*}_{1}\left(t,z,\overline{\overline{\varphi}}\right)$ we have

$$G^{*}_{1}\left(t,z,\overline{\overline{\varphi}}\right)=C_1\left(n,m\right)t^{-\frac{n+m}{2}}\left[1+t^{-\frac{1}{2}}O\left(|z|^{n+m}\right)\right], \tag{23}$$

where $C_1\left(n,m\right)$ is a constant independent of n and m.

Now consider the asymptotics $G^{*}_{2}\left(t,z,\overline{\overline{\varphi}}\right)$ from (11) as $t\to+\infty$. Using the expansion of unit (12), represent $G^{*}_{2}\left(t,z,\overline{\overline{\varphi}}\right)$ in the form

$$G^{*}_{2}\left(t,z,\overline{\overline{\varphi}}\right)=\frac{1}{\left(2\pi\right)^{n+m}}\int_{\delta}^{\infty}\int_{0}^{\pi}...\int_{0}^{\pi}\left[\psi_1\left(\overline{\varphi}\right)+\psi_2\left(\overline{\varphi}\right)\right]\sin^{n+m-2}\varphi_1...\sin^{m-1}\varphi_n\times$$

$$\times\frac{\rho^{n+m-1}}{\left(1+\rho^{2}\right)^{\mu}}e^{-\frac{\chi\rho^{2}T\left(\overline{\varphi}\right)}{1+\eta\rho^{2}}t}e^{i\left(z,\rho T_1\left(\varphi\right)\right)}d\rho d\overline{\varphi}\equiv G^{*}_{2,1}\left(t,z,\overline{\overline{\varphi}}\right)+G^{*}_{2,2}\left(t,z,\overline{\overline{\varphi}}\right). \tag{24}$$

Consider

$$\Gamma_1\left(t,z,\rho,\overline{\overline{\varphi}}\right)=\int_{0}^{\pi}...\int_{0}^{\pi}\psi_1\left(\overline{\varphi}\right)\sin^{n+m-2}\varphi_1...\sin^{m-1}\varphi_n e^{i\left(z,\rho T_1\left(\varphi\right)\right)}e^{-\frac{\chi\rho^{2}T\left(\overline{\varphi}\right)}{1+\eta\rho^{2}}t}d\overline{\varphi}. \tag{25}$$

The integrand function in (25) for $\rho>0$ with respect to $\overline{\varphi}$ has an inner non-degenerate maximum point $\overline{\varphi}=\overline{\varphi}_0$ independent of the parameter ρ. Therefore, for finding asymptotics (25) as $t \to +\infty$ we apply theorem 4.1 from [6] (p. 122-125). Then we get

$$\Gamma_1\left(t,z,\rho,\overline{\overline{\varphi}}\right)=t^{-\frac{n}{2}}\left[\sum_{k=0}^{N}b_k t^{-k}+t^{-(N+1)}O\left(\left(|z|\,\rho\right)^{N+1}\right)\right]$$

uniformly with respect to $\overline{\overline{\varphi}}$, here the expansion coefficients are determined as in expression (14) by substituting ψ_0^* for $\overline{\varphi}_0$, and ∇_{ψ_*} for $\nabla_{\overline{\varphi}}$. Then for $G_{2,1}^*\left(t,z,\overline{\overline{\varphi}}\right)$ as $t\to+\infty$,

$$G_{2,1}^*\left(t,z,\overline{\overline{\varphi}}\right)=t^{-\frac{n}{2}}C_0\left(z,\overline{\overline{\varphi}}\right)+t^{-\frac{n}{2}-1}C_1\left(x\right),$$

where

$$C_0\left(z,\overline{\overline{\varphi}}\right)=\frac{1}{(2\pi)^{n+m}}\int\limits_{\delta}^{\infty}\frac{\rho^{n+m-1}}{(1+\rho^2)^{\mu}}e^{i\left(z,\rho,T_1\left(0,\overline{\overline{\varphi}}\right)\right)}d\rho, \tag{26}$$

$$\left|C_1\left(z\right)\right|\le C\left|z\right|,$$

uniformly with respect to $\overline{\overline{\varphi}}$, where

$$\mu=\left[\frac{n+m}{2}\right]+2, \tag{27}$$

the square brackets means an entire part.

Now estimate $G_{2,2}^*\left(t,z,\overline{\overline{\varphi}}\right)$. Since the integration domain in the expression $G_{2,2}^*\left(t,z,\overline{\overline{\varphi}}\right)$ doesn't contain critical points with respect to ρ and with respect to $\overline{\varphi}$, then for $\psi^*\in\left([\delta,+\infty)\times\left(D_n\backslash\left|\overline{\varphi}-\overline{\varphi}_0\right|<\varepsilon\right)\right)$

$$\left|T_2\left(\psi^*\right)\right|\ge\delta_0>0,$$

where δ_0 is independent of $\overline{\overline{\varphi}}$. Estimating $G_{2,2}^*\left(t,z,\overline{\overline{\varphi}}\right)$ by modulus, we get

$$\left|G_{2,2}^*\left(t,z,\overline{\overline{\varphi}}\right)\right|\le Ce^{-\delta_0 t} \tag{28}$$

uniformly with respect to z and $\overline{\overline{\varphi}}$. From (24), (26) and (28) as $t\to+\infty$ it follows

$$G_2^*\left(t,z,\overline{\overline{\varphi}}\right)=t^{-\frac{n}{2}}C_0\left(z,\overline{\overline{\varphi}}\right)+t^{-\frac{n}{2}-1}C_1\left(z\right), \tag{29}$$

where $C_0\left(z,\overline{\overline{\varphi}}\right)$ and $C_1\left(z\right)$ were determined in (26). From asymptotics (11), (23) and (29) it follows that as $t\to+\infty$

$$G^*\left(t,z,\overline{\overline{\varphi}}\right)=t^{-\frac{n}{2}}C_0\left(z,\overline{\overline{\varphi}}\right)+t^{-\frac{n}{2}-1}C_1\left(z\right). \tag{30}$$

From (9) and (30) as $t\to+\infty$ we get

$$G\left(z,t\right)=\frac{t^{-\frac{n}{2}}}{(2\pi)^{n+m}}\int\limits_0^{\pi}...\int\limits_0^{\pi}\int\limits_0^{2\pi}\sin\overset{m-2}{\varphi_{n+1}}...\sin\underset{n+m-2}{\varphi}\left[C_0\left(z,\overline{\overline{\varphi}}\right)+t^{-1}C_1\left(z\right)\right]d\overline{\overline{\varphi}}. \tag{31}$$

Substituting asymptotics (31) in formula (8), we get the following theorem.

Theorem 1. *Let* $\varphi(z) \in W^{(2\mu)}_{1,n+m}(R_{n+m})$, $f(t,z) \equiv 0$. *Then for the solution* $u_\varphi(t,z)$ *of Cauchy problem (1)-(2) as* $t \to +\infty$ *it holds the asymptotic expansion*

$$u_\varphi(t,z) = t^{-\frac{n}{2}}\left[1+t^{-}\right]A(z), \tag{32}$$

where

$$|A(z)| \le C(n,m)\,|z|^{n+m}\left\|\varphi, W^{(2\mu)}_{1,n+m}(R_{n+m})\right\|,$$

$C(n,m)$ *is a constant independent of* n *and* m; μ *was determined above by formula (27).*

In the sequel, we need the following lemma.

Lemma. *Let*

$$F(t,\alpha,\beta) = \int\limits_0^t (1+\tau)^\alpha (1+|t-\tau|)^\beta\, d\tau,$$

where α,β *are real numbers.*

1) If $\alpha,\beta \ne -1$ *then as* $t \to +\infty$

$$F(t,\alpha,\beta) = C(\alpha,\beta)\,t^{\gamma_1} + O\left(t^{-|\alpha-\beta|}\right),$$

where $\gamma_1 = \max\{\alpha,\beta\}$, $C(\alpha,\beta)$ *was determined by formula (38)*

2) If $\alpha=-1,\ \beta\ne-1$; $\alpha\ne-1, \beta=-1$; $\alpha=\beta=-1$, *then as* $t\to+\infty$

$$F(t,\alpha,\beta) = C_2(\alpha,\beta)\,t^{\gamma_2}\ln t\left(1+O\left(t^{-1}\right)\right),$$

where

$$\gamma_2 = \begin{cases} \beta, & if \quad \alpha=-1, \beta\ne -1 \\ \alpha, & if \quad \alpha\ne-1, \beta=-1 \\ -1, & if \quad \alpha=\beta=-1 \end{cases}$$

$C_2(\alpha,\beta)$ *was determined by formula (41).*

Proof. 1) Represent $F(t,\alpha,\beta)$ in the form

$$F(t,\alpha,\beta) = t^{\alpha+\beta+1}\int\limits_0^1 \left(\frac{1}{t}+\tau\right)^\alpha \left(\frac{1}{t}+|1-\tau|\right)^\beta d\tau.$$

Write the expansion of unit

$$1 \equiv \varphi_1(\tau) + \varphi_2(\tau),$$

where $\varphi_1(\tau)$ and $\varphi_2(\tau)$ are infinitely differentiable functions, $\varphi_1(\tau)\equiv 1$, $\tau\in\left[0,\frac{1}{2}\right]$, $0\le\varphi_1(\tau)\le 1$, $\tau\in\left[\frac{1}{2},\frac{1}{2}+\varepsilon\right]$, $\varphi_1(\tau)\equiv 0$ for $\tau\in\left[\frac{1}{2}+\varepsilon,1\right]$; $\varphi_2(\tau)\equiv 0$ for $\tau\in\left[0,\frac{1}{2}-\varepsilon\right]$, $0\le\varphi_2(\tau)\le 1$ for $\tau\in\left[\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon\right]$, $\varphi_2(\tau)\equiv 1$ for $\tau\in\left[\frac{1}{2}+\varepsilon,1\right]$. Then

$$F(t,\alpha,\beta) = t^{\alpha+\beta+1}\left\{\int\limits_0^{\frac{1}{2}+\varepsilon}\varphi_1(\tau)\left(\frac{1}{t}+\tau\right)^\alpha\left(\frac{1}{t}+|1-\tau|\right)^\beta d\tau + \right.$$

$$+\int\limits_{\frac{1}{2}-\varepsilon}^{1}\varphi_2(\tau)\left(\frac{1}{t}+\tau\right)^{\alpha}\left(\frac{1}{t}+|1-\tau|\right)^{\beta}d\tau\equiv t^{\alpha+\beta+1}\left\{J_1(t,\alpha,\beta)+J_2(t,\alpha,\beta)\right. . \quad (33)$$

Consider each summand in (33) as $t\to+\infty$ separately. At first we consider

$$J_1(t,\alpha,\beta)=\int\limits_0^{\frac{1}{2}+\varepsilon}\varphi_1(\tau)\left(t^{-1}+\tau\right)^{\alpha}\left(t^{-1}+|1-\tau|\right)^{\beta}d\tau.$$

In the expression $J_1(t,\alpha,\beta)$ integrate by parts and therewith integrate $\left(\frac{1}{t}+\tau\right)^{\alpha}$. This provides the quickest change of this function as $t\to+\infty$. Taking into account the properties of the function $\varphi_1(\tau)$, we get

$$J_1(t,\alpha,\beta)=\frac{t^{-\alpha+1}}{\alpha+1}\left(t^{-1}+1\right)^{\beta}+J_1^{(1)}(t),$$

where

$$J_1^{(1)}(t,\alpha,\beta)=\frac{-1}{\alpha+1}\int\limits_0^{\frac{1}{2}+\varepsilon}\left(t^{-1}+\tau\right)^{\alpha+1}\frac{d}{d\tau}\frac{\varphi_1(\tau)}{\left(t^{-1}+|1-\tau|^{\beta}\right)}d\tau. \quad (34)$$

Expanding $\left(t^{-1}+\tau\right)^{\alpha}$ in the vicinity of the point $\tau=1$ for large values of t, we get

$$J_1(t,\alpha,\beta)=\frac{1}{\alpha+1}t^{-(\alpha+1)}-\frac{\beta}{\alpha+1}t^{-(\alpha+2)}+J_1^{(1)}(t,\alpha,\beta)+O\left(t^{-(\alpha+3)}\right). \quad (35)$$

Since $J_1^{(1)}(t,\alpha,\beta)$ has the same form as $J_1(t,\alpha,\beta)$, then applying asymptotic formula (35) to it and changing α by $\alpha+1$ as $t\to+\infty$ we get

$$J_1(t,\alpha,\beta)=\frac{t^{-(\alpha+1)}}{\alpha+1}+O\left(t^{-(\alpha+2)}\right). \quad (36)$$

Now consider the asymptotics $J_2(t,\alpha,\beta)$ as $t\to\infty$

$$J_2(t,\alpha,\beta)=\int\limits_{\frac{1}{2}-\varepsilon}^{1}\varphi_2(\tau)\left(t^{-1}+\tau\right)^{\alpha}\left(t^{-1}+|1-\tau|\right)^{\beta}d\tau.$$

Arguing as in obtaining the asymptotics $J_1(t,\alpha,\beta)$ for large values of t, while integrating by parts integrating $\left(t^{-1}+|1-\tau|\right)^{\beta}$ and taking into account $\varphi_2^{(l)}(\tau)=0$ as $\tau=\frac{1}{2}-\varepsilon$, $l=0,1,2,...$, get

$$J_2(t,\alpha,\beta)=\frac{t^{-(\beta+1)}}{\beta+1}+O\left(t^{-(\beta+2)}\right). \quad (37)$$

From (33), (36), (37) as $t\to+\infty$ we get

$$F(t,\alpha,\beta)=\frac{t^{\beta}}{\alpha+1}+\frac{t^{\alpha}}{\beta+1}+O\left(t^{\beta-1}\right)+O\left(t^{\alpha-1}\right).$$

Denote $\gamma = \max\{\alpha,\beta\}$. Then

$$F(t,\alpha,\beta) = C(\alpha,\beta)\, t^{\gamma} \left\{1 + O\left(t^{-q}\right)\right\},$$

where

$$C(\alpha,\beta) = \begin{cases} \frac{1}{\beta+1}, & if \quad \alpha > \beta, \\ \frac{1}{\alpha+1}, & if \quad \alpha < \beta, \\ \frac{2}{\alpha+1}, & if \quad \alpha = \beta, \end{cases} \tag{38}$$

$$q = \begin{cases} |\alpha-\beta|, & if \quad \alpha \neq \beta, \\ 1, & if \quad \alpha = \beta. \end{cases}$$

2) Consider the proof of item 2. If $\alpha = -1$, $\beta \neq -1$, then behaving as in proving item 1, we can show that as $t \to +\infty$

$$F(t,-1,\beta) = t^{\beta} \ln t \left[1 + O\left(\frac{1}{t}\right)\right]. \tag{39}$$

Since $F(t,\alpha,\beta) = F(t,\beta,\alpha)$, then asymptotics $F(t,\alpha,-1)$ as $t \to +\infty$ is obtained from (39) by replacing β by α.

3) For $\alpha = -1$, $\beta = -1$, $t \to +\infty$

$$F(t,-1-1) = 2t^{-1} \ln t \left[1 + O\left(t^{-1}\right)\right]. \tag{40}$$

Combining formulae (39) and (40), we get the proof of item 2 of the lemma, where

$$C_2(\alpha,\beta) = \begin{cases} 1, & if \quad \alpha \neq \beta, \\ 2, & if \quad \alpha = \beta = -1. \end{cases} \tag{41}.$$

The lemma is proved.

Now, consider the solution $u_f(t,z)$ of Cauchy problem (1)-(2) for $\varphi(z) \equiv 0$, and $f(t,z) \neq 0$. Then by formula (7), we have

$$u_f(t,z) = \int_0^t \int_{R_{n+m}} G(t-\tau, z-\xi)\, f(\tau,\xi)\, d\tau d\xi,$$

where $G(t,z)$ was determined by formula (9). By the lemma, as in theorem 1 the following theorem is proved.

Theorem 2. *Let $f(t,z) \in W_{1,n+m}^{2(\mu-1)}(R_{n+m})$ for each $t > 0$ and as $t \to +\infty$, $f(t,z) = t^{\alpha} f(z)$, where $f(z) \in W_{1,n+m}^{2(\mu-1)}(R_{n+m})$.*

Then as $t \to +\infty$ for solving Cauchy problem (1)-(2) it holds the following asymptotic expansion

1) if $\alpha \neq -1$, $n \neq 2$ then

$$u_f(t,z) = t^{\gamma_1} \left[1 + O\left(t^{-\left|\alpha+\frac{n}{2}\right|}\right)\right] B(z),$$

2) if $\alpha = -1$, $n \neq 2$; $\alpha \neq -1$, $n = 2$; $\alpha = -1$, $n = 2$ then

$$u_f(t,z) = t^{\gamma_2} \ln t \left[1 + O\left(t^{-1}\right)\right] B(z,)$$

where

$$|B(z)| \le C(n,m)\,|z|^{n+m}\left\| f, W_{1,n+m}^{2(\mu-1)}(R_{n+m})\right\|,$$

$C(n,m)$ *is some constant,* $\gamma_1 = \max\left\{\alpha, -\frac{n}{2}\right\}$

$$\gamma_2 = \begin{cases} -\frac{n}{2}, & if \;\; \alpha=-1, \;\; n\neq 2, \\ \alpha, & if \;\; \alpha\neq -1, \;\; n=2, \\ -1, & if \;\; \alpha=-1, \;\; n=2. \end{cases}$$

Now consider the behavior of $D_t u_f(t,z)$, $\Delta_{n+m} u_f(t,z)$ and $\Delta_n u_f(t,z)$ as $t \to +\infty$. Differentiation of $u_f(t,z)$ with respect to t leads to multiplication of the integrand function in the expression $G(t,z)$ by $-\frac{\chi\rho^2 T(\overline{\varphi})}{1+\eta\rho^2}$ that is bounded on all E_{n+m}

$$\frac{\partial}{\partial t}G^*\left(t,z,\overline{\overline{\varphi}}\right) = -\frac{\chi}{(2\pi)^{n+m}}\left\{\int_0^\infty\int_0^\pi\ldots\int_0^\pi \sin^{n+m-1}\varphi_1 \ldots \sin^{m-1}\varphi_n \frac{\rho^{n+m-1}}{(1+\rho^2)^\mu}\times\right.$$

$$\times\frac{1}{(1+\eta\rho^2)}e^{-\frac{\chi\rho^2T(\overline{\varphi})}{1+\eta\rho^2}t}e^{i(z,\rho T_1(\varphi))}d\overline{\varphi}d\rho - \int_0^\infty\int_0^\pi\ldots\int_0^\pi \sin^{n+m+1}\varphi_1 \ldots \sin^{m+1}\varphi_n\times$$

$$\left.\times\frac{\rho^{n+m-1}d\overline{\varphi}d\rho}{(1+\rho^2)^\mu(1+\eta\rho^2)}\right\} \equiv \frac{\partial}{\partial t}G_I^*\left(t,z,\overline{\overline{\varphi}}\right) + \frac{\partial}{\partial t}G_{II}^*\left(t,z,\overline{\overline{\varphi}}\right).$$

Note that $\frac{\partial}{\partial t}G_I^*\left(t,z,\overline{\overline{\varphi}}\right)$ and $\frac{\partial}{\partial t}G_{II}^*\left(t,z,\overline{\overline{\varphi}}\right)$ have the same form as $G^*\left(t,z,\overline{\overline{\varphi}}\right)$ from (10). And the principal part of the asymptotics $G^*\left(t,z,\overline{\overline{\varphi}}\right)$ is defined by dimension of n. Consequently, under the conditions of theorem 1, for $\frac{\partial}{\partial t}u_\varphi\left(t,z,\overline{\overline{\varphi}}\right)$ as $t\to+\infty$ it holds asymptotic expansion whose principal part has the same order as in theorem 1.

Asymptotic expansion as $t\to+\infty$ $\Delta_{n+m}u_\varphi(t,z)$ ($m \ge 0$ is an integer) is conducted as in theorem 1, the difference is that the smoothness of the initial function $\varphi(x)$ should be greater by two units than in theorem 1, therewith the principal part of the asymptotics remains as in theorem 1.

References

[1]. Barenblatt G.I., Jeltov Yu. P., Kochina I.N. *On basic representations of theory of fluid filtration in fissured rocks.* Prikladnaya matematika i mechanika, 1960, vol. 24, issue 5, pp. 852-864 (Russian).

[2].Guseynov F.B. *Asymptotic expantsion of Cauchy problem at large time for Barenblatt-Jeltov Kochina equation.* Transaction of National Academy of Sciences of Azerbaijan, 2003, vol 23, No 4, pp. 43-50.

[3]. Iskenderov B.A., Mamedov J.Y. *Behavior of the solution of Cauchy problem for Barenblatt-Jeltov-Kochina equation at great values of time.* Transactions of National Academy of Sciences of Azerbaijan, 2007, vol. 27, No 1, pp. 71-78.

[4]. Galpern S.A. *Cauchy problem for general systems of linear partial equatins.* Trudy Moskovskovo matematicheskovo obshestva. 1960, vol. 9, pp. 401-423 (Russian).

[5]. Kostyuchenko A. G., Eskin G.I. *Cauchy problem for Sobolev-Galpern equations.* Trudy Moskovskovo mathematicheskovo obshestva, 1961, vol. 10, pp. 273-284 (Russian).

[6] Fedoryuk M.V. *Asymptotics, integrals and series.* Moscow, "Nauka" 1987, 544 p. (Russian).

Bala A. Iskenderov, Gulnar Kh. Akhundova, Seymur E. Suleymanov
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B. Vahabzade str., AZ 1141, Baku, Azerbaijan
Agjabadi branch of the Institute of teachers
Tel.: (99412) 539 47 20 (off).

Received September 06, 2011; Revised November 11, 2011.