Rauf V. GUSEYNOV, Nihan A. ALIYEV, Shahla Yu. SALMANOVA

FIRST ORDER PARTIAL EQUATIONS HAVING A UNIQUE SOLUTION

Abstract

The paper is devoted to investigation of solutions of boundary value problems for partial elliptic type equations of first order when a boundary condition is not necessary for the existence and uniqueness of the solution. We cite examples of boundary value problems for first order partial equations for which boundary conditions are given not on the whole of the boundary but only on its some part.

Let **D** be a bounded, convex with respect to x_2 plane domain with Lyapunov line boundary Γ. Project D on the axis x_1 parallet to x_2, divide the boundary Γ into two parts Γ_1 and Γ_2 with the equations $x_2 = \gamma_k(x_1)$; $k = 1, 2$. $x_1 \in [a_1, b_1]$, where $[a_1, b_1] = np^2_{x_1} D = np^2_{x_1}\Gamma = np^2_{x_1}\overline{\Gamma}_1 = np^2_{x_1}\overline{\Gamma}_2$.

Let $a(x) = [\gamma_2(x_1) - x_2][x_2 - \gamma_1(x_1)]$, $x_1 \in [a_1, b_1]$, $x_2 \in [\gamma_1(x_1), \gamma_2(x_1)]$.

Consider the equation

$$lu \equiv \frac{\partial}{\partial x_2}[a(x)u(x)] + i\frac{\partial}{\partial x_1}[a(x)u(x)] = f(x), \quad x \in D \subset R^2. \tag{1}$$

The fundamental solution of the conjugated equation

$$l^* v \equiv -a(x)\left[\frac{\partial v(x)}{\partial x_2} + i\frac{\partial v(x)}{\partial x_1}\right] = g(x), \tag{2}$$

where $g(x)$ is an arbitrary continuous function, is of the form:

$$V(x-\xi,\xi) = \frac{-1}{a(\xi)}\frac{1}{2\pi}\frac{1}{x_2-\xi_2+i(x_1-\xi_1)}. \tag{3}$$

Indeed,

$$-a(x)\frac{\partial V(x-\xi,\xi)}{\partial x_2} - ia(x)\frac{\partial V(x-\xi,\xi)}{\partial x_1} =$$
$$= -a(x)\frac{-1}{a(\xi)}\frac{\partial}{\partial x_2}\left(\frac{1}{2\pi}\frac{1}{x_2-\xi_2+i(x_1-\xi_1)}\right) -$$
$$-ia(x)\frac{-1}{a(\xi)}\frac{\partial}{\partial x_1}\left(\frac{1}{2\pi}\frac{1}{x_2-\xi_2+i(x_1-\xi_1)}\right) =$$
$$= \frac{a(x)}{a(\xi)}\left[\frac{\partial}{\partial x_2}\left(\frac{1}{2\pi}\frac{1}{x_2-\xi_2+i(x_1-\xi_1)}\right) +\right.$$
$$\left. +i\frac{\partial}{\partial x_1}\left(\frac{1}{2\pi}\frac{1}{x_2-\xi_2+i(x_1-\xi_1)}\right)\right] = \frac{a(x)}{a(\xi)}\delta(x-\xi) = \delta(x-\xi).$$

Now, multiply the both sides of equation (1) by fundamental solution (3) and integrate it with respect to domain D

$$\int_D \frac{\partial[a(x)u(x)]}{\partial x_2} V(x-\xi,\xi)\,dx + i\int_D \frac{\partial[a(x)u(x)]}{\partial x_1} V(x-\xi,\xi)\,dx = \int_D f(x)V(x-\xi,\xi)\,dx.$$

We apply the Ostrogradsky-Gauss formula and get the Green second formula that yields the following main relation:

$$-\int\limits_{\Gamma} a(x)\,u(x)\,V(x-\xi,\xi)\left[\cos(v,x_2)+i\cos(v,x_1)\right]dx+\int\limits_{D} f(x)\,V(x-\xi,\xi)\,dx=$$

$$=-\int\limits_{\Gamma} a(x)\,u(x)\left[\frac{\partial V(x-\xi,\xi)}{\partial x_2}+i\frac{\partial V(x-\xi,\xi)}{\partial x_1}\right]dx=\left\{\begin{array}{l} u(\xi),\xi\in D,\\ \frac{1}{2}u(\xi),\xi\in\Gamma.\end{array}\right. \quad (4)$$

The similar relation is in [1].

The first expression of this relation corresponding to $\xi\in D$ gives us the representation of the solution of equation (1), the second expression corresponding to $\xi\in\Gamma$ gives us necessary conditions.

Thus we get:

Theorem 1. *If D is a bounded convex with respect to x_2 plane domain with Lyapunov line boundary Γ, and $f(x)$ is a continuous function in D, then each solution of equation (1) determined in D satisfies main relation (4).*

Coming back to main relation (4), taking into account $a(x)=0$, $x\in\Gamma$ we get

$$u(\xi)=\int\limits_{D} f(x)\,V(x-\xi,\xi)\,dx,\quad \xi\in D. \quad (5)$$

This establishes

Theorem 2. *Under the conditions of theorem 1, equation (1) without additional restrictions has a unique solution representable in the form (5).*

Let D be a circle of unique radius with the center at the origin of coordinates. The function $a(x)$ is of the form:

$$a(x)=x_2+\sqrt{1-x_1^2},\ \ x_1\in[-1,1]. \quad (6)$$

Then from main relation (4) we get:

$$u(\xi)=\int\limits_{|x|<1} f(x)\,V(x-\xi,\xi)\,dx-\int\limits_{\Gamma_2} a(x)\,u(x)\,V(x-\xi,\xi)\times$$

$$\times\left[\cos(v,x_2)+i\cos(v,x_1)\right]dx, \quad (7)$$

$$\frac{1}{2}u\left(\xi_1,\sqrt{1-\xi_1^2}\right)=\int\limits_{|x|<1} f(x)\,V\left(x_1-\xi_1,x_2-\sqrt{1-\xi_1^2},\xi_1,\sqrt{1-\xi_1^2}\right)dx-$$

$$-\int\limits_{\Gamma_2} a(x)\,u(x)\,V\left(x_1-\xi_1,x_2-\sqrt{1-\xi_1^2},\xi_1,\sqrt{1-\xi_1^2}\right)\times$$

$$\times\left[\cos(v,x_2)+i\cos(v,x_1)\right]dx,\xi_1\in[-1,1]. \quad (8)$$

As is known, for an ordinary linear differential equation, the amount of boundary conditions coincides with the order of differential equation under consideration (both for the Cauchy problem and a boundary value problem). Concerning many-dimensional problems, if the amount of spatial variables is greater than a unit, then the amount of initial conditions coincides with the highest order of time variable derivative, and the amount of boundary conditions coincides with the half of the highest order of spatial variable derivative contained in the equation under consideration. One condition is given for the Laplace equation; either Dirichlet or Neumann condition, or the special case- the Poincare condition. For a polyharmonic equation ($2m$ order equation) m boundary conditions are given [2] [3]. All these conditions are local.

In the work [4] on investigating the process in the nuclear reactor, a mathematical model consisting of a first order linear integro-differential equation is obtained. Further, if one transmits light flux to the reactor, the boundary condition is given on the part of the reactor that remains in the dark, i.e. in some sense on the half of the boundary.

Therefore, we give the boundary condition for elliptic type equation of first order in the nonlocal form [1], [5]. Thus, giving the boundary condition in the nonlocal form, we remove the shortcomings that appear between boundary value problems for ordinary linear differential equations and partial equations.

Cite one boundary condition that corresponds to the operator $\Delta^{\frac{1}{4}}$.

$$\alpha_1(x_1)\,u\left(x_1,\sqrt{1-x_1^2}\right)+\alpha_2(x_1)\,u\left(-x_1,\sqrt{1-x_1^2}\right)+$$

$$+\alpha_3(x_1)\,u\left(x_1,-\sqrt{1-x_1^2}\right)+\alpha_4(x_1)\,u\left(-x_1,-\sqrt{1-x_1^2}\right)=\alpha(x_1),\quad x_1\in[0,1].$$

It is easy to see that these conditions are written so that the Carleman condition is satisfied [6].

Let D be a unique circle, $a(x)$ is of the form (6). For equation (1) give a condition in the following form:

$$a_1(x_1)\,u\left(x_1,\sqrt{1-x_1^2}\right)+b_1(x_1)\,u\left(-x_1,\sqrt{1-x_1^2}\right)=c_1(x_1),\quad x_1\in[0,1],\qquad(9)$$

where $a_1(x_1), b_1(x_1), c_1(x_1)$ are the given continuous functions.

Now, represent necessary conditions (8) in the form:

$$\frac{1}{2}u\left(\xi_1,\sqrt{1-\xi_1^2}\right)=\int\limits_{|x|<1}f(x)\,V\left(x_1-\xi_1,x_2-\sqrt{1-\xi_1^2},\xi_1,\sqrt{1-\xi_1^2}\right)dx-$$

$$-\int\limits_0^1 a\left(x_1,\sqrt{1-x_1^2}\right)u\left(x_1,\sqrt{1-x_1^2}\right)\times$$

$$\times V\left(x_1-\xi_1,\sqrt{1-x_1^2}-\sqrt{1-\xi_1^2},\xi_1,\sqrt{1-\xi_1^2}\right)\times$$

$$\times\left[1+\frac{ix_1}{\sqrt{1-x_1^2}}\right]dx_1-\int\limits_0^1 a\left(-x_1,\sqrt{1-x_1^2}\right)u\left(x_1,\sqrt{1-x_1^2}\right)\times$$

$$\times V\left(-x_1-\xi_1,\sqrt{1-x_1^2}-\sqrt{1-\xi_1^2},\xi_1,\sqrt{1-\xi_1^2}\right)\left[1-\frac{ix_1}{\sqrt{1-x_1^2}}\right]dx_1,\qquad(10)$$

$$\frac{1}{2}u\left(-\xi_1,\sqrt{1-\xi_1^2}\right)=\int\limits_{|x|<1}f(x)\,V\left(x_1+\xi_1,x_2-\sqrt{1-\xi_1^2},-\xi_1,\sqrt{1-\xi_1^2}\right)dx-$$

$$-\int_0^1 a\left(x_1,\sqrt{1-x_1^2}\right)u\left(x_1,\sqrt{1-x_1^2}\right)\times$$

$$\times V\left(x_1+\xi_1,\sqrt{1-x_1^2}-\sqrt{1-\xi_1^2},-\xi_1,\sqrt{1-\xi_1^2}\right)\times$$

$$\times\left[1+\frac{ix_1}{\sqrt{1-x_1^2}}\right]dx_1-\int_0^1 a\left(-x_1,\sqrt{1-x_1^2}\right)u\left(-x_1,\sqrt{1-x_1^2}\right)\times$$

$$\times V\left(-x_1+\xi_1,\sqrt{1-x_1^2}-\sqrt{1-\xi_1^2},-\xi_1,\sqrt{1-\xi_1^2}\right)\times$$

$$\times\left[1-\frac{ix_1}{\sqrt{1-x_1^2}}\right]dx_1,\xi_1\in[0,1] \tag{11}$$

Calculate the following expressions:

$$V\left(x_1-\xi_1,\sqrt{1-x_1^2}-\sqrt{1-\xi_1^2},\xi_1,\sqrt{1-\xi_1^2}\right)=$$

$$=\frac{-1}{a\left(\xi_1,\sqrt{1-\xi_1^2}\right)}\cdot\frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-x_1^2}-\sqrt{1-\xi_1^2}+i\left(x_1-\xi_1\right)}=$$

$$=\frac{-1}{2\sqrt{1-\xi_1^2}}\cdot\frac{1}{2\pi}\cdot\frac{1}{\frac{1-x_1^2-\left(1-\xi_1^2\right)}{\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}}+i\left(x_1-\xi_1\right)}=\frac{-1}{2\pi\left(2\sqrt{1-\xi_1^2}\right)}\times$$

$$\times\frac{\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}}{-\left(x_1-\xi_1\right)\left(x_1+\xi_1\right)+i\left(x_1-\xi_1\right)\left[\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right]}=$$

$$=\frac{\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}}{2\pi\left(2\sqrt{1-\xi_1^2}\right)\left[\left(x_1+\xi_1\right)-i\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\right]}\frac{1}{x_1-\xi_1}$$

and

$$V\left(-x_1+\xi_1,\sqrt{1-x_1^2}-\sqrt{1-\xi_1^2},-\xi_1,\sqrt{1-\xi_1^2}\right)=$$

$$=\frac{-1}{2\sqrt{1-\xi_1^2}}\cdot\frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-x_1^2}-\sqrt{1-\xi_1^2}-i\left(x_1-\xi_1\right)}=$$

$$=\frac{-1}{2\sqrt{1-\xi_1^2}}\cdot\frac{1}{2\pi}\cdot\frac{\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}}{-\left(x_1-\xi_1\right)\left[\left(x_1+\xi_1\right)+i\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\right]}=$$

$$=\frac{\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}}{2\pi\left(2\sqrt{1-\xi_1^2}\right)\left[\left(x_1+\xi_1\right)+i\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\right]}\cdot\frac{1}{x_1-\xi_1}.$$

Thus, taking into account (9)-(11), build-up the following linear combination:

$$A(\xi_1)\,u\left(\xi_1,\sqrt{1-\xi_1^2}\right)+B(\xi_1)\,u\left(-\xi_1,\sqrt{1-\xi_1^2}\right)=$$

$$=2A(\xi_1)\int\limits_{|x|<1}f(x)\,V\left(x_1-\xi_1,x_2-\sqrt{1-\xi_1^2},\xi_1,\sqrt{1-\xi_1^2}\right)dx-$$

$$-2A(\xi_1)\int\limits_0^1 a\left(-x_1,\sqrt{1-x_1^2}\right)u\left(-x_1,\sqrt{1-x_1^2}\right)\times$$

$$\times V\left(-x_1-\xi_1,\sqrt{1-x_1^2},-\sqrt{1-\xi_1^2},\xi_1,\sqrt{1-\xi_1^2}\right)\times\left[1-\frac{ix_1}{\sqrt{1-x_1^2}}\right]dx_1+$$

$$+2B(\xi_1)\int\limits_{|x|<1}f(x)\,V\left(x_1+\xi_1,x_2-\sqrt{1-\xi_1^2},-\xi_1,\sqrt{1-\xi_1^2}\right)dx-$$

$$-2B(\xi_1)\int\limits_0^1 a\left(x_1,\sqrt{1-x_1^2}\right)u\left(x_1,\sqrt{1-x_1^2}\right)\times$$

$$\times V\left(x_1+\xi_1,\sqrt{1-x_1^2}-\sqrt{1-\xi_1^2},-\xi_1,\sqrt{1-\xi_1^2}\right)\left[1+\frac{ix_1}{\sqrt{1-x_1^2}}\right]dx_1$$

$$-2\int\limits_0^1\left[A(\xi_1)-A(x_1)\right]a\left(x_1,\sqrt{1-x_1^2}\right)u\left(x_1,\sqrt{1-x_1^2}\right)\times$$

$$\times V\left(x_1-\xi_1,\sqrt{1-x_1^2}-\sqrt{1-\xi_1^2},\xi_1,\sqrt{1-\xi_1^2}\right)\times\left[1+\frac{ix_1}{\sqrt{1-x_1^2}}\right]dx_1-$$

$$-2\int\limits_0^1\left[B(\xi_1)-B(x_1)\right]a\left(-x_1,\sqrt{1-x_1^2}\right)u\left(-x_1,\sqrt{1-x_1^2}\right)\times$$

$$\times V\left(-x_1+\xi_1,\sqrt{1-x_1^2}-\sqrt{1-\xi_1^2},\xi_1,\sqrt{1-\xi_1^2}\right)\times$$

$$\times\left[1-\frac{ix_1}{\sqrt{1-x_1^2}}\right]dx_1-\frac{1}{\pi}\int\limits_0^1\left\{A(x_1)\,u\left(x_1,\sqrt{1-x_1^2}\right)\times\right.$$

$$\times\frac{\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\left(\sqrt{1-x_1^2}+ix_1\right)}{\sqrt{1-\xi_1^2}\left[x_1+\xi_1-i\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\right]}+B(x_1)\,u\left(-x_1,\sqrt{1-x_1^2}\right)\times$$

$$\left.\times\frac{\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\left(\sqrt{1-x_1^2}-ix_1\right)}{\sqrt{1-\xi_1^2}\left[x_1+\xi_1+i\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\right]}\right\}\frac{dx_1}{x_1-\xi_1}. \qquad (12)$$

Further, since

$$\frac{\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\left(\sqrt{1-x_1^2}+ix_1\right)}{\sqrt{1-\xi_1^2}\left[x_1+\xi_1-i\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\right]}=i+$$

$$+\frac{-\left(x_1+\xi_1\right)\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}+i\xi_1\right)+i\sqrt{1-x_1^2}\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)}{\sqrt{1-\xi_1^2}\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\left[x_1+\xi_1-i\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\right]}\left(x_1-\xi_1\right),$$

and

$$\frac{\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\left(\sqrt{1-x_1^2}-ix_1\right)}{\sqrt{1-\xi_1^2}\left[x_1+\xi_1-i\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\right]}=-i-$$

$$-\frac{\left(x_1+\xi_1\right)\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}-i\xi_1\right)+i\sqrt{1-x_1^2}\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)}{\sqrt{1-\xi_1^2}\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\left[x_1+\xi_1+i\left(\sqrt{1-x_1^2}+\sqrt{1-\xi_1^2}\right)\right]}\left(x_1-\xi_1\right).$$

assuming in (12)

$$A\left(x_1\right)=a_1\left(x_1\right),\quad B\left(x_1\right)=-b_1\left(x_1\right),$$

for $a_1\left(\xi_1\right)u\left(\xi_1,\sqrt{1-\xi_1^2}\right)-b_1\left(\xi_1\right)u\left(-\xi_1,\sqrt{1-\xi_1^2}\right)$ we get nonsingular expressions. Considering the obtained regular relations together with boundary conditions (9) under conditions $a_1\left(x_1\right)\neq 0$, $b_1\left(x_1\right)\neq 0$ we get the Fredholm property of the stated boundary value problems (1), (9).

References

[1]. Aliyev N., Jahanshahi M. *Sufficient conditions for reduction of the BVP including a mixed PDE with non-local boundary conditions to Fredholm integral equations.* International Journal of Mathematical Education in Science and Technology, 1997, 28, No 3, pp. 419-425.

[2]. Meiman N.N. *On a boundary value problem for polyharmonic equations.* DAN SSSR, 1941, 33, No 4, pp. 275-278 (Russian).

[3]. Vekua I.N. *New solution methods for elliptic equations.*M. -L. Gostechizdat, 1948 (Russian).

[4]. Vladimirov V.S. *Equations of mathematical physics.* M., Nauka, 1971, 512 p. (Russian).

[5]. Jahanshahi M., Aliyev N. *Determining of an analytic function on its analytic domain by Cauchy-Riemann equation with special kind of boundary conditions.* Southeast Asian Bulleten Mathematics 2004, 28, No 1, pp. 33-39.

[6]. Carleman T. *Sur la theoric des equations integrales et ses applications in Verhandi Internel.* Mathematicer-Kongresses, vol. 1, Orelle Fussli Zurich-Leipzig, 1932, pp. 138-151.

Rauf V. Guseynov, Nihan A. Aliyev, Shahla Yu. Salmanova
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B.Vahabzade str., AZ 1141, Baku, Azerbaijan
Tel.: (99412) 539 47 20 (off.).

Received May 17, 2011; Revised September 08, 2011.