**Vagif S. GULIYEV, Parviz S. SHUKUROV**

# ADAMS TYPE RESULT FOR SUBLINEAR OPERATORS GENERATED BY RIESZ POTENTIALS ON GENERALIZED MORREY SPACES

**Abstract**

*In this paper the authors study the boundedness for a large class of sublinear operators $T_\alpha$, $\alpha \in (0,n)$ generated by Riesz potential operator on generalized Morrey spaces $M_{p,\varphi}$. We prove the boundedness of the sublinear operator $T_\alpha$, $\alpha \in (0,n)$ satisfies the condition (1.2) generated by Riesz potential operator from one generalized Morrey space $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $1 < p < q < \infty$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ for $1 < q < \infty$. In all the cases the conditions for the boundedness are given it terms of Zygmund-type integral inequalities on $\varphi$, which do not assume any assumption on monotonicity of $\varphi$ in $r$. Conditions of these theorems are satisfied by many important operators in analysis, in particular fractional maximal operator, Riesz potential operator and Marcinkiewicz operator.*

## 1. Introduction

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [21] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [21, 23].

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at $x$ of radius $r$, ${}^{\complement}B(x,r)$ denote its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$.

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The fractional maximal operator $M_\alpha$ and the Riesz potential $I_\alpha$ are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|dy, \qquad 0 \le \alpha < n,$$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n.$$

It is well known that operators $M_\alpha$ and $I_\alpha$ plays an important role in harmonic analysis (see, for example [29, 31]).

We denote by $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\lambda}} \equiv \|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $1 \le p < \infty$ and $0 \le \lambda \le n$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where $\Theta$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} \equiv \|f\|_{WM_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak $L_p$-space of measurable functions $f$ for which

$$\|f\|_{WL_p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} = \sup_{t>0} t\,|\{y \in B(x,r):\, |f(y)| > t\}|^{1/p}\,.$$

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then $I_\alpha$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and for $p = 1 < q < \infty$, $I_\alpha$ is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = n\left(1 - \frac{1}{q}\right)$. S. Spanne (published by J. Peetre [23]) and D.R. Adams [1] studied boundedness of the Riesz potential in Morrey spaces. Their results, can be summarized as follows.

**Theorem 1.1.** *(Spanne, but published by Peetre [23]) Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. *Moreover, let* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $\frac{\lambda}{p} = \frac{\mu}{q}$. *Then for* $p > 1$ *the operator* $I_\alpha$ *is bounded from* $M_{p,\lambda}$ *to* $M_{q,\lambda}$ *and for* $p = 1$ $I_\alpha$ *is bounded from* $M_{1,\lambda}$ *to* $WM_{q,\lambda}$.

**Theorem 1.2.** *(Adams [1]) Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. *Then for* $p > 1$ *the operator* $I_\alpha$ *is bounded from* $M_{p,\lambda}$ *to* $M_{q,\lambda}$ *and for* $p = 1$ $I_\alpha$ *is bounded from* $M_{1,\lambda}$ *to* $WM_{q,\lambda}$.

Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \le \upsilon_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x),$$

hence Theorems 1.1 and 1.2 also implies boundedness of the fractional maximal operator $M_\alpha$, where $v_n$ is the volume of the unit ball in $\mathbb{R}^n$.

Suppose that $T \equiv T_0$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin supp f$

$$|Tf(x)| \le c_0 \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \tag{1.1}$$

where $c_0$ is independent of $f$ and $x$. Similarly, we assume that $T_\alpha$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin supp f$

$$|T_\alpha f(x)| \le c_1 \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \tag{1.2}$$

for some $\alpha \in (0,n)$, where $c_1$ is independent of $f$ and $x$.

In [14] we prove the boundedness of the sublinear operators $T$ satisfying condition (1.1) generated by Calderón-Zygmund operators from one generalized Morrey space $M_{p,\varphi_1}$ to another $M_{p,\varphi_2}$, $1 < p < \infty$, and from the space $M_{1,\varphi_1}$ to the weak space $WM_{1,\varphi_2}$.

In this work, we shall prove the boundedness of the sublinear operators $T_\alpha$, $\alpha \in (0,n)$ satisfying condition (1.2) generated by Riesz potential operator from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $1 < p < q < \infty$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ for $1 < q < \infty$.

We point out that the condition (1.2) was first introduced by Soria and Weiss in [26]. The conditions (1.1) and (1.2) are satisfied by many interesting operators in harmonic analysis, such as the Calderón–Zygmund operators, Carleson's maximal operators, Hardy–Littlewood maximal operators, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci–Stein's oscillatory singular integrals, the Bochner–Riesz means and so on.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant $C$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.

## 2. Generalized Morrey spaces

**Definition 2.1.** *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} \equiv \|f\|_{M_{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))}.$$

*Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which*

$$\|f\|_{WM_{p,\varphi}} \equiv \|f\|_{WM_{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the spaces $M_{p,\lambda}$ and $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}},$$

$$WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

In [10]-[13], [16], [20] and [22] there were obtained sufficient conditions on $\varphi_1$ and $\varphi_2$ for the boundedness of the fractional maximal operator $M_\alpha$ and Riesz potential operator $I_\alpha$ from $M_{p,\varphi_1}$ to $M_{q,\varphi_2}$, $1 < p < q < \infty$ (see also [2]-[6]). In [22] the following condition was imposed on $\varphi(x,r)$:

$$c^{-1}\varphi(x,r) \le \varphi(x,t) \le c\,\varphi(x,r) \tag{2.1}$$

whenever $r \le t \le 2r$, where $c(\ge 1)$ does not depend on $t$, $r$ and $x \in \mathbb{R}^n$, jointly with the condition:

$$\int_r^\infty t^{\alpha p} \varphi(x,t)^p \frac{dt}{t} \le C\, r^{\alpha p} \varphi(x,r)^p, \tag{2.2}$$

for the sublinear operator $T_\alpha$ satisfying condition (1.2), where $C(>0)$ does not depend on $r$ and $x \in \mathbb{R}^n$.

## 3. Sublinear operators in the spaces $M_{p,\varphi}$

### 3.1. Spanne type result

In [7] the following statements was proved by sublinear operator $T_\alpha$ satisfying condition (1.2), containing the result in [20, 22].

**Theorem 3.3.** *Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\varphi(x,r)$ satisfy conditions (2.1) and (2.2). Let $T_\alpha$ be a sublinear operator satisfying condition (1.2) and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Then the operator $T_\alpha$ is bounded from $M_{p,\varphi}$ to $M_{q,\varphi}$.*

The following statements, containing results obtained in [20], [22] was proved in [10,12] (see also [2]-[6], [11,13]).

**Theorem 3.4.** *Let $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $(\varphi_1, \varphi_2)$ satisfy the condition*

$$\int_t^\infty r^\alpha \varphi_1(x,r) \frac{dr}{r} \le C\, \varphi_2(x,t), \tag{3.1}$$

*where $C$ does not depend on $x$ and $t$. Then the operators $M_\alpha$ and $I_\alpha$ are bounded from $M_{p,\varphi_1}$ to $M_{q,\varphi_2}$ for $p > 1$ and from $M_{1,\varphi_1}$ to $WM_{q,\varphi_2}$ for $p = 1$.*

**Theorem 3.5.** *Let $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and $(\varphi_1, \varphi_2)$ satisfy the condition*

$$\int_t^\infty \frac{\operatorname*{ess\,inf}_{r<s<\infty} \varphi_1(x,s) s^{\frac{n}{q}}}{r^{\frac{n}{q}+1}} dr \le C\, \varphi_2(x,t), \tag{3.2}$$

*where $C$ does not depend on $x$ and $t$. Let $T_\alpha$ be a sublinear operator satisfying condition (1.2) bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $1 < p < q < \infty$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $1 < q < \infty$. Then the operator $T_\alpha$ is bounded from $M_{p,\varphi_1}$ to $M_{q,\varphi_2}$ for $1 < p < q < \infty$ and from $M_{1,\varphi_1}$ to $WM_{q,\varphi_2}$ for $1 < q < \infty$. Moreover, for $p > 1$*

$$\|T_\alpha f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},$$

*and for $p = 1$*

$$\|T_\alpha f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

**Corollary 3.1.** *Let $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $(\varphi_1, \varphi_2)$ satisfy condition (3.2). Then the operators $M_\alpha$ and $I_\alpha$ are bounded from $M_{p,\varphi_1}$ to $M_{q,\varphi_2}$ for $p > 1$ and from $M_{1,\varphi_1}$ to $WM_{q,\varphi_2}$ for $p = 1$.*

### 3.2. Adams type result

The following is a result of Adams type.

**Theorem 3.6.** *Let $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $q > p$ and let $\varphi(x,t)$ satisfies the conditions*

$$\int_r^\infty \varphi(x,t)^{\frac{1}{p}} \frac{dt}{t} \le C\varphi(x,r)^{\frac{1}{p}}, \tag{3.3}$$

$$\int_r^\infty t^\alpha\, \varphi(x,t)^{\frac{1}{p}} \frac{dt}{t} \le C r^{-\frac{\alpha p}{q-p}}, \tag{3.4}$$

*where $C$ does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Let also $T_\alpha$ be a sublinear operator satisfying condition (1.2) and the condition*

$$|T_\alpha(f\chi_{B(x_0,r)})(x)| \lesssim r^\alpha Mf(x) \tag{3.5}$$

*holds for any ball $B(x_0,r)$.*

*Then the operator $T_\alpha$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $p > 1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ for $p = 1$.*

**Proof.** Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $q > p$ and $f \in M_{p,\varphi^{\frac{1}{p}}}$.

For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at $x_0$ and of radius $r$. We represent $f$ as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad t > 0, \tag{3.6}$$

and have

$$\|T_\alpha f\|_{L_q(B)} \leq \|T_\alpha f_1\|_{L_q(B)} + \|T_\alpha f_2\|_{L_q(B)}.$$

For $T_\alpha f_2(x)$ we have

$$|T_\alpha f_2(x)| \leq \int_{\complement_{B(x,2r)}} |x-y|^{\alpha-n}|f(y)|dy \lesssim \int_{\complement_{B(x,2r)}} |f(y)|dy \int_{|x-y|}^{\infty} t^{\alpha-n-1}dt \lesssim$$

$$\lesssim \int_{2r}^{\infty} \left( \int_{2r<|x-y|<t} |f(y)|dy \right) t^{\alpha-n-1}dt \lesssim \int_r^\infty t^{\alpha-\frac{n}{p}-1}\|f\|_{L_p(B(x,t))}dt. \tag{3.7}$$

Then from conditions (3.4), (3.5) and (3.7) we get

$$|T_\alpha f(x)| \lesssim r^\alpha\, Mf(x) + \int_r^\infty t^{\alpha-\frac{n}{p}-1}\|f\|_{L_p(B(x,t))}dt \leq$$

$$\leq r^\alpha\, Mf(x) + \|f\|_{M_{p,\varphi}} \int_r^\infty t^\alpha \varphi(x,t)^{\frac{1}{p}} \frac{dt}{t} \lesssim r^\alpha\, Mf(x) + r^{-\frac{\alpha p}{q-p}}\, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}. \tag{3.8}$$

Hence choose $r = \left( \frac{\|f\|_{M_{p,\varphi^{1/p}}}}{Mf(x)} \right)^{\frac{q-p}{\alpha q}}$ for every $x \in \mathbb{R}^n$, we have

$$|T_\alpha f(x)| \lesssim (Mf(x))^{\frac{p}{q}}\, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}}.$$

Hence the statement of the theorem follows in view of the boundedness of the maximal operator $M$ in $M_{p,\varphi^{\frac{1}{p}}}$ provided by Theorem 4.2 in [12] in virtue of condition (3.3).

$$\|T_\alpha f\|_{M_{q,\varphi^{\frac{1}{q}}}} = \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|T_\alpha f\|_{L_q(B(x,t))} \lesssim$$

$$\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|Mf\|_{L_p(B(x,t))}^{\frac{p}{q}} =$$

$$= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \left( \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{p}} t^{-\frac{n}{p}} \|Mf\|_{L_p(B(x,t))} \right)^{\frac{p}{q}} =$$

$$= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \|Mf\|_{M_{p,\varphi^{\frac{1}{p}}}}^{\frac{p}{q}} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}},$$

if $1 < p < q < \infty$ and

$$\|T_\alpha f\|_{WM_{q,\varphi^{\frac{1}{q}}}} = \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|T_\alpha f\|_{WL_q(B(x,t))} \lesssim$$

$$\lesssim \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|Mf\|_{WL_1(B(x,t))}^{\frac{1}{q}} =$$

$$= \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \left( \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-1} t^{-n} \|Mf\|_{WL_1(B(x,t))} \right)^{\frac{1}{q}} =$$

$$= \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \|Mf\|_{WM_{1,\varphi}}^{\frac{1}{q}} \lesssim \|f\|_{M_{1,\varphi}},$$

if $1 < q < \infty$.

**Corollary 3.2.** *Let $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $q > p$, and let $\varphi(x,t)$ satisfies the conditions (3.3) and (3.4). Then the operators $M_\alpha$ and $I_\alpha$ are bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $p > 1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ for $p = 1$.*

Note that Corollary 3.2 was proved in [12] and [24]. In the case $\varphi(x,t) = t^{\lambda-n}$, $0 < \lambda < n$ from Corollary 3.2 we get Adams theorem 1.2.

## 4. Some applications

In this section, we shall apply Theorem 3.5 to several particular operators such as the Marcinkiewicz operator, Schrödinger type operators $V^\gamma(-\Delta+V)^{-\beta}$, $V^\gamma\nabla(-\Delta+V)^{-\beta}$ and fractional powers of the some analytic semigroups.

### 4.1. Marcinkiewicz operator

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that $\Omega$ satisfies the following conditions.

(a) $\Omega$ is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(\mu x) = \Omega(x), \quad \text{for any} \quad \mu > 0, x \in \mathbb{R}^n \setminus \{0\}.$$

(b) $\Omega$ has mean zero on $S^{n-1}$, that is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0.$$

(c) $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, $0 < \gamma \le 1$, that is there exists a constant $M > 0$ such that,

$$|\Omega(x') - \Omega(y')| \le M|x' - y'|^\gamma \quad \text{for any} \quad x', y' \in S^{n-1}.$$

In 1958, Stein [27] defined the Marcinkiewicz integral of higher dimension $\mu_\Omega$ as

$$\mu_\Omega(f)(x) = \left( \int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

The Marcinkiewicz operator is defined by (see [32])

$$\mu_{\Omega,\alpha}(f)(x) = \left( \int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,\alpha,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y)dy.$$

Note that $\mu_\Omega f = \mu_{\Omega,0} f$.

Let $H$ be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\}$. Then, it is clear that $\mu_{\Omega,\alpha}(f)(x) = \|F_{\Omega,\alpha,t}(x)\|$.

By Minkowski inequality and the conditions on $\Omega$, we get

$$\mu_{\Omega,\alpha}(f)(x) \le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left( \int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy.$$

Thus, $\mu_{\Omega,\alpha}$ satisfies condition (1.2). It is known that $\mu_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $1 < p < q < \infty$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $1 < q < \infty$ (see [32]), then from Theorem 3.5 we get

**Corollary 4.3.** *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\varphi(x,t)$ satisfies the conditions (3.3), (3.4), and $\Omega$ satisfies conditions (a)–(c). Then $\mu_{\Omega,\alpha}$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $p > 1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ for $p = 1$.*

### 4.2. Schrödinger type operators $V^\gamma(-\Delta+V)^{-\beta}$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$

In this section we consider the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$, where the nonnegative potential $V$ belongs to the reverse Hölder class $B_\infty(\mathbb{R}^n)$ for some $q_1 \ge n$. The generalized Morrey $M_{p,\varphi_1} \to M_{q,\varphi_2}$ estimates for the operators $V^\gamma(-\Delta+V)^{-\beta}$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$ are obtained.

The investigation of Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class has attracted attention of a number of authors (cf. [9, 25, 33]). Shen [25] studied the Schrödinger operator $-\Delta + V$, assuming the nonnegative potential $V$ belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q \ge n/2$ and he proved the $L_p$ boundedness of the operators $(-\Delta+V)^{i\gamma}$, $\nabla^2(-\Delta+V)^{-1}$, $\nabla(-\Delta+V)^{-\frac{1}{2}}$ and $\nabla(-\Delta+V)^{-1}$. Kurata and Sugano generalized Shens results to uniformly elliptic operators in [15]. Sugano [30] also extended some results of Shen to the operator $V^\gamma(-\Delta+V)^{-\beta}$, $0 \le \gamma \le \beta \le 1$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta - \gamma \ge \frac{1}{2}$. Later, Lu [19] and Li [17] investigated the Schrödinger operators in a more general setting.

We investigate the generalized Morrey $M_{p,\varphi_1}$-$M_{q,\varphi_2}$ boundedness of the operators

$$\mathcal{T}_1 = V^\gamma(-\Delta+V)^{-\beta}, \quad 0 \le \gamma \le \beta \le 1,$$

$$\mathcal{T}_2 = V^\gamma\nabla(-\Delta+V)^{-\beta}, \quad 0 \le \gamma \le \frac{1}{2} \le \beta \le 1, \ \beta - \gamma \ge \frac{1}{2}.$$

Note that the operators $V(-\Delta+V)^{-1}$ and $V^{\frac{1}{2}}\nabla(-\Delta+V)^{-1}$ in [17] are the special case of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively.

It is worth pointing out that we need to establish pointwise estimates for $\mathcal{T}_1$, $\mathcal{T}_2$ and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{R}^n$ in [17]. And we prove the generalized Morrey estimates by using $M_{p,\varphi_1} - M_{q,\varphi_2}$ boundedness of the fractional maximal operators.

Let $V \ge 0$. We say $V \in B_\infty$, if there exists a constant $C > 0$ such that

$$\|V\|_{L_\infty(B)} \le \frac{C}{|B|} \int_B V(x)dx$$

holds for every ball $B$ in $\mathbb{R}^n$ (see [17]).

The following two pointwise estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$ which proven in [33], Lemma 3.2 with the potential $V \in B_\infty$ are valid.

**Theorem B.** *Suppose $V \in B_\infty$ and $0 \le \gamma \le \beta \le 1$. Then for any $f \in C_0^\infty(\mathbb{R}^n)$ there exists a constant $C > 0$ such that*

$$|\mathcal{T}_1 f(x)| \lesssim M_\alpha f(x),$$

*where $\alpha = 2(\beta - \gamma)$.*

**Theorem C.** *Suppose $V \in B_\infty$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta - \gamma \ge \frac{1}{2}$. Then for any $f \in C_0^\infty(\mathbb{R}^n)$ there exists a constant $C > 0$ such that*

$$|\mathcal{T}_2 f(x)| \lesssim M_\alpha f(x),$$

*where $\alpha = 2(\beta - \gamma) - 1$.*

The above theorems will yield the generalized Morrey estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$.

**Corollary 4.4.** *Assume that $V \in B_\infty$ and $0 \le \gamma \le \beta \le 1$. Let $1 \le p < q < \infty$, $2(\beta-\gamma) = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and the conditions (3.3) and (3.4) be satisfied for $\alpha = 2(\beta-\gamma)$. Then $\mathcal{T}_1$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $p > 1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ for $p = 1$.*

**Corollary 4.5.** *Assume that $V \in B_\infty$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta - \gamma \ge \frac{1}{2}$. Let $1 \le p < q < \infty$, $2(\beta - \gamma) - 1 = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and the conditions (3.3) and (3.4) be satisfied for $\alpha = 2(\beta - \gamma) - 1$. Then $\mathcal{T}_2$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $p > 1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ for $p = 1$.*

### 4.3. Fractional powers of the some analytic semigroups

The theorems of the previous sections can be applied to various operators which are estimated from above by Riesz potentials. We give some examples.

Suppose that $L$ is a linear operator on $L_2$ which generates an analytic semigroup $e^{-tL}$ with the kernel $p_t(x, y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x, y)| \le \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{4.1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where $c_1$, $c_2 > 0$ are independent of $x$, $y$ and $t$.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator $L$ are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential $I_\alpha$. See, for example, Chapter 5 in [28].

**Theorem 4.7.** *Let condition (4.1) be satisfied. Moreover, let $1 \le p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\varphi$ satisfies the conditions (3.3) and (3.4). Then $L^{-\alpha/2}$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $p > 1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ for $p = 1$.*

**Proof.** Since the semigroup $e^{-tL}$ has the kernel $p_t(x, y)$ which satisfies condition (4.1), it follows that

$$|L^{-\alpha/2} f(x)| \lesssim I_\alpha(|f|)(x)$$

(see [8]). Hence by the aforementioned theorems we have

$$\|L^{-\alpha/2}f\|_{M_{q,\varphi_2}} \lesssim \|I_\alpha(|f|)\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}.$$

Property (4.1) is satisfied for large classes of differential operators (see, for example [4]). In [4] also other examples of operators which are estimates from above by Riesz potentials are given. In these case Theorem 3.5 is also applicable for proving boundedness of those operators from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$.

## References

[1]. Adams D.R. *A note on Riesz potentials*, Duke Math. 1975, 42, pp.765-778.

[2]. Burenkov V.I., Guliyev H.V., Guliyev V.S. *Necessary and sufficient conditions for the boundedness of the fractional maximal operator in the local Morrey-type spaces,* Dokl. Akad. Nauk 2006, 74 (1), pp. 540-544.

[3]. Burenkov V.I., Guliyev H.V., Guliyev V.S., *Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces,* J. Comput. Appl. Math. 2007, 208(1), pp. 280-301.

[4]. Burenkov V.I., Guliyev V.S., *Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces,* Potential Anal. 2009, 30 (3), pp. 211-249.

[5]. Burenkov V.I., Gogatishvili A., Guliyev V.S., Mustafayev R.Ch. *Boundedness of the fractional maximal operator in local Morrey-type spaces,* Complex Var. Elliptic Equ. 2010, 55(8-10), pp. 739-758.

[6]. Burenkov V.I., Gogatishvili A., Guliyev V.S., Mustafayev R.Ch. *Boundedness of the fractional maximal operator in local Morrey-type spaces,* Potential Anal. 2011, 35, No 1, pp. 67-87.

[7]. Ding Y., Yang D., Zhou Z., *Boundedness of sublinear operators and commutators on* $L^{p,\omega}(\mathbb{R}^n)$, Yokohama Mathematical, 1998, 46, pp. 15-27.

[8]. Duong, X.T., Yan, L.X. On commutators of fractional integrals. Proc. Amer. Math. Soc. 2004, 132 (12), pp. 3549-3557.

[9]. Fefferman C. *The uncertainty principle,* Bull. Amer. Math. Soc. 1983, 9, pp.129206.

[10]. Guliyev V.S. *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$. Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 1994, 329 p. (in Russian)

[11]. Guliyev V.S. *Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications*, Casioglu, Baku, 1999, 332 p. (in Russian)

[12]. Guliyev V.S. *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. 2009, Art. ID 503948, 20 p.

[13]. Guliyev V.S., Hasanov J.J., Stefan Samko, *Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces*, Math. Scand. 2010, 197 (2), pp. 285-304.

[14]. Guliyev V.S., Aliyev S.S., Karaman T., Shukurov P., *Boundedness of sublinear operators and commutators on generalized Morrey spaces.* Integral Equations and Operator Theory, 2011, 71 (3), pp. 1-29.

[15]. Kurata K., Sugano S. *A remark on estimates for uniformly elliptic operators on weighted $L_p$ spaces and Morrey spaces,* Math. Nachr. 2000, 209, pp. 137-150.

[16]. Lin Y. *Strongly singular Calderón-Zygmund operator and commutator on Morrey type spaces*, Acta Math. Sin. (Engl. Ser.) 2007, 23 (11), pp. 2097-2110.

[17]. Li H.Q. *Estimations $L_p$ des operateurs de Schrödinger sur les groupes nilpotents,* J. Funct. Anal. 1999, 161, pp. 152-218.

[18]. Yu Liu *The weighted estimates for the operators $V^\alpha(-\Delta_G + V)^{-\beta}$ and $V^\alpha \nabla_G(-\Delta_G + V)^{-\beta}$ on the stratified Lie group* $\mathbb{G}$, J. Math. Anal. Appl. 2009, 349, pp. 235-244.

[19]. Lu G.Z. *A FeffermanPhong type inequality for degenerate vector fields and applications,* Panamer. Math. J. 1996, 6, pp.37-57.

[20]. Mizuhara T. *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo 1991, pp. 183-189.

[21]. Morrey C.B. *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. 1938, 43, pp. 126-166.

[22]. Nakai E. *Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. 1994, 166, pp. 95-103.

[23]. Peetre J. *On the theory of $M_{p,\lambda}$*, J. Funct. Anal. 1969, 4, pp.71-87.

[24]. Persson L., Samko N. *Weighted Hardy and potential operators in the generalized Morrey spaces*, J. Math. Anal. Appl. 2011, in press.

[25]. Shen Z.W. *$L_p$ estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) 1995, 45, pp. 513-546.

[26]. Soria F., Weiss G. *A remark on singular integrals and power weights*, Indiana Univ. Math. J. 1994, 43, pp.187-204.

[27]. Stein E.M. *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. 1958, 88, pp. 430-466.

[28]. Stein, E.M. *Singular integrals and differentiability of functions.* Princeton University Press, Princeton, NJ, 1970.

[29]. Stein E.M. *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton NJ, 1993.

[30]. Sugano S. *Estimates for the operators $V^\alpha(-\Delta+V)^{-\beta}$ and $V^\alpha\nabla(-\Delta+V)^{-\beta}$ with certain nonnegative potentials $V$*, Tokyo J. Math. 1998, 21, pp. 441-452.

[31]. Torchinsky A. *Real Variable Methods in Harmonic Analysis*, Pure and Applied Math. 123, Academic Press, New York, 1986.

[32]. Torchinsky A. and Wang S. *A note on the Marcinkiewicz integral*, Colloq. Math. 1990, 60/61, pp. 235-243.

[33]. Zhong J.P. *Harmonic analysis for some Schrödinger type operators,* PhD thesis, Princeton University, 1993.

**Vagif S. Guliyev, Parviz S. Shukurov**
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, B.Vahabzade str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 538 62 17 (apt.).

Received July 12, 2011; Revised October 05, 2011