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OPTIMAL STABILIZATION FOR A WEAK NONLINEAR HYPERBOLIC EQUATION WITH PHASE RESTRICTION

Abstract

In the paper, we consider an optimal stabilization problem for a weak nonlinear hyperbolic equation with phase restrictions. Using the penalty method, we get the necessary condition of optimality for an approximate optimal control problem.

The optimal control problems for the systems described by nonlinear oscillatory and wave equations when no phase restrictions exist, have been studied sufficiently well (see, [1]-[4]).

However, only the paper [5] in which the model equation from the book [7] is considered, has been devoted to optimal stabilization problems for hyperbolic type nonlinear equations with pointwise phase restrictions. In the present paper, we consider a problem of indicated type. The penalty method is used for its solution, and its convergence is proved. A necessary condition of optimality is obtained for the approximate optimal control problem. The solution of the approximate problem for sufficiently small value of the penalty parameter is taken in the place of approximate solution of the problem.

We consider the equation

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) = f_1(x,t,u) + f_2(x,t)\upsilon, \quad (x,t)\in Q_T, \qquad (1)$$

with boundary conditions

$$u(x,0) = u_0(x), \frac{\partial u(x,0)}{\partial t} = u_1(x), \quad x\in\Omega, \qquad (2)$$

$$u = 0, \quad (x,t)\in\Sigma, \qquad (3)$$

where $\Omega\subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$ $Q_T = \Omega\times(0,T)$ is a cylinder $0<T<\infty$, $\sum = \partial\Omega\times(0,T)$ is a lateral surface of the cylinder Q_T, $u_0\in H_0^1(\Omega)$, $u_1\in L_2(\Omega)$, $f_1(x,t,u)$ is a continuous function in $Q_T\times R$ and has a bounded derivative with respect to u for all $(x,t,u)\in Q_T\times R$, $f_2(x,t)$ is a continuous function in Q_T, the functions $a_{ij}(x,t)$, $\frac{\partial a_{ij}(x,t)}{\partial t} i,j = 1,n$ are continuous in Q_T and for all $(x,t)\in Q_T$ satisfy the condition

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \ge \alpha\sum_{i=1}^{n}\xi_i, \quad \alpha = const > 0, \quad \forall\xi = (\xi_1,...,\xi_n)\in R^n.$$

Then, by means of the Galerkin method [see 6,7], one can prove that for any control υ from the space $V = L_2(Q_T)$, problem (1)-(3) has a unique solution $u = u(\upsilon)$ from the space

$$Y = \left\{u \middle| u\in L_\infty\left([0,T];H_0^1(\Omega)\right), \frac{\partial u}{\partial t}\in L_\infty\left([0,T];L_2(\Omega)\right)\right\}.$$

The control is selected from the convex closed subset V_{ad} of the space V. The system's state should belong to the convex closed subset Y_{ad} of the space Y. The final state of the system is zero, i.e. the following conditions are fulfilled:

$$u(x,T) = 0, \quad \frac{\partial u(x,T)}{\partial t} = 0, x \in \Omega. \tag{4}$$

The control $v \in V_{ad}$ is assumed to be admissible if the appropriate state of the system (1)-(3) is the element of the set Y_{ad} and satisfies the final conditions (4). Everywhere in the sequel, it is supposed that the set U of admissible controls is not empty. The optimal stabilization problem consists of finding such an admissible control that minimizes the functional

$$I(v) = \int_{Q_T} f(x,t,v,u(v))\, dxdt, \tag{5}$$

where F is a given function in $\overline{Q}_T \times R \times R$.

It is assumed that F is a continuous function with respect to all the arguments, convex with respect to the fourth argument and coercive with respect to the third and fourth argument, growing not rapidly than the quadratic function with respect to these arguments, and continuously differentiable with respect to them.

Theorem 1. *The optimal stabilization problem is solvable.*

Proof. Let $\{v_k\} \in V_{ad}$ be a minimizing sequence, i.e.

$$\lim_{k\to\infty} I(v_k) = \inf_{v\in V_{ad}} I(v). \tag{6}$$

Then for the solution of problem (1)-(3) $u_k = u(v_k)$, $k = 1, 2, ...$, corresponding to the controls $v_k \in V_{ad}$, the following condition is fulfilled:

$$u_k \in Y_{ad} \text{ and } u_k(x,T) = 0, \quad \frac{\partial u_k(x,T)}{\partial t} = 0, \ k = 1, 2, \dots . \tag{7}$$

From the conditions on the function $F(x,t,v,u)$ in functional (5) it follows that the sequence $\{v_k\}$ is bounded in $L_2(Q_T)$, i.e.

$$\|v_k\|_{L_2(Q_T)} \le c, k = 1, 2, ..., \tag{8}$$

here and in the sequel, c denotes various constants independent of admissible controls and estimated quantities.

Taking into account (8) and the conditions on the data of problem (1)-(3), by means of the results of the works [6,7], for its solution we get the estimation

$$\|u_k\|_Y \le c, \quad k = 1, 2, \dots . \tag{9}$$

Hence and from (8) it follows that from $\{v_k, u_k\}$ we can select such a subsequence, denote it by $\{v_k, u_k\}$, that as $k \to \infty$ it holds

$$v_k \to v_0 \text{ weakly in } L_2(Q_T),$$

$$u_k \to u_0 \ * -\text{weakly in } L_\infty\left(0,T; H_0^1(\Omega)\right),$$

$$\frac{\partial u_k}{\partial t} \to \frac{\partial u_0}{\partial t} \quad * -\text{weakly in} \quad L_\infty\left(0,T;H_0^1\left(\Omega\right)\right),$$

and by the imbedding theorem [6]

$$u_k \to u_0 \quad \text{strongly in} \quad L_2\left(Q_T\right).$$

As V_{ad} and Y_{ad} are convex, closed sets in V and Y respectively, and they are weakly closed, then $\upsilon_0 \in V_{ad}$ and $u_0 \in Y_{ad}$, moreover $u_0 = u\left(\upsilon_0\right)$ i.e. u_0 is a solution of problem (1)-(3) that corresponds to the control $\upsilon_0 \in V_{ad}$. Furthermore, it follows from (7) that

$$u_0\left(x,T\right) = 0, \frac{\partial u_0\left(x,T\right)}{\partial t} = 0.$$

Under the conditions imposed on $F\left(x,t,\upsilon,u\right)$, functional (5) is weakly lower semicontinuous, i.e.

$$\lim_{k\to\infty} I\left(\upsilon_k\right) = \lim_{k\to\infty} \int\limits_{Q_T} F\left(x,t,\upsilon_k,u\left(\upsilon_k\right)\right) dxdt \geq$$

$$\geq \int\limits_{Q_T} F\left(x,t,\upsilon_0,u_0\right) dxdt = I\left(\upsilon_0\right). \tag{9}$$

Thus, it follows from (6) and (9) that $\upsilon_0 \in V_{ad}$ is an optimal control in problem (1)-(5), $u_0 = u\left(\upsilon_0\right)$ is the appropriate solution of problem (1)-(3).

The theorem is proved.

For deriving necessary conditions of optimality, in the present paper we use a modified variant of the penalty method [5,8]. Define the functional

$$I_k\left(\upsilon,u\right) = \int\limits_{Q_T} F\left(x,t,\upsilon,u\right) dxdt +$$

$$+\frac{1}{2\varepsilon_k}\int\limits_{Q_T}\left[\frac{\partial^2 u}{\partial t^2} - Lu - f_1\left(x,t,u\right) - f_2\left(x,t\right)\upsilon\right]^2 dxdt +$$

$$+\frac{1}{2\varepsilon_k}\int\limits_{Q_T}\left[u^2\left(x,T\right) + \left(\frac{\partial u\left(x,T\right)}{\partial t}\right)^2\right] dx, \tag{10}$$

where $\varepsilon_k > 0$ and $\varepsilon_k \to 0$ as $k \to \infty$,

$$Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\left(x,t\right)\frac{\partial u}{\partial x_j}\right), \quad \text{moreover,} \quad \upsilon \in V_{ad}, \frac{\partial^2 u}{\partial t^2} - Lu \in L_2\left(Q_T\right).$$

The approximate problem consists of minimization of the functional I_k on the set $V_{ad} \times Y_{ad}$ subject to the initial conditions (2).

Theorem 2. *The approximate problem is solvable.*

Proof. Let $\{\upsilon_m, u_m\}$ be a minimizing sequence, i.e.

$$\lim_{m\to\infty} I_k\left(\upsilon_m,u_m\right) = \inf_{V_{ad}\times Y_{ad}} I_k\left(\upsilon,u\right), \quad k = 1,2,\ldots.$$

From definition of functional (10) it follows that

$$\|\upsilon_m\|_{L_2(Q_T)} \leq c, \|u_m\|_{L_2(Q_T)} \leq c.$$

The form of the functional (10) yields

$$\frac{\partial^2 u_m}{\partial t^2} - Lu_m = f_1(x,t,u_m) + f_2(x,t)\,\upsilon_m + g_m(x,t),$$

where $g_m(x,t)$ is such a function that $\|g_m\|_{L_2(Q_T)} \leq c$.

Hence, taking into account the conditions on the data of problem (1)-(3) by the standard reasonings from theory of boundary value problems [6,7] we get

$$\|u_m\|_{L_\infty\left(0,T;H_0^1(\Omega)\right)} + \left\|\frac{\partial u_m}{\partial t}\right\|_{L_\infty(0,T;L_2(\Omega))} \leq c.$$

Then from the sequence $\{\upsilon_m, u_m\}$ we can select such a subsequence, denote it again by $\{\upsilon_m, u_m\}$, that as $m \to \infty$

$$\upsilon_m \to \overline{\upsilon} \text{ weakly in } L_2(Q_T),$$

$$u_m \to \overline{u} * -\text{weakly in } L_\infty\left(0,T;H_0^1(\Omega)\right),$$

$$\frac{\partial u_m}{\partial t} \to \frac{\partial \overline{u}}{\partial t} * -\text{weakly in } L_\infty(0,T;L_2(\Omega)),$$

$$u_m \to \overline{u} \text{ strongly in } L_2(Q_T).$$

The functional $I_k(\upsilon,u)$ is weakly lower-semicontinuous in $V \times Y$. Therefore,

$$\underline{\lim}_{m\to\infty} I_k(\upsilon_m,u_m) \geq I_k(\overline{\upsilon},\overline{u}),$$

i.e. $\{\overline{\upsilon},\overline{u}\} \in V_{ad} \times Y_{ad}$ delivers minimum to the functional (10).

Now, establish the form of the convergence that corresponds to the weak form of the approximate solution of optimization problems.

Definition [5]. *The point u is said to be a weak approximate solution of the problem of minimization of the functional J on the subset U of topological space if for sufficiently small vicinity O of this set and sufficiently small positive number δ, the inclusion $u \in O$ and the inequality $J(u) \leq \inf_{v\in U} J(v) + \delta$ are valid.*

According to the given definition, the weakly approximate solution, generally speaking, lies beyond the set U characterizing the restrictions on the system, but sufficiently close to some of its elements. Thus, the given restrictions are assumed to be fulfilled only with some degree of accuracy. And the value of the functional in the considered point turns out to be sufficiently close to its low boundary on the given set.

Let the pair (υ_k, u_k) be a solution of the approximate problem. The following theorem is valid.

Theorem 3. *As $k \to \infty$, it holds the convergence*

$$\upsilon_k \to \upsilon_0 \quad \textit{weakly in } V, u \to u(\upsilon_0) \textit{ weakly in } Y,$$

$$u_k|_{t=T} \to 0 \quad \textit{strongly} \ \ \textit{in } L_2(\Omega), \ \left.\frac{\partial u_k}{\partial t}\right|_{t=T} \to 0 \ \textit{strongly in } L_2(\Omega),$$

and the inclusions $\upsilon_0 \in V_{ad}, \quad u(\upsilon_0) \in Y_{ad}$ *are true.*

Proof. Since for $\upsilon = \upsilon_0, \quad u = u(\upsilon_0)$, the characterizing "penalty" addends in expression (10) equal zero, then

$$I_k(\upsilon_k, u_k) \le I_k(\upsilon_0, u(\upsilon_0)) = I(\upsilon_0) = c.$$

Then from this estimation taking into account the conditions on the function $F(x,t,\upsilon,u)$ we get $\|\upsilon_k\|_{L_2(Q_T)} \le c$,

$$\frac{1}{\varepsilon_k}\left\|\frac{\partial^2 u_k}{\partial t^2} - Lu_k - f_1(x,t,u_k) - f_2(x,t)\upsilon_k\right\|^2_{L_2(Q_T)} \le c,$$

$$\frac{1}{\varepsilon_k}\|u_k(x,T)\|_{L_2(\Omega)} \le c, \quad \frac{1}{\varepsilon_k}\left\|\frac{\partial u_k(x,T)}{\partial t}\right\|_{L_2(Q_T)} \le c.$$

Hence it follows that

$$\left\|\frac{\partial^2 u_k}{\partial t^2} - Lu_k - f_1(x,t,u_k) - f_2(x,t)\upsilon_k\right\|_{L_2(Q_T)} \le c\sqrt{\varepsilon_k},$$

$$\|u_k(x,T)\|_{L_2(\Omega)} \le c\sqrt{\varepsilon_k}, \quad \left\|\frac{\partial u_k(x,T)}{\partial t}\right\|_{L_2(\Omega)} \le c\sqrt{\varepsilon_k}.$$

Therefore, as

$$\upsilon_k \to \upsilon_0 \ \text{ weakly in } \ L_2(Q),$$

$$u_k \to u(\upsilon_0) \ * -\text{weakly in } \ Y,$$

$$u_k|_{t=T} \to 0 \ \text{ strongly in } \ L_2(Q),$$

$$\left.\frac{\partial u_k}{\partial t}\right|_{t=T} \to 0 \quad \text{strongly in } \ L_2(Q),$$

and by convexity and closeness of the sets V_{ad} and Y_{ad} they are weakly closed, i.e. $\upsilon_0 \in V_{ad}$, $u(\upsilon_0) \in Y_{ad}$. The theorem is proved.

As is known, the necessary condition of minimum at the point υ of the functional J on the set U is the variation inequality

$$\left\langle J'(\upsilon), w - \upsilon\right\rangle \ge 0, \quad \forall w \in U. \tag{11}$$

The following theorem is valid.

Theorem 4. *The functional* $I_k(\upsilon,u)$ *at the point* $y = (\upsilon_k, u_k)$ *has the Gato derivative*

$$I'_k(y) = I'_{k\upsilon}(\upsilon_k, u_k), \ I'_{ku}(\upsilon_k, u_k),$$

characterized by the equalities

$$I'_{k\upsilon}(\upsilon_k, u_k) = F_\upsilon(\upsilon_k, u_k) + p_k,$$

$$I'_{ku}(\upsilon_k, u_k) = F_u(\upsilon_k, u_k) + r_k,$$

where $F_\upsilon(\upsilon_k,u_k)$ and $F_u(\upsilon_k,u_k)$ are the partial derivatives of the function $F(x,t,\upsilon,u)$ with respect to the third and fourth argument at the point (υ_k,u_k), p_k and r_k satisfy the relation

$$\frac{\partial^2 u_k}{\partial t^2}-Lu_k=f_1(x,t,u_k)+f_2(x,t)\upsilon_k+\varepsilon_k p_k,\quad (x,t)\in Q_T, \tag{12}$$

$$\frac{\partial^2 p_k}{\partial t^2}-Lp_k-\frac{\partial f_1(x,t,u_k)}{\partial u}p_k=r_k,\quad (x,t)\in Q_T, \tag{13}$$

$$p_k(x,T)=-\frac{1}{\varepsilon_k}\frac{\partial u_k(x,T)}{\partial t},\ \ \frac{\partial p_k(x,T)}{\partial t}=\frac{1}{\varepsilon_k}u_k(x,T),\quad x\in\Omega, \tag{14}$$

$$p_k=0,\quad (x,t)\in\Sigma. \tag{15}$$

Proof. Calculate the first variation of the functional $I_k(\upsilon,u)$ at the point $y=(\upsilon_k,u_k)$. By definition, for any $w\in V_{ad}$ and for any $z\in Y_{ad}$ we have:

$$\delta I_{k\upsilon}(\upsilon_k,u_k;w-\upsilon_k)=\frac{d}{d\lambda}I_k(\upsilon_k+\lambda(w-\upsilon_k),u_k)\Big|_{\lambda=0}=$$

$$=\int\limits_{Q_T}F_\upsilon(x,t,\upsilon_k,u_k)(w-\upsilon_k)\,dxdt+$$

$$+\frac{1}{\varepsilon_k}\int\limits_{Q_T}\left(\frac{\partial^2 u_k}{\partial t^2}-Lu_k-f_1(x,t,u_k)+f_2(x,t)\upsilon_k\right)(w-\upsilon_k)\,dxdt, \tag{16}$$

$$\delta I_{ku}(\upsilon_k,u_k;z-\upsilon_k)=\frac{d}{d\lambda}I_k(\upsilon_k,u_k+\lambda(z-\upsilon_k))\Big|_{\lambda=0}=$$

$$=\int\limits_{Q_T}F_u(x,t,\upsilon_k,u_k)(z-\upsilon_k)\,dxdt+$$

$$+\frac{1}{\varepsilon_k}\int\limits_{Q_T}\left(\frac{\partial^2 u_k}{\partial t^2}-Lu_k-f_1(x,t,u_k)-f_2(x,t)\upsilon_k\right)\times$$

$$\times\left(\frac{\partial^2(z-u_k)}{\partial t^2}-L(z-u_k)-\frac{\partial f_1(x,t,u_k)}{\partial u}(z-u_k)\right)dxdt+$$

$$+\frac{1}{\varepsilon_k}\int\limits_{\Omega}\left[u_k(x,T)(z(x,T)-u_k(x,T))+\right.$$

$$\left.+\frac{\partial u_k(x,T)}{\partial t}\left(\frac{\partial z(x,T)}{\partial t}-\frac{\partial u_k(x,T)}{\partial t}\right)\right]dx. \tag{17}$$

If by p_k we denote

$$p_k=\frac{1}{\varepsilon_k}\left(\frac{\partial^2 u_k}{\partial t^2}-Lu_k-f_1(x,t,u_k)-f_2(x,t)\upsilon_k\right),$$

it follows from (17) that

$$\delta I_{ku}(\upsilon_k,u_k;z-\upsilon_k)=\int\limits_{Q_T}F_u(x,t,\upsilon_k,u_k)(z-u_k)\,dxdt+$$

$$+\int\limits_{Q_T}\left[\frac{\partial^2 p_k}{\partial t^2}-Lp_k-\frac{\partial f_1(x,t,u_k)}{\partial u}p_k\right](z-u_k)\,dxdt+$$

$$+\int\limits_{Q_T}\left(\frac{1}{\varepsilon_k}u_k(x,T)-\frac{\partial p_k(x,T)}{\partial t}\right)(z(x,T)-u_k(x,T))\,dx+$$

$$+\int\limits_{Q_T}\left(\frac{1}{\varepsilon_k}\frac{\partial u_k(x,T)}{\partial t}+p_k(x,T)\right)\left(\frac{\partial z(x,T)}{\partial t}-\frac{\partial u_k(x,T)}{\partial t}\right)dx. \qquad (17')$$

If by r_k we denote

$$r_k\equiv\frac{\partial^2 p_k}{\partial t^2}-Lp_k-\frac{\partial f_1(x,t,u_k)}{\partial u}p_k,$$

then from (17') we get

$$\delta I_{ku}(\upsilon_k,u_k;z-\upsilon_k)=\int\limits_{Q_T}\left[F_u(x,t,\upsilon_k,u_k)+r_k\right](z-u_k)\,dxdt+$$

$$+\int\limits_{\Omega}\left(\frac{1}{\varepsilon_k}u_k(x,T)-\frac{\partial p_k(x,T)}{\partial t}\right)(z(x,T)-u_k(x,T))\,dx+$$

$$+\int\limits_{\Omega}\left(\frac{1}{\varepsilon_k}\frac{\partial u_k(x,T)}{\partial t}+p_k(x,T)\right)\left(\frac{\partial z(x,T)}{\partial t}-\frac{\partial u_k(x,T)}{\partial t}\right)dx. \qquad (18)$$

Here, taking into account definition 1[5] of the weak approximate solution of the problem, we get the validity of conditions (14). Therefore, it follows from (16) and (18) that the functional $I_k(\upsilon,u)$ at the point $y=(\upsilon_k,u_k)$ has the Gato derivative

$$I'_k(y)=\left(I'_{k\upsilon}(\upsilon_k,u_k),I'_{ku}(\upsilon_k,u_k)\right),$$

moreover,

$$I'_{k\upsilon}(\upsilon_k,u_k)=F_\upsilon(x,t,\upsilon_k,u_k)+p_k,$$

$$I'_{ku}(\upsilon_k,u_k)=F_u(x,t,\upsilon_k,u_k)+r_k.$$

The theorem is proved.

Using theorem 4and relation (11), we get a necessary condition of optimality for the approximate problem.

Theorem 5. *The solution of the approximate problem is characterized by the system including variational inequalities*

$$\int\limits_{Q_T}\left[F_\upsilon(x,t,\upsilon_k,u_k)+p_k\right](w-\upsilon_k)\,dxdt\geq 0,\quad \forall w\in V_{ad},$$

$$\int\limits_{Q_T}\left[F_u(x,t,\upsilon_k,u_k)+r_k\right](z-u_k)\,dxdt\geq 0,\quad \forall z\in Y_{ad},$$

equation (12) with boundary conditions

$$u_k(x,0) = u_0(x), \quad \frac{\partial u_k(x,0)}{\partial t} = u_1(x), \quad x \in \Omega$$

$$u_k = 0, \quad (x,t) \in \Sigma$$

and equation (13) with boundary conditions (14) and (15). Since in equation (13) the right side belongs to $L_2(Q_T)$ and in conditions (14) $\frac{\partial u_k(x,T)}{\partial t} \in L_2(\Omega)$, under the solution of boundary value problem (13)-(15) we understand the weak classic solution, i.e. the solution from $L_2(Q_T)$.

Thus, according to the results obtained above, the solution of the approximate problem for a sufficiently large number k may be selected as a weakly approximate solution of the optimal stabilization problem.

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