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TWO–WEIGHTED INEQUALITY FOR PARABOLIC SINGULAR INTEGRAL OPERATORS IN VECTOR-VALUED LEBESGUE SPACES

Abstract

In this paper, the author establishes the boundedness in weighted $L_{p,\omega}(\mathbb{R}^{n+1};E)$ spaces on $\mathbb{R}^{n+1}$ with parabolic singular integral operators. The conditions of these theorems are satisfied by many important operators in analysis. Sufficient conditions on weighted functions ω and ω_1 are given so that certain parabolic singular integral operator is bounded from the weighted Lebesgue spaces $L_{p,\omega}(\mathbb{R}^{n+1};E)$ to $L_{p,\omega_1}(\mathbb{R}^{n+1};E)$.

In this paper, the author establishes the boundedness in weighted $L_{p,\omega}(\mathbb{R}^{n+1};E)$ spaces on $\mathbb{R}^{n+1}$ with parabolic singular integral operators. The conditions of these theorems are satisfied by many important operators in analysis. Sufficient conditions on weighted functions ω and ω_1 are given so that certain parabolic singular integral operator is bounded from the weighted Lebesgue spaces $L_{p,\omega}(\mathbb{R}^{n+1};E)$ to $L_{p,\omega_1}(\mathbb{R}^{n+1};E)$. The parabolic singular integral operators by many interesting operators in harmonic analysis, such as the parabolic Calderon-Zygmund operators, parabolic maximal operators, parabolic Hardy-Littlewood maximal operators, and so on. See [13] for details.

Note that singular integral operators with Calderon-Zygmund kernels were proved in [11] and for singular integral operators, defined on homogeneous groups, in [12], [7] (see also [6]).

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x'=(x_1,\dots,x_n)$, $|x'|^2=\sum\limits_{i=1}^{n}x_i^2$ and denote by $x=(x',t)=(x_1,\dots,x_n,t)$ a point in $\mathbb{R}^{n+1}$. An almost everywhere positive and locally integrable function $\omega:\mathbb{R}^{n+1}\to\mathbb{R}^n$ will be called a weight.

Let us now endow $\mathbb{R}^{n+1}$ with the following parabolic metric introduced by Fabes and Riviére in [4]:

$$d(x,y)=\rho(x-y),\ \text{where}\ \rho(x)=\sqrt{\frac{|x'|^2+\sqrt{|x'|^4+4t^2}}{2}}. \tag{1}$$

A ball with respect to the metric d centered at zero and of radius r is the ellipsoid

$$\mathcal{E}_r(0)=\left\{x\in\mathbb{R}^{n+1}:\frac{|x'|^2}{r^2}+\frac{t^2}{r^4}<1\right\}.$$

Obviously, the unit sphere with respect to this metric coincides with the unit sphere in $\mathbb{R}^{n+1}$, i.e.,

$$\partial\mathcal{E}_1(0)\equiv\Sigma_{n+1}=\left\{x\in\mathbb{R}^{n+1}:|x|=\left(\sum_{i=1}^{n}x_i^2+t^2\right)^{\frac{1}{2}}=1\right\}.$$

Let
$$\tilde{d}(x,y)=\tilde{\rho}(x-y),\ \tilde{\rho}(x)=\max\{|x'|,t^{\frac{1}{2}}\}.$$

I be a parabolic cylinder centered at some point x of radius r, that is, $I\equiv I_r(x)=\{y=(y',\tau)\in\mathbb{R}^{n+1}:|x'-y'|<r,\ |t-\tau|<r^2\}$. It is easy to see that for any ellipsoid $\mathcal{E}_r$ there exist cylinders $\underline{I}$ and $\overline{I}$ with measures comparable with r^{n+2} and such that $\underline{I}\subset\mathcal{E}_r\subset\overline{I}$. Obviously, this implies an equivalence of both metrics and the topologies induced by them. Later we shall use this equivalence without making reference to, except if required.

Let E-Banach space norm of the element $a\in E$ be determined by $\|a\|_E$.

We say that a Banach space E is ζ-convex, or convex be Burkholder [3] if there exists a symmetric function of $\zeta(a,b)$ on $E\times E$, convex in each variable and satisfying the conditions

$$\zeta(0,0)>0,\ \ \zeta(a,b)\le\|a,b\|_E,\ \ \|a\|_E=\|b\|_E=1.$$

Suppose that ω is a positive, measurable, and real function defined in $\mathbb{R}^{n+1}$, i.e., is a weight function. By $L_{p,\omega}(\mathbb{R}^{n+1};E)$ we denote a space of measurable E-valued functions $f(x)$ on $x\in\mathbb{R}^{n+1}$ with finite norm

$$\|f\|_{L_{p,\omega}(\mathbb{R}^n;E)}=\left(\int\limits_{\mathbb{R}^n}\|f(x)\|_E^p\omega(x)dx\right)^{1/p},\quad 1\le p<\infty.$$

Theorem 1. *[4] If the E-valued function $f:\mathbb{R}^n\to E$ integrable by Bochner, then*
$$\left\|\int\limits_{\mathbb{R}^n}f(x)dx\right\|_E\le\int\limits_{\mathbb{R}^n}\|f(x)\|_E\,dx.$$

It is worth noting that $\rho(x)$ has been employed in the study of singular integral operators with Calderon-Zygmund kernels of mixed homogeneity (see [4]).

Definition 1. *A function K defined on $\mathbb{R}^{n+1}\times(\mathbb{R}^{n+1}\setminus\{0\})$ is said to be a parabolic Calderon-Zygmund (PCZ) kernel in $\mathbb{R}^{n+1}$ if*

i) $K(x,\cdot)\in C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$ for almost every $x\in\mathbb{R}^{n+1}$;

ii) $K(x,\delta_r(y))=r^{-(n+2)}K(x,y)$ for each $r>0$, $y\in\mathbb{R}^{n+1}\setminus\{0\}$ for almost every $x\in\mathbb{R}^{n+1}$, where $\delta_r(y)=(ry',r^2\tau)$;

iii) $\int\limits_{\Sigma_{n+1}}K(x,y)d\sigma_y=0$ for almost every $x\in\mathbb{R}^{n+1}$, where $d\sigma$ is the element of area of the Σ_{n+1};

iv) for every multi-index β, vrai $\sup\limits_{x\in\mathbb{R}^{n+1}}\sup\limits_{y\in\Sigma_{n+1}}\left|\left(\frac{\partial}{\partial y}\right)^\beta K(x,y)\right|<\infty$.

Let K be a parabolic Calderon-Zygmund kernel and T be the integral operator

$$Tf(x)\equiv\int\limits_{\mathbb{R}^n}\int\limits_0^t K(x,x-y)f(y)dy=\lim_{\varepsilon\to0}\int\limits_{\mathbb{R}^n}\int\limits_0^{t-\varepsilon}K(x,x-y)f(y)dy.\qquad(2)$$

Theorem 2. *Let E be ζ-convex Banach lattice. Let K be a parabolic Calderon-Zygmund kernel and T be the integral operator* (2). *Moreover, let $p\in(1,\infty)$, $\omega(t)$*

be a weight function on $(0,\infty)$, $\omega_1(t)$ *be a positive increasing function on* $(0,\infty)$ *and the following conditions be satisfied:*

(a) *there exists a constant* $b>0$ *such that* $\omega_1(2t)\le b\omega(t)$ *for a.e.,* $t>0$,

$$(b) \qquad \mathcal{A}\equiv\sup_{\tau>0}\left(\int\limits_{2\tau}^{\infty}\omega_1(t)t^{-p}d\tau\right)\left(\int\limits_0^{\tau}\omega^{1-p'}(t)dt\right)^{p-1}<\infty.$$

Then there exists a constant $c=c(n,p,\omega,\omega_1,K)$ *such that for all* $f\in L_{p,\omega}(\mathbb{R}^{n+1}_+;E)$

$$\int\limits_{\mathbb{R}^{n+1}_+}\|Tf(x)\|^p_E\omega_1(t)dx'dt\le c\int\limits_{\mathbb{R}^{n+1}_+}\|f(x)\|^p_E\omega(t)dx'dt. \qquad (3)$$

Proof. Suppose that $f\in L_{p,\omega}(\mathbb{R}^{n+1}_+;E)$ and ω_1 are positive increasing functions on $(0,\infty)$ that satisfy the condition (a), (b).

Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t)=\omega_1(0+)+\int\limits_0^t\psi(\tau)d\tau, \quad t>0,$$

where $\omega_1(0+)=\lim_{t\to0}\omega_1(t)$ and $\psi(t)\ge0$ on $(0,\infty)$. In fact there exists a sequence of increasing absolutely continuous functions ϖ_n such that $\varpi_n(t)\le\omega_1(t)$ and $\lim\limits_{n\to\infty}\varpi_n(t)=\omega_1(t)$ for any $t\in(0,\infty)$ (see [1], [6], [8], [9]).

We have

$$\int\limits_{\mathbb{R}^{n+1}_+}\|Tf(x)\|^p_E\omega_1(t)dx'dt=\omega_1(0+)\int\limits_{\mathbb{R}^{n+1}_+}\|Tf(x)\|^p_Edx+$$

$$+\int\limits_{\mathbb{R}^{n+1}_+}\|Tf(x)\|^p_E\left(\int\limits_0^t\psi(\tau)d\tau\right)dx=I_1+I_2.$$

If $\omega_1(0+)=0$, then $J_1=0$. If $\omega_1(0+)\ne0$ by the boundedness of T in $L_p(\mathbb{R}^{n+1};E)$ by the (a), we have

$$I_1\le c\omega_1(0+)\int\limits_{\mathbb{R}^{n+1}_+}\|f(x)\|^p_Edx\le$$

$$\le c\int\limits_{\mathbb{R}^{n+1}_+}\|f(x)\|^p_E\omega_1(2t)dx'dt\le c\int\limits_{\mathbb{R}^{n+1}_+}\|f(x)\|^p_E\omega(t)dx'dt.$$

After changing the order of integration in I_2, we have

$$I_2=\int\limits_0^{\infty}\psi(\lambda)\left(\int\limits_{\mathbb{R}^n}\int\limits_{\lambda}^{\infty}\|Tf(x)\|^p_Edx'dt\right)d\lambda\le$$

$$\leq c\int_0^\infty \psi(\lambda)\left(\int_{\mathbb{R}^n}\int_\lambda^\infty \left\|\int_{\mathbb{R}^n}\int_{\lambda/2}^\infty K(x,x-y)f(y)dy\right\|_E^p dx\right)d\lambda+$$

$$+c\int_0^\infty \psi(\lambda)\left(\int_{\mathbb{R}^n}\int_\lambda^\infty \left\|\int_{\mathbb{R}^n}\int_0^{\lambda/2} K(x,x-y)f(y)dy\right\|_E^p dx\right)d\lambda=$$

$$=I_{21}+I_{22}.$$

Using the boundeedness of T in $L_p(\mathbb{R}^{n+1};E)$, we obtain

$$I_{21}\leq c\int_0^\infty \psi(t)\left(\int_{\mathbb{R}^n}\int_{\lambda/2}^\infty \|f(y',\tau)\|_E^p dy'd\tau\right)dt=$$

$$=c\int_{\mathbb{R}^{n+1}_+}\|f(y)\|_E^p\left(\int_0^{2\tau}\psi(t)dt\right)dy\leq c\int_{\mathbb{R}^{n+1}_+}\|f(y)\|_E^p\omega_1(2\tau)dy'd\tau\leq$$

$$\leq c_1\int_{\mathbb{R}^{n+1}_+}\|f(y)\|_E^p\omega(\tau)dy'd\tau.$$

Let us estimate I_{22}. For $t>\lambda$ and $0\leq\tau\leq\lambda/2$, we have $t/2\leq|t-\tau|\leq 3t/2$, and so

$$I_{22}\leq c\int_0^\infty \psi(\lambda)\left(\int_{\mathbb{R}^n}\int_\lambda^\infty\left(\int_{\mathbb{R}^n}\int_0^{\lambda/2}\frac{\|f(y)\|_E}{\rho(x-y)^{n+2}}dy\right)^p dx\right)d\lambda\leq$$

$$\leq c\int_0^\infty \psi(\lambda)\left(\int_\lambda^\infty\int_{\mathbb{R}^n}\left(\int_0^{\lambda/2}\int_{\mathbb{R}^n}\frac{\|f(y',\tau)\|_E}{(|x'-y'|+t^{1/2})^{n+2}}dy\right)^p dx'dt\right)d\lambda.$$

For $x=(x',t)\in\mathbb{R}^{n+1}$ let

$$I(t,\lambda)=\int_{\mathbb{R}^n}\left(\int_0^{\lambda/2}\int_{\mathbb{R}^n}\frac{\|f(y',\tau)\|_E}{(|x'-y'|+t^{1/2})^{n+2}}dy\right)^p dx'=$$

$$=\int_{\mathbb{R}^n}\left(\int_0^{\lambda/2}\left(\int_{\mathbb{R}^n}\frac{\|f(y',\tau)\|_E}{(|x'-y'|+t^{1/2})^{n+2}}dy'\right)d\tau\right)^p dx'.$$

Using the Minkowski and Young inequalities, we obtain

$$I(t,\lambda)\leq\left[\int_0^{\lambda/2}\left(\int_{\mathbb{R}^n}\|f(y',\tau)\|_E^p dy'\right)^{1/p}\left(\int_{\mathbb{R}^n}\frac{dy'}{(|y'|+t^{1/2})^{n+2}}\right)d\tau\right]^p=$$

$$= \left(\int_0^{\lambda/2} \|f(\cdot,\tau)\|_{L_p(\mathbb{R}^n;E)} d\tau \right)^p \left(\int_{\mathbb{R}^n} \frac{dy'}{(|y'| + t^{1/2})^{n+2}} \right)^p =$$

$$= \frac{c}{t^p} \left(\int_0^{\lambda/2} \|f(\cdot,\tau)\|_{L_p(\mathbb{R}^n;E)} d\tau \right)^p \left(\int_{\mathbb{R}^n} \frac{dy'}{(|y'| + 1)^{n+2}} \right)^p =$$

$$= \frac{c}{t^p} \left(\int_0^{\lambda/2} \|f(\cdot,\tau)\|_{L_p(\mathbb{R}^n;E)} d\tau \right)^p .$$

Integrating in $(0,\infty)$, we get

$$I_{22} \le c \int_0^\infty \psi(\lambda) \left(\int_\lambda^\infty \left(\int_0^{\lambda/2} \|f(\cdot,\tau)\|_{L_p(\mathbb{R}^n;E)} d\tau \right)^p \frac{dt}{t^p} \right) d\lambda =$$

$$= c \int_0^\infty \psi(\lambda) \lambda^{-p-1} \left(\int_0^{\lambda/2} \|f(\cdot,\tau)\|_{L_p(\mathbb{R}^n;E)} d\tau \right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda) \lambda^{-p-1} \left(\int_0^{\lambda/2} \|f(\cdot,\tau)\|_{L_p(\mathbb{R}^n;E)} d\tau \right)^p d\lambda \le$$

$$\le C \int_{\mathbb{R}_+} \|f(\cdot,\tau)\|^p_{L_p(\mathbb{R}^n;E)} \omega(\tau) d\tau = \int_{\mathbb{R}^{n+1}_+} \|f(y)\|^p_E \omega(\tau) dy' d\tau.$$

for $p \in (1,\infty)$ is characterized by the condition $C \le c\mathcal{A}'$, where

$$\mathcal{A}' \equiv \sup_{\tau>0} \left(\int_{2\tau}^\infty \psi(t) t^{-p-1} d\tau \right) \left(\int_0^\tau \omega^{1-p'}(t) dt \right)^{p-1} < \infty.$$

Note that

$$\int_{2t}^\infty \psi(\tau)\tau^{-p-1} d\tau = p \int_{2t}^\infty \psi(\tau) d\tau \int_\tau^\infty \lambda^{-p} d\lambda =$$

$$= p \int_{2t}^\infty \lambda^{-p} d\lambda \int_{2t}^\lambda \psi(\tau) d\tau \le p \int_{2t}^\infty \lambda^{-p} \omega_1(\lambda) d\lambda.$$

By theorem from condition (b) it follows $\mathcal{A}' \le p\mathcal{A} < \infty$. Hence, applying the Hardy inequality, we obtain

$$I_{22} \le c \int_0^\infty \|f(\cdot,\tau)\|^p_{L_p(\mathbb{R}^n;E)} \omega(\tau) d\tau \le c \int_{\mathbb{R}^{n+1}_+} \|f(y)\|^p_E \omega(\tau) dy' d\tau.$$

Combining the estimates of I_1 and I_2, we obtain (3) for $\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (3).

Theorem 3. *Let E be ζ-convex Banach lattice. Let K be a parabolic Calderon-Zygmund kernel and T be the integral operator* (2). *Moreover, let $p \in (1,\infty)$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$. Assume that there exists a constant $b_1 > 0$ such that the inequality $\omega_1(t) \le b_1\,\omega(t)$ holds for almost all $t > 0$. Then there exists a constant $c = c(n,p,K)$ such that for all $f \in L_{p,\omega}(\mathbb{R}^{n+1}_+;E)$,*

$$\|Tf\|_{L_{p,\omega_1}(\mathbb{R}^{n+1}_+;E)} \le c\|f\|_{L_{p,\omega}(\mathbb{R}^{n+1}_+;E)}. \tag{4}$$

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau, \quad t > 0,$$

where $\omega_1(+\infty) = \lim\limits_{t\to\infty} \omega_1(t)$ and $\psi(t) \ge 0$ on $(0,\infty)$.

We have

$$\int_{\mathbb{R}^{n+1}_+} \|Tf(x)\|_E^p \omega_1(t)dx'dt = \omega_1(+\infty) \int_{\mathbb{R}^{n+1}_+} \|Tf(x)\|_E^p dx +$$

$$+ \int_{\mathbb{R}^{n+1}_+} \|Tf(x)\|_E^p \left(\int_t^\infty \psi(\tau)d\tau\right) dx = J_1 + J_2.$$

If $\omega_1(+\infty) = 0$, then $J_1 = 0$. If $\omega_1(+\infty) \ne 0$ by the boundedness of T in $L_p(\mathbb{R}^{n+1}_+;E)$

$$J_1 \le c\omega_1(+\infty) \int_{\mathbb{R}^{n+1}_+} \|f(x)\|_E^p dx \le$$

$$\le c \int_{\mathbb{R}^{n+1}_+} \|f(x)\|_E^p \omega_1(t)dx'dt \le c \int_{\mathbb{R}^{n+1}_+} \|f(x)\|_E^p \omega(t)dx'dt.$$

After changing the order of integration in J_2, using the boundeedness of T in $L_p(\mathbb{R}^{n+1};E)$ and applying the Minkowski inequality, we obtain

$$J_2 = \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^n} \int_0^\lambda \|Tf(x)\|_E^p dx'dt\right) d\lambda =$$

$$= \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^n} \int_0^\lambda \left\| \int_{\mathbb{R}^n} \int_0^t K(x,x-y)f(y)\chi_{\{0\le\tau\le\lambda\}}(y)dy \right\|_E^p dx'dt\right) d\lambda \le$$

$$\leq c\int\limits_0^\infty \psi(\lambda)\left(\int\limits_{\mathbb{R}^n}\int\limits_0^\lambda \|f(x)\|_E^p dx'dt\right)d\lambda =$$

$$= c\int\limits_{\mathbb{R}_+^{n+1}}\left(\int\limits_t^\infty \psi(\lambda)d\lambda\right)\|f(x)\|_E^p dx \leq c\|f\|^p_{L_{p,\omega_1}(\mathbb{R}_+^{n+1};E)} \leq c\|f\|^p_{L_{p,\omega}(\mathbb{R}_+^{n+1};E)}.$$

Combining the estimates of J_1 and J_2, we get (4) for $\omega_1(t) = \omega_1(+\infty)+\int\limits_t^\infty \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (4).

Corollary 1. *Let E be ζ-convex Banach lattice. Let K be a parabolic Calderon-Zygmund kernel and T be the integral operator* (2)*. Moreover, let $p \in (1,\infty)$ and $\omega(t)$ be a positive decreasing function on $(0,\infty)$. Then there exists a constant $c = c(n,p,K)$ such that for all $f \in L_{p,\omega}(\mathbb{R}_+^{n+1};E)$,*

$$\|Tf\|_{L_{p,\omega}(\mathbb{R}_+^{n+1};E)} \leq c\|f\|_{L_{p,\omega}(\mathbb{R}_+^{n+1};E)}.$$

Acknowledgements. The authors express their thanks to Prof. M.S. Jabrailov for helpful comments.

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Received October 03, 2011; Revised December 26, 2011