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DETERMINATION OF THE COEFFICIENT OF PARABOLIC EQUATION IN THE PROBLEM WITH NONLINEAR BOUNDARY CONDITION

Abstract

The goal of the paper is to investigate the well-posedness of the inverse problem on definition of the coefficient at a minor term of a parabolic equation in the nonlinear boundary condition problem under nonlocal additional condition. A theorem on theuniqueness and "conditional" stability of the problem under consideration is proved.

Let R^n be an n-dimensional Euclidean space, $x=(x_1,...,x_n)$ be an arbitrary point of the bounded domain $D\subset R^n$ with a sufficiently smooth boundary ∂D, $\Omega=D\times(0;T]$, $S=\partial D\times[0;T]$, $0<T$ be a fixed number.

The spaces $C^l(\cdot)$, $C^{l+\alpha}(\cdot)$, $C^{l,l/2}(\cdot)$, $C^{l+\alpha,(l+\alpha)/2}(\cdot)$, $l=0,1,2$, $\alpha\in(0,1)$ and the norms in these spaces were determined for example in [1, pp. 12-20]

$$\|\cdot\|_l=\|\cdot\|_{C^l}\,,\quad u_t=\frac{\partial u}{\partial t},\quad u_{x_j}=\frac{\partial u}{\partial x_j},\quad i=\overline{1,n},$$

$\Delta u=\sum\limits_{i=1}^{N}\dfrac{\partial^2 u}{\partial x_i^2}$ is a Laplace operator, $\dfrac{\partial u}{\partial \nu}$ is an internal conormal derivative.

We consider an inverse problem on definition of a pair of functions $\{u(x,t)\,,c(x)\}$ from the conditions

$$u_t-\Delta u+c(x)\,u=f(x,t)\,,\quad (x,t)\in\Omega \tag{1}$$

$$u(x,0)=\varphi(x)\,,\quad x\in\overline{D}=D\cup\partial D \tag{2}$$

$$\frac{\partial u}{\partial\nu}=\psi(x,t,u)\,,\quad (x,t)\in S \tag{3}$$

$$\int\limits_0^T u(x,t)\,dt=h(x)\,,\quad x\in\overline{D}, \tag{4}$$

here $f(x,t)\,,\ \varphi(x)\,,\ \ \psi(x,t,p)\,,\ \ h(x)$ are the given functions.

The coefficient inverse problems were studied in the papers [2-4] (see also the references in these papers).

For the input data of problem (1)-(4), we make the following suppositions:

1^0. $f(x,t)\in C^{\alpha,\alpha,/2}\left(\overline{\Omega}\right)$;

2^0. $\varphi(x)\in C^{2+\alpha}\left(\overline{D}\right)$;

3^0. $\psi(x,t,p)\in C^{\alpha,\alpha,/2}\left(\overline{\Omega}\times R_1\right)$, there exists $m_1>0$, such that for any $(x,t)\in\overline{\Omega}$ and $p_1,\,p_2\in R_1:|\psi(x,t,p_1)-\psi(x,t,p_2)|\le m_1\,|p_1-p_2|$;

4^0. $h(x)\in C^{2+\alpha}\left(\overline{D}\right)$.

Definition 1. *A pair of functions* $\{c(x), u(x,t)\}$ *is said to be a solution of problem (1)-(4) if:*

1) $c(x) \in C(\overline{D})$;

2) $u(x,t) \in C^{2,1}(\Omega) \cap C^{1,0}(\overline{\Omega})$;

3) for these functions, relations (1)-(4) are fulfilled, therewith, condition (3) is determined in the following way:

$$\frac{\partial u(x,t)}{\partial \nu(x,t)} = \lim_{\substack{y\to x\\ y\in\sigma}} \frac{\partial u(y,t)}{\partial \nu(x,t)},$$

where σ *is a closed cone with a vertex* x*, that is contained in* $D \cup \{x\}$.

The uniqueness theorem and also estimation of stability of the solutions of inverse problems occupies a central place in investigation of their well-posedness matters. In the paper, the uniqueness of the solution of problem (1)-(4) is proved under more general suppositions and the estimation characterizing the "conditional" stability of the problem is established.

Let $\{u_i(x,t), c_i(x)\}$ be the solution of problem (1)-(4) corresponding to the data $f_i(x,t)$, $\varphi_i(x)$, $\psi_i(x,t,u_i)$, $h_i(x)$, $i=1,2$.

Definition 2. *Say that the solution of problem (1)-(4) is stable if for any* $\varepsilon > 0$ *there will be found* $\delta(\varepsilon) > 0$ *such that for* $\|f_1 - f_2\| < \delta$, $\|\varphi_1 - \varphi_2\| < \delta$, $\|\psi_1 - \psi_2\| < \delta$, $\|h_1 - h_2\| < \delta$ *the inequality* $\|u_1 - u_2\| + \|c_1 - c_2\| \le \varepsilon$ *is fulfilled.*

Theorem. *Let:*

1. f_i, φ_i, ψ_i, h_i, $i=1,2$ *satisfy conditions* 1^0-4^0*, respectively;*

2. there exist the solutions $\{u_i(x,t), c_i(x)\}$, $i=1,2$, *of problem (1)-(4) in the sense of definition 1, and they belong to the set*

$$K_\alpha = \left\{ (u,c) \middle| \, u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}), \quad c(x) \in C^\alpha(\overline{D}) \right\}.$$

Then there exists a $T^* > 0$ such that for $(x,t) \in \overline{D} \times [0,T^*]$ the solution of problem (1)-(4) is unique, and it is valid the stability estimation

$$\|u_1 - u_2\|_0 + \|c_1 - c_2\|_0 \le$$
$$\le m_2 \left[\|f_1 - f_2\|_0 + \|\varphi_1 - \varphi_2\|_2 + \|\psi_1 - \psi_2\|_0 + \|h_1 - h_2\|_2 \right], \tag{5}$$

where $m_2 > 0$ depends on the data of problem (1)-(4) and the set K_α.

Proof of the theorem. At first prove the validity of estimation (5). In order to get a uniqueness theorem, in the arguments below we should suppose that perturbations of input data everywhere identically equal zero. Allowing for (2) and the conditions of the theorem, from equation (1) for the function $c(x)$ we get

$$c(x) = \left[u(x,T) - \varphi(x) - \Delta h(x) - \int_0^T f(x,t)\,dt \right] \cdot (h(x))^{-1}, \tag{6}$$

Denote $z(x,t) = u_1(x,t) - u_2(x,t)$, $\lambda(x) = c_1(x) - c_2(x)$,

$\delta_1(x,t) = f_1(x,t) - f_2(x,t), \delta_2(x) = \varphi_1(x) - \varphi_2(x),$
$\delta_3(x,t,p) = \psi_1(x,t,p) - \psi_2(x,t,p),\ \delta_4(x) = h_1(x) - h_2(x).$

We can verify that the functions $\lambda(x),\ w(x,t) = z(x,t) - \delta_2(x)$ satisfy the relations of the system:

$$w_t - \Delta w = F(x,t), \quad (x,t) \in \Omega, \tag{7}$$

$$w(x,0) = 0, \quad x \in \overline{D}; \quad \frac{\partial w}{\partial \nu}(x,t) = \Psi(x,t), \quad (x,t) \in S \tag{8}$$

$$\lambda(x) = z(x,T) \cdot (h_1(x))^{-1} - H(x), \quad x \in \overline{D}, \tag{9}$$

where

$$F(x,t) = \delta_1(x,t) - \Delta\delta_2(x) - c_1(x) z(x,t) - \delta_2(x) u_2(x,t),$$

$$\Psi(x,t) = \delta_3(x,t,u_1) - \frac{\partial\delta_2}{\delta\nu}(x,0) + \psi_2(x,t,u_1) - \psi_2(x,t,u_2),$$

$$H(x) = \left\{ \left[\delta_2(x) + \Delta\delta_4(x) + \int_0^T \delta_1(x,t)\,dt \right] h_2(x) + \right.$$

$$\left. + \left[u_2(x,T) - \varphi_2(x) - \Delta h_2(x) - \int_0^T f_2(x,t)\,dt \right] \delta_4(x) \right\} \times$$

$$\times [h_1(x) \cdot h_2(x)]^{-1}.$$

Under the conditions of the theorem, if follows that there exists a classic solution of problem (7), (8) on definition of $w(x,t)$ and it may be represented in the form [51, p. 182]

$$w(x,t) = \int_0^t \int_D \Gamma(x,t;\xi,\tau) F(\xi,\tau)\,d\xi d\tau + \int_0^t \int_{\partial D} \Gamma(x,t;\xi,\tau) \rho(\xi,\tau)\,d\xi_{\partial D} d\tau, \tag{10}$$

where $\Gamma(x,t;\xi,\tau)$ is a fundamental solution of the equation $w_t - \Delta w = 0,\ d\xi = d\xi_1 ... d\xi_n, d\xi_{\partial D}$ is an element of the surface $\partial D, \rho(x,t)$is a continuous bounded solution of the following integral equation [2 p. 183]

$$\rho(x,t) = 2\int_0^t \int_D \frac{\Gamma(x,t;\xi,\tau)}{\partial\nu(x,t)} F(\xi,\tau)\,d\xi d\tau +$$

$$+2\int_0^t \int_D \frac{\Gamma(x,t;\xi,\tau)}{\partial\nu(x,t)} \rho(\xi,\tau)\,d\xi_{\partial D} d\tau - 2\Psi(x,t). \tag{11}$$

Assume

$$\chi = \|u_1 - u_2\|_0 + \|c_1 - c_2\|_0.$$

Estimate the function $|z(x,t)|$. Taking into account that $z(x,t) = w(x,t) + \delta_2(x)$, from (10) we get:

$$|z(x,t)| \le |w(x,t)| + |\delta_2(x)| \le |\delta_2(x)| + \int_0^t \int_D |\Gamma(x,t;\xi,\tau)| \cdot |F(\xi,\tau)|\, d\xi d\tau +$$

$$+ \int_0^t \int_D |\Gamma(x,t;\xi,\tau)| \cdot |\rho(\xi,\tau)|\, d\xi_{\partial D} d\tau, \tag{12}$$

For the expression $\int_D |\Gamma(x,t;\xi,\tau)|\, d\xi$ in the second summand of the right hand side of (12), the following estimation is true:

$$\int_D |\Gamma(x,t;\xi,\tau)|\, d\xi \le m_3. \tag{13}$$

By the requirements imposed on the input data and on the set $F(x,t)$, the integrand function K_α in the second summand of the right side of (12), satisfies the estimation

$$|F(x,t)| \le |\delta_1(x,t)| + |\Delta\delta_2(x)| + |c_1(x)|\, |z(x,t)| +$$

$$+ |\lambda(x)|\, |u_2(x,t)| \le \|f_1 - f_2\|_0 + \|\varphi_1 - \varphi_2\|_2 + m_4 \cdot \chi, \quad (x,t) \in \overline{\Omega} \tag{14}$$

here $m_4 > 0$ depends on the data of problem (1)-(4) and the set K_α.

The expression $\int_{\partial D} |\Gamma(x,t;\xi,\tau)|\, d\xi_{\partial D}$ in the third summand of the right side of (12) satisfies the estimation

$$\int_{\partial D} |\Gamma(x,t;\xi,\tau)|\, d\xi_{\partial D} \le m_5. \tag{15}$$

Taking into account expression (11), the conditions of the theorem, definition of the set K_α and the following estimation [5, p. 20]:

$$\int_D \left| \frac{\partial \Gamma(x,t;\xi,\tau)}{\partial \nu(x,t)} \right| d\xi \le m_6 (t-\tau)^{-\mu}, \quad \frac{1}{2} < \mu < 1,$$

for the function $\rho(x,t)$ we get:

$$|\rho(x,t)| \le m_7 \left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi \right] + m_8 \|\rho\| \cdot t^{1-\mu}, (x,t) \in S$$

where $m_7, m_8 > 0$ depend on the data of problem (1)-(4) and the set K_α.

The last inequality is fulfilled for all $(x,t) \in \partial D \times [0,T]$, therefore the following estimation is true:

$$\|\rho\|_0 \le m_7 \left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi \right] + m_8 t^{1-\mu} \|\rho\|_0 .$$

Let $0 < T_1 \leq T$ be a number such that $m_8 T^{1-\mu} < 1$. Then for all $(x,t) \in \partial D \times [0, T_1]$ we have

$$\|\rho\|_0 \leq m_9 \left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi\right], \tag{16}$$

where $m_9 > 0$ depends on the data of problem (1)-(4) and the set K_α.

Taking into account inequalities (13), (14), (15) and (16) for $|z(x,t)|$, from (12) we get:

$$|z(x,t)| \leq m_{10} \left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0\right] + m_{11} \chi t, \quad (x,t) \in \overline{\Omega} \tag{17}$$

where $m_{10},\ m_{11} > 0$ depend on the data of problem (1)-(4) and the set K_α.

Now estimate the function $|\lambda(x)|$. From (9) it follows

$$|\lambda(x)| \leq |z(x,t)| \cdot \left|h_1(x)^{-1}\right| + |H(x)|.$$

Taking into account the conditions of the theorem, definitions of the set K_α, inequalities (17) and expressions for $H(x)$, from the last inequality we get:

$$|\lambda(x)| \leq m_{12} \left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \|\delta_4\|_2\right] + m_{13} t \chi,\ x \in \overline{\Omega} \tag{18}$$

where $m_{12},\ m_{13} > 0$ depend on the data of the problem and the set K_α.

Inequalities (17) and (18) are satisfied for any values of $(x,t) \in \overline{D} \times [0, T]$.

Consequently, combining these inequalities, we get

$$\chi \leq m_{14} \left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \|\delta_4\|_2\right] + m_{15} t \chi, \tag{19}$$

where $m_{14}, m_{15} > 0$ depend on the data of problem (1)-(4) and the set K_α.

Let $T_2\,(0 < T_2 \leq T)$ be a number such that $m_{15} T_2 < 1$. Then from (19) we get that for $(x,t) \in \overline{D} \times [0, T^*]$, $T^* = \min(T_1, T_2)$, the stability estimation for the solution of problem (1)-(4) is true.

Uniqueness of the solution of problem (1)-(4) follows from estimation (5) for $f_1(x,t) = f_2(x,t)$, $\varphi_1(x) = \varphi_2(x)$, $\psi_1(x,t,u) = \psi_2(x,t,u)$, $h_1(x) = h_2(x)$.

The theorem is completely proved.

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