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ANALOGUE OF J.WALSH PROBLEM IN INTEGRAL METRIC ON CURVES IN A COMPLEX PLANE

Abstract

In this paper we obtain constructive characteristic of analogue of Lipschitz class on curves in a complex plane in intgral metrics. The obtained results are the first attempt to solve the analogue of J.L. Walsh problem related to Jacson-Bernstein classic theorem.

This paper is devoted to J.L. Walsh problem [1] related to Jackson-Bernstein theorem that says:

In order that

$$f \in Lip_{[0,2\pi]}\alpha\,(0<\alpha<1)\,,$$

it is necessary and sufficient that

$$E_n\left(f;[0,2\pi]\right) \le \frac{const}{n^\alpha}, \quad \left(E_n\left(f;[a,b]\right) = \inf_{P_n}\|f-P_n\|_{C[a,b]}\right).$$

More specifically, in connection with this theorem, J.L. Walsh formulated the following problem: what necessary and sufficient conditions should satisfy a closed curve Γ for Jackson-Bernstein theorem be valid on it. This problem was considered by J.L. Walsh, H.G. Russell [1,2], W.E. Sewell [3], A.I. Markushevich [4], S.N. Mergelyan [5], S.Ya. Alper [6] and others.

Note that this problem preserves its actuality in the integral metric $L_p(\Gamma)$ as well. Our paper is devoted to this problem, exactly to Jackson-Bernstein theorem on closed curves in a complex plane in the metric $L_p(\Gamma)$, i.e. to the J.L. Walsh problem in the metric $L_p(\Gamma)$. The first part of this paper, namely, the analogue of Jackson's direct theorem, was published in the Proceedings of the 6th International ISAAC Congress in 2007 [7]. Markov-Bernstein type estimations used in the proof of inverse approximation theorems are cited in [8].

Here we attempt to prove a corresponding inverse theorem and to get a constructive characteristic for the analogue of Lipschits class of order α on the curves in a complex plane in the metric $L_p(\Gamma)$, i.e. for the class $H_p^\alpha(\Gamma)$ $(p>1)$ (the case $p=1$ requires additional arguments) and $0<\alpha<1$.

Before we state our results, we introduce the basic classes of curves on which approximation is conducted, determine some classes of functions and give some concepts and notations. We also give some known facts to be used in the paper.

Let Ω be an arbitrary simply connected domain in a complex plane containing a point $z=\infty$, $\overline{B}$ be a continuum being a complement to $\Omega : d_0 \overset{df}{=} diam\overline{B} > 0$, $\Gamma = \partial\Omega = \partial\overline{B}$ be their common boundary. Further, let $\omega = \varphi(z)$ be a function that conformally and univalently maps Ω onto the exterior of a unit circle and normalized by the conditions

$$\varphi(\infty)=\infty, \quad \lim_{z\to\infty}\frac{\varphi(z)}{z} > 0.$$

Denote by $z=\psi(\omega)=\varphi^{-1}(\omega)$, the inverse of the function $\omega=\varphi(z)$ and by $\Gamma_{1+\sigma}\overset{df}{=}\{t:|\varphi(t)|=1+\sigma\geq 1\}$ a level line of the continuum $\overline{B}$, $d(z,\sigma)\overset{df}{=}\inf\limits_{t\in\Gamma_{1+\sigma}}|z-t|$ for $z\in\Gamma$, $\widetilde{d}(t,\sigma)\overset{df}{=}\inf\limits_{t\in\Gamma}|z-t|$ for $t\in\Gamma_{1+\sigma}$. Assume that Γ is a closed rectifiable Jordan curve of length l and diameter d $\left(d=\sup\limits_{t,\tau\in\Gamma}|t-\tau|\right)$ defined by the equation $t=t(s)$ $(0<s\leq l)$ in angular positions. For $0\leq\delta\leq d$ we denote, $\Gamma_\delta(t)=\{\tau\in\Gamma:|t-\tau|\leq\delta\}$, $t\in\Gamma$, $\theta_t(\delta)=mes t\in\Gamma_\delta(t)$ (Lebesgue measure), $\theta(\delta)\sup\limits_{t\in\Gamma}\theta_t(\delta)$. Obviously, $\theta(\delta)\geq\delta$.

Definition 1 [9]. *The curve* Γ *belongs to the class* S_θ*, if there exists a constant* $C(\Gamma)\geq 2$ *such that* $\theta(\delta)\leq C(\Gamma)\,\delta$.

Class S_θ was introduced by V.V. Salayev [9], and so we call it the Salayev class of curves. Note also that the class S_θ is the largest of the classes of curves to which Plemel's-Privalov theorem applies.

Definition 2. *The curve being an image of a circle for some* K*-quasiconformal mapping of the plane onto itself is said to be a* K*-quasiconformal curve. We denote the class of such curves by* A_k.

Definition 3. *The curve* Γ *belongs to the class of* K*-curves (class of Lavrent'ev) if, whatever the points* z_1 *and* z_2 *on it, the smallest of the arcs connecting these two points has the same order length as the of chord connecting them* $l(z_1,z_2)=l_\Gamma(z_1,z_2)$*, i.e. the following inequality holds:*

$$|z_1-z_2|\geq kl(z_1,z_2)$$

where k *is a positive constant depending on* Γ.

Definition 4 [10, p.392]**.** *We say that the set* E *with a rectifiable Jordan boundary* $\Gamma=\partial E$ *belongs to the class* B_k *for some positive integer* k *if* $\Gamma\in S_\theta$ *and the following conditions are satisfied:*

1) $|\widetilde{z}-z|\asymp d\left(z,\frac{1}{n}\right)$*, where* $\widetilde{z}=\widetilde{z}\left(\frac{1}{n}\right)=\psi\left(\left(1+\frac{1}{n}\right)\varphi(z)\right)$;

2) $\left|\widetilde{\xi}-\xi\right|\preccurlyeq\left|\widetilde{\xi}-z\right|^{k-1}|\widetilde{z}-z|$*, for all* $z,\xi\in\Gamma$;

3) $\left|\widetilde{\xi}-z\right|^k\preccurlyeq(1+n|h|)^k\left|\widetilde{\xi}_h-z\right|$*, for all* $z,\xi\in\Gamma$*, where* $\widetilde{\xi}_h=\widetilde{\xi}_h\left(\frac{1}{n}\right)\overset{df}{=}\psi\left(\left(1+\frac{1}{n}\right)\varphi(\xi)\,e^{-ih}\right)$*,* $h\in[-\pi,\pi]$;

4) $\left|\widetilde{\xi}_h-\xi\right|\preccurlyeq(1+n|h|)^k\left|\widetilde{\xi}-z\right|$*, for all* $z,\xi\in\Gamma$*,* $h\in[-\pi,\pi]$.

Remark: In what follows we use only conditions 1) and 2) of the class B_k, which, in particular, are valid for arbitrary K-quasiconformal curves.

Therefore, it is natural to consider a class of curves N_k consisting of curves of the class S_θ satisfying conditions 1) and 2) of the class B_k. Furthermore, it should be noticed that a set of continua with connected complement and a boundary belonging to the class N_k; coincides with the set B_k; since conditions 3) and 4) as shown by V. I. Belyi [11] are fulfilled for any continuum with connected complement.

It is easy to see that a class of K-curves [10] containing a class of piecewise-smooth curves without cusp is contained in the classes N_k and B_k. Moreover, the classes of curves A_k and B_k are not embedded into each other.

If the function $f(t)$ determined Γ on is measurable and the function $|f(t)|^p$ is integrable (by Lebesgue) on Γ, then $f \in L_p(\Gamma)$.

Obviously, if we define the norm in $L_p(\Gamma)$, $p \geq 1$ like

$$|f(t)|_{L_p(\Gamma)} = \left\{ \int\limits_{\Gamma} |f(t)|^p |dt| \right\}^{1/p},$$

then $L_p(\Gamma)$ turns into a Banach space. Assuming $f \in L_p(\Gamma)$ $(p \geq 1)$, consider

$$z_h = \psi\left(\varphi(z)\, e^{ih}\right), \quad \psi\left(\varphi(z)\, e^{ih}\right) \in \Gamma, \; z \in \Gamma,$$

$$u_p(f,\delta)_\Gamma = u_p(\delta) = \sup_{|h| \leq \delta} \|f(z_h) - f(z)\|_{L_p(\Gamma)}.$$

By $H_p^\alpha(\Gamma)$ $(p \geq 1, 0 < \alpha < 1)$ we denote a class of functions $f \in E_p(G)^2$ $(\Gamma \in \partial G)$, for which

$$u_p(\delta) \preccurlyeq \delta^\alpha.$$

By D_k and D_k^* we denote a class of rectifiable closed curves Γ belonging to N_k and A_k, respectively, for which

$$\left|\psi'(w)\right| \leq C(\Gamma)\, \sigma^{-1} d(\psi(w), \sigma), \sigma > 0, |w| = 1, \tag{1}$$

where $d(\psi(w), \sigma)$ is the distance from the point $z = \psi(w)$ to the level line $\Gamma_{1+\sigma}$. Domains whose boundary belongs to the class D_k will also be denoted by D_k, while those whose boundary belongs to A_k will be denoted by D_k^*. Recall that conditions (1) satisfied by arbitrary convex domains are necessary for obtaining constructive characteristic of the class $H_p^\alpha(\Gamma)$ $(0 < \alpha < 1)$, since, if not, i.e., for

$$\left|\psi'(w)\right| > \sigma^{-1} d(\psi(w), \sigma)$$

we get another constructive characteristic of an absolutely different vast class of functions [12, p.551].

Obviously, by the results of [8], if $\Gamma \in D_k$ or $\Gamma \in D_k^*$, then for any polynomial $P_n(z)$ of degree $\leq n$, the inequalities

$$\left\| d\left(z, \frac{1}{n}\right) P_n'(z) \right\|_{L_p(\Gamma)} \leq C(p,\Gamma) \|P_n\|_{L_p(\Gamma)} \tag{2}$$

$$\forall z \in \Gamma, \quad |P_n(z)| \leq C(p,\Gamma)\, d^{-1/p}\left(z, \frac{1}{n}\right) \|P_n\|_{L_p(\Gamma)} \tag{3}$$

$$\forall z \in \Gamma, \quad \left|P_n'(z)\right| \leq C(p,\Gamma)\, d^{-(1+1/p)}\left(z, \frac{1}{n}\right) \|P_n\|_{L_p(\Gamma)} \tag{4}$$

are valid for all $p \geq 1$.

Theorem 1. *Let $\partial G = \Gamma \in D_k$ (or $\Gamma \in D_k^*$) $f \in L_p(\Gamma)$ $(p \geq 1)$ and let for each positive integer number n there exist a polynomial $P_n(z)$ of degree $\leq n$ such that*

$$\|f - P_n\|_{L_p(\Gamma)} \preccurlyeq \frac{1}{n^\alpha} \quad (0 < \alpha < 1). \tag{5}$$

Then there exists a function $f \in E_p(G)$ (Smirnov's class) analytic in domain G whose angular boundary values interior Γ to are almost everywhere equal to $f(t)$ and

$$u_p(f,\delta)_\Gamma \preccurlyeq \delta^\alpha \quad (0<\alpha<1) \tag{6}$$

i.e. $f \in H_p^\alpha(\Gamma)$ $(0<\alpha<1)$.

Proof. It follows directly from (5) that

$$\|f-P_n\|_{L_p(\Gamma)} \to 0, \quad as \quad n\to\infty,$$

i.e. the sequence $\{P_n(z)\}$ converges to the function $f(z)$ in the sense of the metric of the space $L_p(\Gamma)$. Hence it follows that the sequence $\left\{\|P_n\|_{L_p(\Gamma)}\right\}$ is bounded. Then, by G.Tumarkin's theorem [13, p.268], the sequence $\{P_n(z)\}$ converges uniformly inside G to some analytic function $f(t) \in E_p(G)$ whose angular boundary values interior to Γ coincide with $f(z)$.

Consider the series

$$P_1(z)+\sum_{m=1}^{\infty}\left[P_{2^m}(z)-P_{2^{m-1}}(z)\right]=\sum_{m=0}^{\infty}U_m(z) \tag{7}$$

where

$$U_0(z)=P_1(z), \quad U_m(z)=P_{2^m}(z)-P_{2^{m-1}}(z).$$

By (7) we have

$$|U_m|_{L_p(\Gamma)} \preccurlyeq 2^{-m\alpha}. \tag{8}$$

It follows from the inequality

$$\|S_n(z)-S_m(z)\|_{L_p(\Gamma)} \le \sum_{k=m+1}^{n}\|U_k\|_{L_p(\Gamma)} \le 2^{-m\alpha}(m<n),$$

where

$$S_N(z)=\sum_{m=0}^{N}U_m(z),$$

that $S_N(z)$ is a fundamental sequence in the space $L_p(\Gamma)$. By the completeness of the space $L_p(\Gamma)$, the sequence $\{S_N(z)\}$ converges in the sense of the $L_p(\Gamma)$ metrics to the function $f(z)$.

Furthermore, by (2) and (8), the inequality

$$\left\|d\left(z,2^{-m}\right)U_m'(z)\right\|_{L_p(\Gamma)} \preccurlyeq 2^{-m\alpha} \tag{9}$$

is valid.

Now, let's prove inequality (6). For definiteness, we'll assume that $h>0$. Obviously, we have

$$\|f(z_h)-f(z)\|_{L_p(\Gamma)} \le \sum_{m=0}^{N_0}\|U_m(z_h)-U_m(z)\|_{L_p(\Gamma)}+$$

$$+\left\|f(z_h)-\sum_{m=0}^{N_0}U_m(z_h)\right\|_{L_p(\Gamma)}+\left\|f(z)-\sum_{m=0}^{N_0}U_m(z)\right\|_{L_p(\Gamma)}=K_1+K_2+K_3, \quad (10)$$

where the positive integer N_0 satisfies the following condition:

$$\frac{1}{2^{N_0+1}}\le h\le\frac{1}{2^{N_0}}. \quad (11)$$

Consider the expression K_1:

$$K_1=\sum_{m=0}^{N_0}\|U_m(z_h)-U_m(z)\|_{L_p(\Gamma)}=\sum_{m=0}^{N_0}a_m(h). \quad (12)$$

Further, we have

$$a_m(h)=\|U_m(z_h)-U_m(z)\|_{L_p(\Gamma)}\le$$

$$\le\left\{\int_\Gamma\left(\int_0^h\left|U_m'\left(\psi\left(\varphi(z)e^{it}\right)\right)\psi'\left(\varphi(z)e^{it}\right)\right|dt\right)^p|dz|\right\}^{1/p}.$$

Hence, by Minkovski's generalized inequality, we get:

$$a_m(h)\le\int_0^h\left\{\int_\Gamma\left|U_m'\left(\psi\left(\varphi(z)e^{it}\right)\right)\psi'\left(\varphi(z)e^{it}\right)\right|^p|dz|\right\}^{1/p}dt.$$

The curves $\Gamma\in D_k$ (or $\Gamma\in D_k^*$) satisfy condition (1) and therefore

$$a_m(h)\le 2^m\int_0^h dt\left\{\int_\Gamma\left|d\left(z_1,2^{-m}\right)U_m'(z_t)\right|^p|dz|\right\}^{1/p},\quad z_t=\psi\left(\varphi(z)e^{it}\right).$$

Let $\sigma_j\left(j=\overline{1,k}\right)$ be a part of the curve Γ inside a circle of radius $Md\left(z_j,\frac{1}{2^m}\right)$ $(M>2)$ centered at the point $z_j\left(j=\overline{1,k}\right)$ (z_j are angular points of the curve Γ) and

$$Z=\bigcup_{j=1}^k\sigma_j.$$

Obviously, by virtue of relation

$$a_1^\mu+a_2^\mu\asymp(a_1+a_2)^\mu,\quad a_1,a_2>0,\quad 0\le\mu\le 1 \quad (13)$$

we get

$$a_m(h)\le 2^m\int_0^h\left\{\left(\int_{\Gamma\backslash Z}\left|d\left(z_t,2^{-m}\right)U_m'(z_t)\right|^p|dz|\right)^{1/p}+\right.$$

$$\left.+\left(\int_Z\left|d\left(z_t,2^{-m}\right)U_m'(z_t)\right|^p|dz|\right)^{1/p}\right\}dt=2^m\int_0^h\left(B_1(t)+B_2(t)\right)dt. \quad (14)$$

Using the substitution $z_t = u$, we find

$$B_1^p(t) = \int\limits_{\Gamma\backslash Z'} \left|d\left(u, 2^{-m}\right) U'_m(u_t)\right|^p \left|\frac{\varphi'(u)}{\varphi'(u_t)}\right| |du|, \tag{15}$$

where $Z' = \bigcup\limits_{j=1}^{k} \sigma'_j$ and $\sigma'_j \left(j = \overline{1,k}\right)$ is an image of the arch σ_j for the mapping $z_t = u$.

It is easy to show [13, p.513] that

$$\left|\frac{\varphi'(u)}{\varphi'(u_t)}\right| \leq 1, \quad u \in \Gamma\backslash Z'.$$

From (7) we find

$$B_1^p(t) = \int\limits_{\Gamma\backslash Z'} \left|d\left(u, 2^{-m}\right) U'_m(u_t)\right|^p du.$$

Finally, by relations (2) and (8) we have

$$B_1(t) \preccurlyeq 2^{-m\alpha}. \tag{16}$$

In order to estimate $B_2(t)$ it suffices to consider

$$B_2^*(z) \leq \left\{\int\limits_{\sigma_j} \left|d\left(z_t, 2^{-m}\right) U'_m(z_t)\right|^p |dz|\right\}^{1/p}.$$

By relations (4) and (8) we get

$$B_2^*(t) \leq \left\{\int\limits_{\sigma_j} \left|d\left(z_t, 2^{-m}\right)\right|^{-1} \|U_m(z)\|^p_{L_p(\Gamma)} |dz|\right\}^{1/p} =$$

$$= \left\{\int\limits_{|z-z_j|\leq d(z_j,2^{-m})} \left|d\left(z_t, 2^{-m}\right)\right|^{-1} |dz|\right\}^{1/p} \|U_m(z)\|^p_{L_p(\Gamma)} \preccurlyeq 2^{-m}. \tag{17}$$

Here it was taken into account that for all $z \in \Gamma$, with $\Gamma \in D_k$ the following relation holds [10, p.391]:

$$d(z, \sigma) \asymp \sigma\left(|z - z_j| + \sigma^{2-\alpha_i}\right)^{\frac{1-\alpha_j}{2-\alpha_i}},$$

where z_j is the joint point on the curve Γ nearest to z with internal angle $\alpha_i \pi$. And for all $z_t \in \Gamma$ with $|z_t - z_j| \preccurlyeq d\left(z_j, \frac{1}{2^m}\right)$ we have

$$d\left(z_t, \frac{1}{2^m}\right) \asymp d\left(z_j, \frac{1}{2^m}\right).$$

From (14), by virtue of (16) and (17), we find

$$a_m(h) \preccurlyeq h2^{m(1-\alpha)}.$$

Consequently,

$$K_1 = \sum_{m=0}^{N_0} a_m(h) \preccurlyeq h^{\alpha}. \tag{18}$$

Now let's estimate the expression K_2. Obviously, the following inequality holds:

$$K_2 = \left\{ \int_{\Gamma} \left| f(z_h) - \sum_{m=0}^{N_0} U_m(z_h) \right|^p |dt| \right\}^{1/p} \leq \sum_{n=N_0+1}^{\infty} \left\{ \int_{\Gamma} |U_m(z_h)|^p |dz| \right\}^{1/p} =$$

$$= \sum_{n=N_0+1}^{\infty} \left\{ \int_{\Gamma/Z} |U_m(z_h)|^p |dz| + \int_{Z} |U_m(z_h)|^p |dz| \right\}^{1/p}.$$

Continuing in the same way as in the estimation of K_1, by (3), (8) and (11) we get

$$K_2 \preccurlyeq h^{\alpha}. \tag{19}$$

Notice that the estimation

$$K_3 \preccurlyeq h^{\alpha}. \tag{20}$$

is easily obtained by (5) and (11). Thus, taking into account (18)-(20), from (10) we find

$$\|f(z_h) - f(z_h)\|_{L_p(\Gamma)} \preccurlyeq h^{\alpha} \text{ or } u_p(f,\delta) \preccurlyeq h^{\alpha}.$$

Q.E.D.

Now, we can formulate the theorem that gives constructive characteristic of the class $H_p^{\alpha}(\Gamma)$ $(p > 1, 0 < \alpha < 1)$:

Theorem 2. *Let* $\Gamma \in D_k$ *(or* $\Gamma \in D_k^*$*). In order that function* $f(z) \in L_p(\Gamma)$ $(p \geq 1)$*have the best approximation*

$$\rho_n^{(p)}(f,\Gamma) = \inf_{P_n \in \mathcal{P}_n} \|f - P_n\|_{L_p(\Gamma)} \preccurlyeq \frac{1}{n^{\alpha}} (0 < \alpha < 1),$$

it is necessary and sufficient that $f \in E_p(G)$ *and* $\omega_p(f,\delta) \preccurlyeq \delta^{\alpha} (0 < \alpha < 1)$*, i.e.* $f \in H_p^{\alpha}(\Gamma)$ $(p \geq 1, 0 < \alpha < 1)$.

In particular, if we denote by U_p a class of curves Γ in a complex plane for which an analogue of Jackson-Bernstein theorem in integral metric is true, then the above-proved theorems yield the following

Corollary. *If we denote by* J *a class of curves* Γ *for which the relation (1) is valid, i.e.,*

$$|\psi(\omega)| \leq C(\Gamma)\,\delta^{-1} d(\psi(\omega),\sigma), \sigma > 0, \quad |\omega| = 1$$

then $J \subset U_p$.

References

[1]. Walsh J.L. *Interpolation and approximation by rational functions in the complex domain.* M., IL, 1961.(Russian).

[2]. Walsh J.L. and Russell H. G. *Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions.* Transactions of the American Mathematical Society, 1959, vol. 92, No 2, pp. 355-370.

[3]. Sewell W.E. *Degree of approximation by polynomials in the complex domain.* Princeton University Press, 1942.

[4]. Markushevich A. I. *Theory of analytic functions* M.: Nauka, Vol. I 1967, vol II 1968, (Russian).

[5]. Mergelyan S.N. *Uniform approximation of complex variable functions.* Uspekhi mathem. nauk, 1952, 7:2 (48), pp. 31-122 (Russian).

[6]. Alper S.Ya. *On uniform approximation of complex variable functions in a closed domain.* Izv. AN SSSR, ser.mat. 1955, vol.19, No 3, pp. 423-444 (Russian).

[7]. Mamedkhanov J.I. and Dadashova I.B. *Analogue of Jackson-Bernstein theorem in on closed curves in the complex plane.* Further progress in analysis: Proceedings of the 6th International ISAAC Congress, Ankara, Turkey, 13-18 August 2007, World Scientific, 2009, pp. 260-267.

[8]. Mamedkhanov J.I. and Dadashova I.B., Bernstein S.N. *Type Estimations in the Mean on the Curves in a Complex Plane.* Abstract and Applied Analysis, vol. 2009, Article ID 165194, 19 p. doi:10.1155/2009/165194

[9]. Salayev V.V. *Primal and inverse estimations for Cauchy's singular integral on a closed curve.* Math.Zam., 1976, 19, pp. 365-380 (Russian).

[10]. Dzyadyk V.K. *Introduction to the theory of approximation of functions by polynomials.* M., Nauka, 1977, (Russian).

[11]. Belyi V.I. *Conformal mapping and approximation of analytic functions in quasiconformal boundary domains.* Math.sb. 1977, 102 (144), No 3, pp. 331-361 (Russian).

[12]. Dzyadyk V.K. and Alibekov G. A. *On uniform approximation of a complex variable function on closed sets with angles.* Math.sb. AN SSSR, 1968, 75 (117), No 4, pp. 502-557 (Russian).

[13]. Privalov I.I. *Boundary properties of single-valued analytic functions* M. Nauka, 1950 (Russian).

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