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ON ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF 2-ND ORDER OPERATOR-DIFFERENTIAL EQUATIONS ON A SEMI-AXIS

Abstract

In the paper, asymptotic distribution of eigenvalues of operator-differential equations on a semi-axis is studied. An asymptotic formula for the function of distribution eigen values of the given operator is obtained.

Let H be a separable Hilbert space. Denote by H_1 a Hilbert space of strongly measurable functions $f(x) \quad (0 \leq x \leq \infty)$ with the values from H for which $\int\limits_0^\infty \|f(x)\|_H^2 dx < \infty$. A scalar product of the elements $f(x)$, $g(x) \in H_1$ is defined by the equality

$$(f,g) = \int\limits_0^\infty (f(x), g(x))_H dx.$$

In the space $H_1 = L_2[H; 0 \leq x \leq \infty]$ consider a differential expression

$$l(y) = (-1)^n y^{(2n)} + \sum_{j=2}^{2n} Q_j(x) y^{(2n-j)}, 0 \leq x < \infty \tag{1}$$

with boundary conditions

$$y^{(l_1)}(0) = y^{(l_2)}(0) = ... = y^{(l_n)}(0) = 0. \tag{2}$$

Here $0 \leq l_1 \leq l_2 \leq ... \leq l_n \leq 2n-1$, $y \in H_1$, and the derivatives are understood in the strong sense. Everywhere by $Q(x)$ we'll denote $Q_{2n}(x)$.

Let D' be a totality of all the functions of the form $\sum\limits_{k=1}^{p} \varphi_k(x) f$, where $\varphi_k(x)$ are finite, $2n$ times continuously differentiable scalar functions, and $f_k \in D(Q)$.

Determine the operator L' generated by expression (1) and boundary conditions (2) with domain of definition D'.

Subject to certain conditions, the operator L' is a positive and symmetric operator in H_1. We'll assume that the closure L of the operator L' is self-adjoint and lower bounded operator in H_1.

Under some conditions on the operator coefficients $Q(x)$, $Q_j(x)$, $j = \frac{1}{2,2n-1}$ it is proved that the operator L has a discrete spectrum.

Denote by $\lambda_1, \lambda_2, ..., \lambda_n, ...$ eigenvalues, by $\varphi_1(x), \varphi_2(x), ..., \varphi_n(x), ...$ orthonormed eigenfunctions of the operator L. Denote by $N(\lambda)$ the number of eigenvalues of the operator L, smaller than the given number λ, i.e.

$$N(\lambda) = \sum_{\lambda_n < \lambda} 1.$$

$N(\lambda)$ is said to be a distribution function of eigenvalues of the operator L. The goal of our paper is to study asymptotic behavior of the function $N(\lambda)$ as $\lambda \to \infty$.

Note that A. G. Kostyuchenko and B. M. Levitan [1] have first studied asymptotic behavior of eigenvalues of Sturm-Liouville operator with a self-adjoint operator coefficient. In the paper [2], E. Abdukadyrov generalized the results of [1].

The Green function and asymptotic behavior of the function $N(\lambda)$ for higher order operator-differential equations given on the axis and semi-axis was studied in the papers of M. Bayramoglu [3], H.I. Aslanov [4], A.A. Abudov and H.I. Aslanov [5].

Enumerate the main suppositions under which the asymptotic behavior of the Green function is investigated and an asymptotic formula for eigen values of the operator L is obtained.

1. The operators $Q(x)$ for almost all $x \in [0,\infty]$ are self-adjoint in H, and almost for all x there exists a general domain of definition H dense in $D(Q)$, on which the operators $Q(x)$ are uniformly lower bounded, i.e. there exist $d > 0$ such that for all x and for all $f \in D(Q)$ $(Q(x), f, f) > d(f,f)$.

2. For $|x-\xi| \leq 1$ it holds

$$\left\| [Q(\xi) - Q(x)] Q^{-a}(x) \right\|_H < A|x-\xi|,$$

$$\left\| Q^{-\frac{1}{2n}}(x) Q^{\frac{1}{2n}}(\xi) \right\|_H < C_1,$$

$$\left\| Q^{\frac{1}{2n}}(x) Q^{-\frac{1}{2n}}(\xi) \right\|_H < C_2,$$

where $0 < a < \frac{2n+1}{2n}$, A, C_1, C_2 are constant numbers.

3. For $|x-\xi| > 1$ it holds the inequality

$$\left\| Q(\xi) \exp\left[-\frac{Jm\omega_1}{2} |x-\xi| Q^{\frac{1}{2n}}(x) \right] \right\|_H < B, \quad B = const$$

where $Jm\omega_1 = \min\limits_{i=1,2,..,n} \left\{ Lm\omega_i > 0, \ \ \omega_i^{2n} = -1 \right\}$.

4. For all $x \in [0,\infty]$ it holds the inequality

$$\left\| Q_j(x) Q^{\frac{1-j}{2n}+\varepsilon}(x) \right\| < C, \quad j = \overline{2, 2n-1}, \quad \varepsilon > 0.$$

5. Almost for all $x \in [0,\infty]$ the operator $Q(x)$ is inverse for a completely continuous operator.

Denote by $\beta_1(x) \leq \beta_2(x) \leq \ldots \leq \beta_n(x) \leq \ldots$ the operators $Q(x)$ in the increasing order, for which we'll assume that they are measurable functions. Furthermore, suppose that the series $\sum\limits_{k=1}^{\infty} \beta_k^{\frac{1-4n}{2n}}(x)$ converges almost everywhere, and its sum $F(x) \in L_1[0,\infty]$.

In the paper [4] we have investigated the Green function $G(x,\eta,\mu)$ of the operator L and obtained the following asymptotic formula

$$G(x,\eta,\mu) = g(x,\eta,\mu)[E + r(x,\eta,\mu)] \tag{3}$$

where $\|r(x,\eta,\mu)\|_H = O(1)$ as $\mu \to \infty$ uniformly with respect to (x,η).

Here the function $g(x,\eta,\mu)$ is the Green function of the equation

$$(-1)^n y^{(2n)} + \{Q(x) + \mu\} y = 0,$$

with "frozen" coefficients at the point "ξ" on the axis. It is of the form:

$$g(x,\eta,\mu) = \frac{[Q(x)+\mu E]^{\frac{1-2n}{2n}}}{2ni} \sum_{\alpha=1}^{n} \omega_\alpha e^{i\omega_\alpha [Q(x)+\mu E]^{\frac{1}{2n}} |x-\eta|}. \quad (4)$$

Here ω_k, $k = 1, 2, ..., n$ are the roots from (-1) of degree $2n$ lying in the upper half-plane.

It holds the following main

Theorem. *It conditions 1)-5) are fulfilled, then as $\mu \to \infty$ it holds the formula*

$$\int_0^\infty \frac{N(\lambda)\, d\lambda}{(\lambda+\mu)^3} \sim \frac{C_n}{8} \int_0^\infty \frac{dx}{\{\beta_{kj}(x)+\mu\}^{\frac{4n-1}{2n}}}, \quad (5)$$

where $C_n = \frac{i}{n^2}\left[\sum_{\alpha=1}^{n} \omega_\alpha + \sum_{\substack{\alpha_1 \neq \alpha_2 \\ \alpha_1,\alpha_2=1}}^{n} \frac{\omega_{\alpha_1}\omega_{\alpha_2}}{\omega_{\alpha_1}+\omega_{\alpha_2}}\right]$.

Proof. As $G(x,\eta,\mu)$ is the Green function of the operator L, we can write

$$\varphi_n(x) = (\lambda_n + \mu) \int_0^\infty G(x,\eta,\mu)\, \varphi_n(\eta)\, d\eta. \quad (6)$$

From equality (3) we get

$$\varphi_n(x) \sim (\lambda_n+\mu) \int_0^\infty g(x,\eta,\mu)\, \varphi_n(\eta)\, d\eta \quad \text{as} \quad \mu \to \infty$$

or

$$\frac{\varphi_n(x)}{\lambda_n+\mu} \sim \int_0^\infty g(x,\eta,\mu)\, \varphi_n(\eta)\, d\eta. \quad (7)$$

Denote $a_n = \int_0^\infty g(x,\eta,\mu)\, \varphi_n(\eta)\, d\eta$. Then we have

$$\frac{\|\varphi_n(x)\|_H^2}{(\lambda_n+\mu)^2} \sim \|a_n\|_H^2 .$$

Hence

$$\sum_{n=1}^{N} \frac{1}{(\lambda_n+\mu)^2} \sim \int_0^\infty \left(\sum_{n=1}^{N} \|a_n\|_H^2 \right) dx. \quad (8)$$

The numbers a_n are the Fourier coefficients for the operator-valued function $g(x,\eta,\mu)$ by the orthonormed system of vectors $\{\varphi_n(x)\}$.

Then, from the Parseval equality we have:

$$\sum_{n=1}^{N} \|a_n\|_H^2 = \int_0^{\infty} \sum_{m=1}^{\infty} r_{mm}^2 (x,\eta,\mu)\, d\eta \tag{9}$$

where $r_{ii}(x,\eta,\mu)$ are diagonal elements of the matrix corresponding to the operator $g(x,\eta,\mu)$ in the orthonormed basis made of eigen vectors $\beta_k(x)$ of the operator $Q(x)$, i.e.

$$r_{mm}(x,\eta,\mu) = \frac{[\beta_m(x)+\mu]^{\frac{1-2n}{2n}}}{2ni} \sum_{\alpha=1}^{n} \omega_\alpha e^{i\omega_\alpha [\beta_m(x)+\mu]^{\frac{1}{2n}} |x-\eta|}.$$

Then, from (9) we get:

$$\sum_{m=1}^{\infty} \|a_n\|^2 = \int_0^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{\{\beta_m(x)+\mu\}^{\frac{1-2n}{2n}}}{2ni} \sum_{\alpha=1}^{n} \omega_\alpha e^{i\omega_\alpha [\beta_m(x)+\mu]^{\frac{1}{2n}} |x-\eta|} \right\}^2 d\eta =$$

$$= \sum_{m=1}^{\infty} \frac{\{\beta_m(x)+\mu\}^{\frac{1-2n}{2n}}}{-4n^2} \left\{ \int_0^{\infty} \sum_{\alpha=1}^{n} \omega_\alpha^2 \left[\sum_{\alpha=1}^{n} \omega_\alpha^2 e^{2i\omega_\alpha [\beta_m(x)+\mu]^{\frac{1}{2n}} |x-\eta|} + \right.\right.$$

$$\left.\left. +2 \sum_{\substack{\alpha_1,\alpha_2 \neq 1 \\ \alpha_1=\alpha_2}} \omega_{\alpha_1}\omega_{\alpha_2} e^{i(\omega_{\alpha_1}+\omega_{\alpha_2})\{\beta_m(x)+\mu\}^{\frac{1}{2n}} |x-\eta|} \right\} d\eta =$$

$$= \sum_{m=1}^{\infty} \frac{\{\beta_m(x)+\mu\}^{\frac{1-2n}{2n}}}{-4n^2} \left\{ \sum_{\alpha=1}^{n} \omega_\alpha^2 \sum_{\alpha=1}^{n} \omega_\alpha^2 \int_0^{\infty} e^{2i\omega_\alpha \{\beta_m(x)+\mu\}^{\frac{1}{2n}} |x-\eta|} d\eta + \right.$$

$$\left. +2 + \sum_{\substack{\alpha_1,\alpha_2 \neq 1 \\ \alpha_1=\alpha_2}} \omega_{\alpha_1}\omega_{\alpha_2} \int_0^{\infty} e^{i(\omega_{\alpha_1}+\omega_{\alpha_2})[\beta_m(x)+\mu]^{\frac{1}{2n}} |x-\eta|} d\eta \right\} =$$

$$= \sum_{m=1}^{\infty} \frac{\{\beta_m(x)+\mu\}^{\frac{1-2n}{2n}}}{-4n^2} \left\{ \sum_{\alpha=1}^{n} \frac{\omega_\alpha^2}{2i\omega_\alpha [\beta_m(x)+\mu]} + \right.$$

$$\left. +2 \sum_{\substack{\alpha_1=\alpha_2=1 \\ \alpha_1=\alpha_2}} \frac{\omega_{\alpha_1}\omega_{\alpha_2}}{(\omega_{\alpha_1}+\omega_{\alpha_2})[\beta_m(x)+\mu]^{1/2}} \right\} =$$

$$= \frac{1}{8} \sum_{m=1}^{\infty} \frac{\{\beta_m(x)+\mu\}^{\frac{1-2n}{2n}}}{\{\beta_m(x)+\mu\}^{1/2n}} \left\{ \frac{i}{n^2} \left[\sum_{\alpha=1}^{n} \omega_\alpha + 4 \sum_{\substack{\alpha_1,\alpha_2 \neq 1 \\ \alpha_1=\alpha_2}} \frac{\omega_{\alpha_1}\omega_{\alpha_2}}{\omega_{\alpha_1}+\omega_{\alpha_2}} \right] \right\} =$$

$$= \frac{C_n}{8} \sum_{m=1}^{\infty} [\beta_m(x)+\mu]^{\frac{1-4n}{2n}}$$

where $C_n = \frac{i}{n^2}\left(\sum\limits_{\alpha=1}^{n}\omega_\alpha + \sum\limits_{\substack{\alpha_1,\alpha_2\neq 1\\ \alpha_1=\alpha_2}}\frac{\omega_{\alpha_1}\omega_{\alpha_2}}{\omega_{\alpha_1}+\omega_{\alpha_2}}\right)$.

So,

$$\sum_{n=1}^{\infty}\|a_n\|_H^2 = \frac{C_n}{8}\sum_{m=1}^{\infty}\left[\beta_m(x)+\mu\right]^{\frac{1-4n}{2n}}.$$

By integrating with respect to x in the interval $[0,\infty)$ (taking into account the summability of the function $F(x) = \sum\limits_{m=1}^{\infty}\left[\beta_m(x)+\mu\right]^{\frac{1-4n}{2n}}$ in the interval $[0,\infty)$), we get

$$\int_0^\infty\left(\sum_{n=1}^{\infty}\|a_n\|_H^2\right)dx = \frac{C_n}{8}\sum_{m=1}^{\infty}\int_0^\infty\frac{dx}{\left[\beta_m(x)+\mu\right]^{\frac{4n-1}{2n}}}. \tag{10}$$

Taking into attention (8), we get

$$\sum_{n=1}^{\infty}\frac{1}{(\lambda_n+\mu)^2} \sim \frac{C_n}{8}\sum_{m=1}^{\infty}\int_0^\infty\frac{dx}{\left[\beta_m(x)+\mu\right]^{\frac{4n-1}{2n}}}. \tag{11}$$

It holds the known identity ([6], p. 209)

$$\int_0^\infty\frac{N(\lambda)}{(\lambda+\mu)^3} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(\lambda_n+\mu)^2}. \tag{12}$$

Then from (11) and (12) we can write the relation

$$\int_0^\infty\frac{N(\lambda)}{(\lambda+\mu)^3} \sim \frac{C_n}{16}\sum_{m=1}^{\infty}\int_0^\infty\frac{dx}{\left\{\beta_m(x)+\mu\right\}^{\frac{4n-1}{2n}}}. \tag{13}$$

The theorem is proved.

For obtaining the asymptotic function $N(\lambda)$, we use the following Tauberian theorem of Titchmarch ([6]. pp. 422).

Theorem. *Let $f(x)$ be a non-negative and non-decreasing function, and $x\to\infty$*

$$\int_0^\infty\frac{f(y)\,dy}{(x+y)^\alpha} \sim \int_{-\infty}^{\infty}\frac{d\xi}{\{q(\xi)+x\}^\beta}, \quad \textit{where} \ \ \beta>0,\ \alpha-\beta\geq 1.$$

If $q(x)$ satisfies the condition

$$\frac{c_2}{x^\beta}\int\limits_{\{q(\xi)<x\}}d\xi \leq \int\limits_{\{q(\xi)>x\}}\frac{d\xi}{\{q(\xi)\}^\beta} \leq \frac{c_1}{x^\beta}\int\limits_{\{q(\xi)<x\}}d\xi, \quad c_1,c_2 = const>0$$

then

$$f(x) \sim \frac{C\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\int\limits_{\{q(\xi)<x\}}\{x-q(\xi)\}^{\alpha-\beta-1}d\xi.$$

In order to get an asymptotic formula for the function $N(\lambda)$ from formula (13) with the help of Titchmarch theorem, the following condition should be fulfilled:

a) There exist positive constants C_1 and C_2 such that the following inequality is fulfilled

$$\frac{c_1}{t^{\frac{4n-1}{2n}}}\sum_{m=1}^{\infty}\int\limits_{\{\beta_m(X)\le t\}} dx \le \sum_{m=1}^{\infty}\int\limits_{\{\beta_m(X)>t\}}\frac{dx}{\beta_m^{\frac{4n-1}{2m}}(x)} \le \frac{c_1}{t^{\frac{4n-1}{2n}}}\sum_{m=1}^{\infty}\int\limits_{\{\beta_m(X)\le t\}} dx.$$

It we assume that condition a) is fulfilled, then from (13) we get the following asymptotic formula for the function $N(\lambda)$ as $\lambda\to\infty$

$$N(\lambda)\sim\frac{C_n n^2}{2(2n-1)\Gamma\left(\frac{1}{2n}\right)\Gamma\left(1-\frac{1}{2n}\right)}\sum_{m=1}^{\infty}\int\limits_{\{\beta_m(x)<\lambda t\}}\{\lambda-\beta_m(x)\}^{\frac{1}{2n}}dx.$$

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