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ON RELATION BETWEEN THE NORM AND THE QUASINORM OF THE SUM OF POLYNOMIALS DEPENDENT ON DIFFERENT VARIABLES

Abstract

In the paper, the matter of approximation of a function of two variables by the sums of polynomials of one variable is investigated. At first the relation is established in the form of the estimates between the norms and the quasinorms of the sum of polynomials of one variable. Then the existence in theorem approximation of continuous functions of two variables on a rectangle by the sums of polynomials of one variable is given. Furthermore, a sufficient condition is established for extremal property of the sum of polynomials in approximation of a function of two variables. The lower estimations of such a best approximation are also obtained in the paper.

Consider the sum of polynomials

$$P_{m+n}(x,y)=P_m(x)+Q_n(y)=\sum_{k=0}^{m}a_kx^k+\sum_{q=0}^{n}b_qy^q$$

determined on the rectangle $T=[a,b;c,d]$

Denote by

$$M(P_{m+n})=\|P_{m+n}(x,y)\|_{C(T)}=\max_{(x,y)\in T}|P_{m+n}(x,y)|$$

the norm, by

$$L(P_{m+n})=\sum_{k=0}^{m}|a_k|+\sum_{q=0}^{n}|b_q|$$

the quasinorm of the polynomial $P_{m+n}(x,y)$.

Theorem 1. *There exist the constants A and B such that*

$$M(P_{m+n})\le AL(P_{m+n}) \tag{1}$$

and

$$L(P_{m+n})\le BM(P_{m+n}). \tag{2}$$

Proof. We have

$$M(P_{m+n})\le\max_{x\in[a,b]}|P_m(x)|+\max_{y\in[c,d]}|Q_n(y)|\le$$

$$\le\sum_{k=0}^{m}|a_k|\max_{x\in[a,b]}|x|^k+\sum_{q=0}^{n}|b_q|\max_{y\in[c,d]}|y|^q\,.$$

Denoting $A=\max\limits_{(x,y)\in T}(1,|x|,...,|x|^m,|y|,...,|y|^n)$ we continue the estimation

$$M(P_{m+n})\le A\sum_{k=0}^{m}|a_k|+A\sum_{q=0}^{n}|b_q|=AL(P_{m+n}).$$

Inequality (1) is proved.

Further, according to the inequality established in (1) we have

$$\sum_{k=0}^{m} |a_k| \leq B_1 \|P_m\|_{C[a,b]}$$

and

$$\sum_{q=0}^{n} |b_q| \leq B_2 \|Q_n\|_{C[a,b]} .$$

Then

$$L(P_{m+n}) \leq B_1 \|P_m\|_{C[a,b]} + B_2 \|Q_n\|_{C[a,b]} \leq$$
$$\leq B_1 \|P_{m+n}\|_{C(T)} + B_2 \|P_{m+n}\|_{C(T)} = (B_1 + B_2) \|P_{m+n}\|_{C(T)} .$$

Denoting $B = B_1 + B_2$, we complete the proof of theorem 1

$$L(P_{m+n}) \leq BM(P_{m+n}).$$

On the existence of extremal element for a continuous function of two variables in the class of sums of polynomials of one variable

Consider the approximation of a continuous function of two variables $f(x,y)$ on a rectangle $[a,b;c,d]$ by the sums of polynomials of one variable $P_{m+n}(x,y) = P_m(x) + Q_n(y)$.

Theorem 2. *In approximation of a continuous function of two variables on a rectangle $[a,b;c,d]$ in the class $\{P_{m+n}(x,y)\}$ there exists a best approximating element.*

The proof follows from the theorem on approximation of a function of two variables by polynomials of two variables with a condition on the coefficients of polynomials (although this theorem may be proved separately).

Let $f(x,y)$ be a continuous function of variables x and y given on a rectangle $[a,b;c,d]$. Denote by $H^*_{m,n}$ the set of all polynomials of two variables $P(x,y)$ whose power is at most m with respect to the variable x and at most n with respect to y, besides, their k coefficients are fixed in a definite way.

For each polynomial $P(x,y) \in H^*_{m,n}$

$$\triangle(P) = \max_{\substack{a\leq x\leq b \\ c\leq y\leq d}} |f(x,y) - P(x,y)|$$

that is called a deviation of the polynomial $P(x,y)$ from the function $f(x,y)$ in $[a,b;c,d]$. We denote

$$E^*_{m,n} = \inf\{\Delta(P)\},$$

where $P(x,y)$ ranges over the class $H^*_{m,n}$. The number $E^*_{m,n}$ is called the least deviation of polynomials of the class $H^*_{m,n}$ from the function $f(x,y)$.

Theorem 3 [1]. *For each continuous function $f(x,y)$ in $[a,b;c,d]$ in the class $H^*_{m,n}$ there exists a polynomial of best approximation $Q(x,y)$, i.e. such a polynomial for which $\Delta(Q) = E^*_{m,n}$.*

For applying this theorem, we must take the conditions on the coefficients of a polynomial of two variables $P(x,y)=\sum_{k=0}^{m}\sum_{q=0}^{n}a_{kq}x^k y^q$ in the form

$$a_{kq}=\begin{cases} 0, & \text{for} \quad k\neq 0, \quad q=\overline{1,n} \\ 0, & \text{for} \quad q\neq 0, \quad k=\overline{1,m} \\ a_{kq} & \text{for} \quad q=0 \text{ and } k\neq 0 \end{cases}$$

Sufficient condition for the best approximation sum

Theorem 4. *Let $f(x,y)$ be a continuous function in $T=[a,b;c,d]$ and $P_{m+n}(x,y)$ be a polynomial of the form $P_m(x)+Q_n(y)$.*

In order $P_{m+n}(x,y)$ be the best approximation in the class $\{P_m(x)+Q_n(y)\}$ for the function $f(x,y)$ in T, the existence of $n+2$ points in T that are arranged on a straight line parallel to the axis y (or $m+2$ points arranged on a straight line parallel to the axis x) wherein the difference

$$f(x,y)-[P_m(x)+Q_n(y)]$$

takes the values

$$\max|f(x,y)-P_m(x)-Q_n(y)|$$

with alternately interlacing signs, is enough.

Proof. Let there exist some fixed point $x_0\in[a,b]$ and the points $y_1<y_2<\ldots<y_{n+2}$ from $[c,d]$ arranged on the segment along the axis y be such that

$$f(x_0,y_i)-P^0_{m+n}(x_0,y_i)=(-1)^k\Delta(P_{m+n}),\quad i=\overline{1,n+2}.$$

Show that

$$P^0_{m+n}=P^0_m(x)+Q^0_n(y)$$

is a polynomial of best approximation in the class $\{P_m(x)+Q_n(y)\}$ for the function $f(x,y)$ in T. Assume the contrary that $P^0_m(x)+Q^0_n(y)=P^0_{m+n}(x,y)$ is not a polynomial of best approximation of the form $P_m(x)+Q_n(y)$ in T for f.

By theorem 1, there exists the sum

$$P^*_m(x)+Q^*_n(y)=Q^*_{m+n}(x,y)$$

that is the best approximator for f i.e. such that

$$\left|f(x,y)-P^*_{m+n}(x,y)\right|=E_{m,n}<\Delta(P^0_{m+n})$$

on the rectangle T.

Consider the difference

$$P_{m,n}(x,y)=Q_{m+n}(x,y)-P^*_{m+n}(x,y)$$

that may be written as follows

$$P(x,y) = \left[f(x,y) - P^0_{m+n}(x,y)\right] - f(x,y) - Q^*_{m,n}(x,y).$$

By the condition, at the points (x_0, y_i), $i = \overline{1, n+2}$ the difference $f(x,y) - Q_{m+n}(x,y)$ takes the values $+E_f$ or $-E_f$.

Let at the point (x_0, y_i)

$$f(x_0, y_i) - Q^*_{m+n}(x_0, y_i) = +E_f.$$

Then

$$f(x_0, y_i) - P_{m+n}(x_0, y_i) > |f(x_0, y_i) - Q_{m+n}(x_0, y_i)| = E_f > 0,$$

i.e. at this point the difference is positive: then by the condition at the point (x_0, y_{i+1})

$$f(x_0, y_{i+1}) - Q_{m+n}(x_0, y_{i+1}) = -E_f$$

and

$$f(x_0, y_{i+1}) - P(x_0, y_{i+1}) < -E_f < 0. \tag{3}$$

Thus, at $n+2$ points, the difference $P(x,y)$ changes the sign and since $P(x,y)$ is a polynomial of power $n+2$ (with respect to y), it is possible only in the case

$$P(x_0, y) \equiv 0.$$

Then

$$Q(x_0, y) = P(x_0, y).$$

Take one of the points y_i, $i = \overline{1, n+2}$. On one hand, on the rectangle T

$$P(x_0, y_i) = Q(x_0, y_i) = \Delta(P)$$

on the other hand, by (3)

$$|f(x_0, y_i - Q(x_0, y_i)| < \Delta(P).$$

This contraction completes the proof of theorem 4.

On estimation of the best lower approximation.

Let $f(x,y)$ be a continuous function in a rectangle $T = [a, b; c, d]$. Consider the set $\{P_{m,n}\}$ of all polynomials of two variables of power m with respect to x, and n with respect to y :

$$P_{m,n} = \sum_{k=0}^{m} \sum_{\nu=0}^{n} a_{k\nu} x^k y^\nu$$

and the set $\{P_{m+n}(x,y)\}$ of all sum of two polynomials of power m with respect to x and of power n with respect to y:

$$P_{m+n} = P_m(x) + Q_n(y).$$

Consider the best approximations

$$E_{m,n} = E_{m,n}\left[f, P_{m,n}\right] = \inf_{P \in P_{m,n}} \max_{(x,y) \in T} |f(x,y) - P(x,y)|$$

and

$$E_{m+n} = E_{m+n}\left[f, P_{m+n}\right] = \inf_{P \in P_{m+n}} \max_{(x,y)\in T} \left|f(x,y) - P(x,y)\right|.$$

Theorem 5. *If for some fixed $y^* \in [c,d]$ the difference $f(x,y) - P_{m+n}(x,y)$ takes at the sequential points $x_1 < x_2 < ... < x_{m+2}$ of the segment $[a,b]$ of the axis x the non-zero values $\mu_1, \mu_2, ..., \mu_{m+2}$ with alternately interlacing signs, then*

$$E_{m+n} \geq \sigma = \min\left\{|\mu_1|, |\mu_2|, ..., |\mu_{m+2}|\right\}. \tag{4}$$

Theorem 5 is proved by the inverse assumption, similar to theorem 4.

Proof. In [1] it is established

Theorem 6 [1]. *Let $f(x,y)$ be continuous in $[a,b;c,d]$ and $P(x,y)$ be a polynomial of class $P_{m,n}$. If for some fixed $y^* \in [c,d]$ the difference $f(x,y^*) - P(x,y^*)$ accepts at the sequential points $x_1 < x_2 < ... < x_{m+2}$ of the segment $[a,b]$ on the axis x the non-zero values $\mu_1, \mu_2, ..., \mu_{m+2}$ with alternately interlacing signs, then*

$$E_{m,n} \geq \sigma = \min\left\{|\mu_1|, |\mu_2|, ..., |\mu_{m+2}|\right\}.$$

Further, since $\{P_{m+n}\} \subset \{P_{m,n}\}$, then

$$E_{m+n} \geq E_{m,n}$$

this together with theorem 5 yields (4).

Denote by $\left\{P^*_{m+n}\right\}$ the set of all polynomials of the form $P_{m+n}(x,y)$ for each of which there is a set of numbers $\left\{\mu_1, \mu_2, ..., \mu_{m+2}\right\}$ satisfying the condition of theorem 5, and denote by $\left\{P^*_{m,n}\right\}$ the set of polynomials of two variables for which there exists a set of collection of numbers $\left\{\left(\mu_1, \mu_2, ..., \mu_{m+2}\right)\right\}$ satisfying the condition of theorem 6.

Corollary 1. *The estimations*

$$E_{m,n} \geq \sup_{P^*_{m,n} \in \left\{P^*_{m,n}\right\}} \min\left\{|\mu_1|, |\mu_2|, ..., |\mu_{m+2}|\right\},$$

$$E_{m+n} \geq \sup_{P^*_{m+n} \in \left\{P^*_{m+n}\right\}} \min\left\{|\mu_1|, |\mu_2|, ..., |\mu_{m+2}|\right\}$$

are valid,

$$E_{m+n} \geq \sup_{P^*_{m,n} \in \left\{P^*_{m,n}\right\}} \min\left\{|\mu_1|, |\mu_2|, ..., |\mu_{m+2}|\right\}.$$

is obvious.

References

[1]. Bezborodnikov M.F. *To the question on the best approximation of functions of many groups of variables by polynomials.* Izvestia vysshykh uchebnykh zavedeniy, 1962, No 4 (29), pp. 3-12. (Russian)

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