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ON THE EXISTENCE OF THE SOLUTION OF NONLOCAL CONDITION OPTIMAL CONTROL PROBLEM FOR HYPERBOLIC TYPE EQUATION

Abstract

In the paper, a nonlocal condition optimal control problem is considered for a second order hyperbolic equation. At first, the existence of the solution of initial boundary value problem is proved for each control. Then a theorem on the existence of optimal control is proved.

1. Problem Statement. Assume that the controlled process is described by the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}-\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_j}\right)=f(x,t,u(x,t),\vartheta(x,t)), \qquad (1.1)$$

with initial conditions

$$u(x,0)=\varphi_0(x),\ \frac{\partial u(x,0)}{\partial t}=\varphi_1(x) \qquad (1.2)$$

and nonlocal condition

$$\sum_{i,j=1}^{m}a_{ij}(x,t)\frac{\partial u}{\partial x_j}\cos(v,x_i)=\int_{\Omega}K(x,y,t,u(y,t))\,dy \ \text{ on } \ S_T. \qquad (1.3)$$

Here $u(x,t)$ describes the state of the process, $\vartheta(x,t)$ is a control function, $Q_T=\{(x,t)\,|x\in\Omega,0<t<T\}$, where Ω is a bounded domain in R^m with smooth boundary $\partial\Omega$, $S_T=\{(x,t)\,|x\in\partial\Omega,0<t<T\}$ is a lateral surface of the cylinder Q_T, v is an external normal to S_T.

As a class of admissible controls U_{ad} we take a set of measurable and bounded r - dimensional vector-functions $\vartheta(x,t)$ in Q_T such that almost for all (x,t), the values of these functions belong to the compact set $V\subset R^r$, $V\neq\varnothing$.

We state a problem: find an admissible control from U_{ad} that together with appropriate solution of problem (1.1)-(1.3) delivers minimum to the functional

$$J(\vartheta)=\int_{Q_T}f(x,t,u(x,t),\vartheta(x,t))\,dxdt. \qquad (1.4)$$

For the given control function $\vartheta(x,t)$, under the solution of problem (1.1)-(1.3) we understand the function $u(x,t)\in W_2^1(Q_T)$ that for any function $\Phi(x,t)\in W_2^1(Q_T)$ such that $\Phi(x,T)=0$ satisfies the integral identity

$$\int_{Q_T}\left(-\frac{\partial u(x,t)}{\partial t}\frac{\partial\Phi(x,t)}{\partial t}+\sum_{i,j=1}^{m}a_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_j}\frac{\partial\Phi(x,t)}{\partial x_i}\right)dxdt-$$

$$-\int_0^T\int_{\partial\Omega}\Phi(x,t)\int_{\Omega}K(x,y,t,u(y,t))\,dydsdt=$$

$$=\int_{Q_T}f(x,t,u(x,t),\vartheta(x,t))\,\Phi(x,t)\,dxdt+\int_{\Omega}\varphi_1(x)\,\Phi(x,0)\,dx, \qquad (1.5)$$

and the fulfilment of the condition $u(x,0)=\varphi_0(x)$ is understood in the sense $\lim\limits_{t\to+0}\int\limits_{\Omega}(u(x,t)-\varphi_0(x))^2\,dx=0$. Such a solution is called a generalized solution of problem (1.1)-(1.3).

We'll assume that the following conditions are fulfilled:

1^0. $a_{ij}(x,t),\dfrac{\partial a_{ij}(x,t)}{\partial t}\in C\left(\overline{Q}_T\right)$, $i,j=1,2,...,m$, moreover $\forall\xi\in R^m$ and for all $(x,t)\in\left(\overline{Q}_T\right)$

$\sum\limits_{i,j=1}^{m}a_{ij}(x,t)\,\xi_i\xi_j\geq\mu\sum\limits_{i=1}^{m}\xi_i^2$, $\mu=const>0$, $a_{ij}(x,t)=a_{ji}(x,t)$;

2^0. $\varphi_0(x)\in W_2^1(\Omega)\,,\varphi_1(x)\in L_2(\Omega)$;

3^0. The functions $f(x,t,u,\vartheta)$ and $f_0(x,t,u,\vartheta)$ are continuous on $\overline{Q}_T\times R\times V$, the function $f(x,t,u,\vartheta)$ satisfies the Lipshits condition u uniformly with respect to $(x,t)\in\overline{Q}_T$ and $\vartheta\in V$; the function $f_0(x,t,u,\vartheta)$ satisfies the condition $|f_0(x,t,u,\vartheta)|\leq a_0+b_0|u|^2$, where $a_0,b_0=const>0$; the function $K(x,y,t,u)$ is continuous on $\overline{\partial\Omega}\times\overline{Q}_T\times R$ and has continuous derivatives $\dfrac{\partial K}{\partial t}$, $\dfrac{\partial K}{\partial u}$, moreover $K(x,y,t,0)=0$, $\dfrac{\partial K(x,y,t,0)}{\partial t}=0$, $\left|\dfrac{\partial K(x,y,t,u)}{\partial u}\right|\leq M$, $K(x,y,t,u)$ and $\dfrac{\partial K(x,y,t,u)}{\partial t}$ satisfy the Lipschits condition with respect to u, $M=const>0$;

4^0. For each point $(x,t,u)\in\overline{Q}_T\times R$, the set

$$R^+(x,t,u)=\left\{(\eta,\xi)\in R^2|\eta\geq f_0(x,t,u,\vartheta)\,,\xi=f(x,t,u,\vartheta)\,,\vartheta\in V\right\}$$

is closed and convex in R^2.

The following theorem holds true.

Theorem 1. *Subject to conditions* $1^0,2^0,3^0$, *the mixed problem (1.1)-(1.3) for each* $\vartheta(x,t)\in U_{ad}$ *has a unique solution. And for the aggregate of the solutions of problem (1.1)-(1.3) corresponding to all admissible controls, the following estimation is valid*

$$\|u\|_{W_2^1(Q_T)}\leq const. \qquad (1.6)$$

Proof. Use the Galerkin method. Let $\{\varphi_k(x)\}$ be a fundamental system in $W_2^1(\Omega)$, and the following orthonormality property be fulfilled:

$$\int_{\Omega}\varphi_k(x)\,\varphi_l(x)\,dx=\delta_k^l.$$

We look for the approximate solution $u^N(x,t)$ of problem (1.1)-(1.3) in the form

$$u^N(x,t)=\sum_{k=1}^{N}C_k^N(t)\,\varphi_k(x)$$

from the relations

$$\int_{\Omega}\frac{\partial^2 u^N(x,t)}{\partial t^2}\varphi_l(x)\,dx+\int_{\Omega}\sum_{i,j=1}^{m}a_{ij}(x,t)\,\frac{\partial u^N(x,t)}{\partial x_j}\frac{\partial\varphi_l(x)}{\partial x_i}dx-$$

$$-\int_{\partial\Omega}\varphi_l(x)\int_{\Omega}K\left(x,y,t,u^N(y,t)\right)dyds=$$

$$=\int_{\Omega}f\left(x,t,u^N(x,t),\vartheta(x,t)\right)\varphi_l(x)\,dx,\quad l=1,...,N, \tag{1.7}$$

$$C_k^N(0)=\alpha_k^N,\quad \left.\frac{dC_k^N(t)}{dt}\right|_{t=0}=\beta_k^N, \tag{1.8}$$

where α_k^N and β_k^N are the coefficients of the sums $\varphi_0^N(x)=\sum\limits_{k=1}^{N}\alpha_k^N\varphi_k(x)$ and $\varphi_1^N(x)=\sum\limits_{k=1}^{N}\beta_k^N\varphi_k(x)$ approximating as $n\to\infty$ the functions $\varphi_0(x)$ and $\varphi_1(x)$ in the norms $W_2^1(\Omega)$ and $L_2(\Omega)$, respectively.

It is clear that system (1.7) is a system of second order ordinary differential equations with respect to t for the unknowns $C_k^N(k)$, $k=1,2,...,N$, solved with respect to $\frac{d^2C_k^N}{dt^2}$. Thus, $\forall N$ system (1.7) is uniquely solvable under initial conditions (1.8) [1], moreover, $\frac{d^2C_k^N}{dt^2}\in L_2(0,T)$.

Show that for $u^N(x,t)$ the following estimation is valid:

$$\int_{\Omega}\left(\left(u^N(x,t)\right)^2+\sum_{i=1}^{m}\left(\frac{\partial u^N(x,t)}{\partial x_i}\right)^2+\left(\frac{\partial u^N(x,t)}{\partial t}\right)^2\right)dx\le C(T)\,,\forall t\in[0,T]\,. \tag{1.9}$$

Indeed, by multiplying each of the equalities (1,7) by its own $\frac{dC_l^N}{dt}$, we arrive at the equality

$$\int_{\Omega}\frac{\partial^2 u^N(x,t)}{\partial t^2}\frac{\partial u^N(x,t)}{\partial t}dx+\int_{\Omega}\sum_{i,j=1}^{m}a_{ij}(x,t)\,\frac{\partial u^N(x,t)}{\partial x_j}\frac{\partial^2 u^N(x,t)}{\partial t\partial x_i}dx-$$

$$-\int_{\partial\Omega}\frac{\partial u^N(x,t)}{\partial t}\int_{\Omega}K\left(x,y,t,u^N(y,t)\right)dyds=$$

$$=\int_{\Omega}f\left(x,t,u^N(x,t),\vartheta(x,t)\right)\frac{\partial u^N(x,t)}{\partial t}dx.$$

By integrating it with respect to t from 0 to t, $t\in[0,T]$, we get

$$\int_{\Omega}\left(\left(\frac{\partial u^N(x,t)}{\partial t}\right)^2+\sum_{i,j=1}^{m}a_{ij}(x,t)\,\frac{\partial u^N(x,t)}{\partial x_j}\frac{\partial u^N(x,t)}{\partial x_i}\right)dx-$$

$$-2\int_0^t\int_{\partial\Omega}\frac{\partial u^N(x,t)}{\partial t}\int_\Omega K\left(x,y,t,u^N(y,t)\right)dydsdt=$$

$$=\int_\Omega\left(\left(\frac{\partial u^N(x,0)}{\partial t}\right)^2+\sum_{i,j=1}^m a_{ij}(x,0)\frac{\partial u^N(x,0)}{\partial x_j}\frac{\partial u^N(x,0)}{\partial x_i}\right)dx+$$

$$+\int_0^t\int_\Omega\sum_{i,j=1}^m\frac{\partial a_{ij}(x,t)}{\partial t}\frac{\partial u^N(x,t)}{\partial x_j}\frac{\partial u^N(x,t)}{\partial x_i}dxdt+$$

$$+2\int_0^t\int_\Omega f\left(x,t,u^N(x,t),\vartheta(x,t)\right)\frac{\partial u^N(x,t)}{\partial t}dxdt.$$

Further, assuming $y^N(t)=\int_\Omega\left(\left(\frac{\partial u^N(x,t)}{\partial t}\right)^2+\sum_{i,j=1}^m a_{ij}\frac{\partial u^N(x,t)}{\partial x_i}\frac{\partial u^N(x,t)}{\partial x_j}\right)dx$, we get

$$y^N(t)=y^N(0)+\int_0^t\int_\Omega\sum_{i,j=1}^m\frac{\partial a_{ij}(x,t)}{\partial t}\frac{\partial u^N(x,t)}{\partial x_j}\frac{\partial u^N(x,t)}{\partial x_i}dxdt+$$

$$+2\int_0^t\int_{\partial\Omega}\frac{\partial u^N(x,t)}{\partial t}\int_\Omega K\left(x,y,t,u^N(y,t)\right)dydsdt+$$

$$+2\int_0^t\int_\Omega f\left(x,t,u^N(x,t),\vartheta(x,t)\right)\frac{\partial u^N(x,t)}{\partial t}dxdt. \qquad (1.10)$$

Transform the integral along the lateral surface of the cylinder S_t in the following way:

$$\int_0^t\int_{\partial\Omega}\frac{\partial u^N(x,t)}{\partial t}\int_\Omega K\left(x,y,t,u^N(y,t)\right)dydsdt=i_1+i_2+i_3+i_4,$$

where

$$i_1=-\int_0^t\int_{\partial\Omega}u^N(x,t)\int_\Omega\frac{\partial K\left(x,y,t,u^N(y,t)\right)}{\partial t}dydsdt,$$

$$i_2=-\int_0^t\int_{\partial\Omega}u^N(x,t)\int_\Omega\frac{\partial K\left(x,y,t,u^N(y,t)\right)}{\partial u^N}\frac{\partial u^N(y,t)}{\partial t}dydsdt,$$

$$i_3=\int_{\partial\Omega}u^N(x,t)\int_\Omega K\left(x,y,t,u^N(y,t)\right)dyds,$$

$$i_4=-\int_{\partial\Omega}u^N(x,t)\int_\Omega K\left(x,y,0,u^N(y,0)\right)dyds.$$

Using the known inequality [2]

$$\int_{\partial\Omega} |W(x,t)|\, ds \le \alpha \int_{\Omega} \left(|W(x,t)| + |\nabla W(x,t)|\right) dx \tag{1.11}$$

and the Cauchy- Bunyakowsky inequality, we get

$$|i_1| = \left| -\int_0^t \int_{\partial\Omega} u^N(x,t) \int_{\Omega} \frac{\partial K\left(x,y,t,u^N(y,t)\right)}{\partial t} dydsdt \right| \le$$

$$\le L \int_0^t \left(\int_{\partial\Omega} \left|u^N(x,t)\right| ds \int_{\Omega} \left|u^N(y,t)\right| dy \right) dt \le$$

$$\le L\alpha \int_0^t \int_{\Omega} \left(\left|u^N(x,t)\right| \int_{\Omega} \left|u^N(x,t)\right| dx + \left|\nabla u^N(x,t)\right| \int_{\Omega} \left|u^N(x,t)\right| dx \right) dxdt \le$$

$$\le \frac{L\alpha}{2} \int_0^t \int_{\Omega} \left(\left|u^N(x,t)\right|^2 + \left|\nabla u^N(x,t)\right|^2 \right) dxdt +$$

$$+ L\alpha \left(mes\Omega\right)^2 \int_0^t \int_{\Omega} \left|u^N(x,t)\right|^2 dxdt,$$

where L is Lipschits coefficient, $mes\Omega$ is the measure of the domain Ω.

Now estimate i_2:

$$|i_2| \le M \int_0^t \int_{\partial\Omega} \left|u^N(x,t)\right| \int_{\Omega} \left| \frac{\partial u^N(y,t)}{\partial t} \right| dydsdt \le$$

$$\le \frac{M\alpha}{2} \int_0^t \int_{\Omega} \left(\left|u^N(x,t)\right|^2 + \left|\nabla u^N(x,t)\right|^2 \right) dxdt +$$

$$+ M\alpha \left(mes\ \Omega\right)^2 \int_0^t \int_{\Omega} \left| \frac{\partial u^N(x,t)}{\partial t} \right|^2 dxdt.$$

Further estimate i_3:

$$|i_3| \le \int_{\partial\Omega} \left|u^N(x,t)\right| \int_{\Omega} \left|K\left(x,y,t,u^N(y,t)\right)\right| dyds \le$$

$$\le L \int_{\partial\Omega} \left|u^N(x,t)\right| \int_{\Omega} \left|u^N(y,t)\right| dyds \le$$

$$\leq L\alpha\int\limits_{\Omega}\left(\left|u^N(x,t)\right|\int\limits_{\Omega}\left|u^N(x,t)\right|dx+\left|\nabla u^N(x,t)\right|\int\limits_{\Omega}\left|u^N(x,t)\right|dx\right)dx\leq$$

$$\leq\frac{L\alpha}{2}\int\limits_{\Omega}\left(\left|u^N(x,t)\right|^2+\left|\nabla u^N(x,t)\right|^2\right)dx+$$

$$+L\alpha\,(mes\ \Omega)^2\int\limits_{\Omega}\left|u^N(x,t)\right|^2dx. \tag{1.12}$$

Introduce the denotation

$$Z^N(t)\equiv\int\limits_{\Omega}\left(\left|u^N(x,t)\right|^2+\left|\nabla u^N(x,t)\right|^2+\left|\frac{\partial u^N(x,t)}{\partial t}\right|^2\right)dx.$$

Let $A=L\alpha\left(\frac{1}{2}+(mes\ \Omega)^2\right)<1.$

Here and in the sequel, c will denotes different constants.

Then by means of (1.12) we can estimate i_4:

$$|i_4|\leq Z^N(0).$$

It is clear that

$$\int\limits_{\Omega}\left(u^N(x,t)\right)^2dx\leq2\int\limits_{\Omega}\left(u^N(x,0)\right)^2dx+2t\int\limits_0^t y^N(t)\,dt. \tag{1.13}$$

Now, putting together (1.10) and (1.13), we get

$$y^N(t)+\int\limits_{\Omega}\left(u^N(x,t)\right)^2dx\leq y^N(0)+2\int\limits_{\Omega}\left(u^N(x,0)\right)^2dx+2t\int\limits_0^t y^N(t)\,dt+$$

$$+\int\limits_0^t\int\limits_{\Omega}\sum_{i,j=1}^m\frac{\partial a_{ij}(x,t)}{\partial t}\frac{\partial u^N(x,t)}{\partial x_j}\frac{\partial u^N(x,t)}{\partial x_i}dxdt$$

$$+2\int\limits_0^t\int\limits_{\partial\Omega}\frac{\partial u^N(x,t)}{\partial t}\int\limits_{\Omega}K\left(x,y,t,u^N(y,t)\right)dydsdt+$$

$$+2\int\limits_0^t\int\limits_{\Omega}f\left(x,t,u^N(x,t),\vartheta(x,t)\right)\frac{\partial u^N(x,t)}{\partial t}dxdt.$$

Hence, under the conditions on the coefficients $a_{ij}(x,t)$ and on $f(x,t,u,\vartheta)$, allowing for the estimations i_1, i_2, i_3, i_4 we have:

$$Z^N(t)\leq cZ^N(0)+(c+ct)\int\limits_0^t Z^N(t)\,dt+c\int\limits_0^t\int\limits_{\Omega}\left(f\left(x,t,0,\vartheta(x,t)\right)\right)^2dxdt.$$

Applying the Gronwall lemma to this inequality, we get

$$Z^N(t) \le c(T), \quad t \in [0,T].$$

Hence estimation (1.9) follows.

Integrating with respect to t, from (1.9) we can get the estimation

$$\left\| u^N(x,t) \right\|_{W_2^1(Q_T)} \le const \tag{1.9'}$$

It should be noted that estimation (1.19) was obtained uniformly for $\vartheta \in U_{ad}$.

By (1.9), from the sequence $\left\{ u^N(x,t) \right\}$ we can choose a subsequence converging weakly in $W_2^1(Q_T)$ to some element $u(x,t) \in W_2^1(Q_T)$. Then by the imbedding theorem $W_2^1(Q_T) \subset L_2(Q_T)$ the same subsequence $u^N(x,t)$ converges strongly in $L_2(Q_T)$ to the element $u(x,t)$. Therefore, by the conditions on $f(x,t,u,\vartheta)$

$$f(x,t,u(x,t),\vartheta(x,t)) \text{ strongly in } L_2(Q_T).$$

Show that $u(x,t)$ is a generalized solution of problem (1.1)-(1.3). In order to prove the validity of identity (1.5) for the limit function $u(x,t)$, multiply each of the equations of (1.7) by its own function $\chi_l(t) \in W_2^1(0,T)$, $\chi_l(T) = 0$, sum up the obtained equality on all l from 1 to N, integrate with respect to t from 0 to T, then in the first term integrate by parts carrying over $\frac{\partial}{\partial t}$ from u^N on $\eta \equiv \sum_{l=1}^{N} \chi_l(t)\varphi_l(x)$.

This gives us the identity

$$\int_{Q_T} \left(-\frac{\partial u^N(x,t)}{\partial t}\frac{\partial \eta(x,t)}{\partial t} + \sum_{i,j=1}^{m} a_{ij}\frac{\partial u^N(x,t)}{\partial x_j}\frac{\partial \eta(x,t)}{\partial x_i} \right) dxdt -$$

$$-\int_0^T \int_{\partial\Omega} \eta(x,t) \int_{\Omega} K\left(x,y,t,u^N(y,t)\right) dydsdt =$$

$$= \int_{Q_T} f\left(x,t,u^N(x,t),\vartheta(x,t)\right)\eta(x,t)\,dxdt + \int_{\Omega} \frac{\partial u^N(x,t)}{\partial t}\eta(x,0)\,dx, \tag{1.14}$$

valid $\forall \eta$ of the form $\sum_{l=1}^{N} \chi_l(t)\varphi_l(x)$. Denote the aggregate of such η by m_N. In (1.14) we can pass to limit by the subsequence chosen above for the fixed η from any m_N. This leads to identity (1.5) for the limit function $u(x,t)$ for any $\eta \in m_N$. Since $u(x,t) \in W_2^1(Q_T)$, (1.5) will be fulfilled for $u(x,t)$ at $\forall \eta(x,t) \in W_2^1(Q_T)$, $\eta(x,T) = 0$.

So, we proved that the limit function $u(x,t)$ is a generalized solution of problem (1.1)-(1.3) from $W_2^1(Q_T)$.

The uniqueness of the solution of problem (1.1)-(1.3) is proved in standard way.

Since the norm in the Hilbert space is weakly lower semi-continuous, it follows from (1.9) that for the limit function $u(x,t)$ the following estimation is valid:

$$\int\limits_{\Omega}\left((u(x,t))^2+\sum_{i=1}^{m}\left(\frac{\partial u(x,t)}{\partial x_i}\right)^2+\left(\frac{\partial u(x,t)}{\partial t}\right)^2\right)dx\le c(T),\quad \forall t\in[0,T].$$

Hence the estimation (1.6) follows.
The theorem is proved.

2. On the existence of optimal control

Theorem 2. *Let conditions 1^0-4^0 be fulfilled. Then an optimal control exists in problem (1.1)-(1.4).*

Proof. Denote by γ the lower bound of the functional $J(\vartheta)$ in the set U_{ad}:

$$\gamma=\inf_{\vartheta\in U_{ad}}J(\vartheta).$$

From the condition $U_{ad}\neq\varnothing$ it follows that $\gamma<+\infty$. Show that $\gamma>-\infty$ Assume that $\{\vartheta_k(x,t)\}$ is a minimizing sequence of admissible controls. Denote by $u_k(x,t)$ the solution of problem (1.1)-(1.3) corresponding to $\vartheta_k(x,t)$.

Then $\gamma=\lim\limits_{k\to\infty}J(\vartheta_k)=\lim\limits_{k\to\infty}\int\limits_{Q_T}f_0(x,t,u_k(x,t),\vartheta_k(x,t))\,dxdt$.

Since by theorem 1, $\|u_k\|_{W_2^1(Q_T)}\le const$, then from the sequence $\{u_k(x,t)\}$ we can choose such a sub sequence (denote it also by $\{u_k(x,t)\}$) that

$$u_k\to u_0\quad\text{weakly in}\quad W_2^1(Q_T)\quad\text{as}\quad k\to\infty. \tag{2.1}$$

Then by the theorem on compactness of the imbedding [2], as $k\to\infty$ we have:

$$u_k\to u_0\quad\text{strongly in}\quad L_2(Q_T). \tag{2.2}$$

According to (2.1), as $k\to\infty$

$$\frac{\partial u_k}{\partial t}\to\frac{\partial u_0}{\partial t}\quad\text{weakly in}\quad L_2(Q_T), \tag{2.3}$$

$$\frac{\partial u_k}{\partial x_i}\to\frac{\partial u_0}{\partial x_i}\quad\text{weakly in}\quad L_2(Q_T)\ \ i=1,2,...,m.$$

From the conditions imposed on the function $f_0(x,t,u,\vartheta)$, and from the conditions $\|u_k(x,t)\|_{W_2^1(Q_T)}\le const$ it follows that $-\infty<\gamma<+\infty$. Further, from (2.2) it follows that the sequence $\{u_k(x,t)\}$ as $k\to\infty$ converges to the measure $u_0(x,t)$. Consequently, from this sequence we can choose such a subsequence (denote it also by $\{u_k(x,t)\}$) that as $k\to\infty$ $u_k(x,t)\to u_0(x,t)$ almost everywhere in Q_T. From the condition imposed on $f(x,t,u,\vartheta)$ we get that the sequence $\{f(x,t,u_k(x,t),\vartheta_k(x,t))\}$ is bounded in $L_2(Q_T)$ and we can assume that as $k\to\infty$

$$f(x,t,u_k(x,t),\vartheta_k(x,t))\to Z(x,t)\ \text{weakly in}\ L_2(Q_T). \tag{2.4}$$

Then by Mazur's theorem [3] one can construct such a convex combination

$$\psi_s(x,t)=\sum_{l=1}^{k}\alpha_{ls}f\left(x,t,u_{n_s+l}(x,t),\vartheta_{n_s+l}(x,t)\right)\left(\alpha_{ls}\geq 0,\sum_{l=1}^{k}\alpha_{ls}=1\right), \tag{2.5}$$

that as $s\to\infty$ it strongly converges to $Z(x,t)$ in $L_2(Q_T)$ (generally speaking, k depends on s). Hence it follows that there is a sequence $\{\psi_s(x,t)\}$ that converges to $Z(x,t)$ almost everywhere in Q_T.

Assume

$$\lambda_s(x,t)=\sum_{l=1}^{k}\alpha_{ls}f_0\left(x,t,u_{n_s+l}(x,t),\vartheta_{n_s+l}(x,t)\right) \tag{2.6}$$

and denote $\underline{\lim}_{s\to\infty}\lambda_s(x,t)=Z_0(x,t)$. From the conditions on the function $f_0(x,t,u,\vartheta)$ it follows that $Z_0(x,t)$ is integrable and finite almost everywhere in Q_T.

By the Fatou lemma it is clear that

$$\int_{Q_T}Z_0(x,t)\,dxdt\leq\underline{\lim}_{s\to\infty}\int_{Q_T}\lambda_s(x,t)\,dxdt=$$

$$=\underline{\lim}_{s\to\infty}\sum_{l=1}^{k}\alpha_{ls}\int_{Q_T}f_0\left(x,t,u_{n_s+l}(x,t),\vartheta_{n_s+l}(x,t)\right)dxdt=\underline{\lim}_{s\to\infty}\sum_{l=1}^{k}\alpha_{ls}J\left(\vartheta_{n_s+l}\right).$$

On the other hand

$$\underline{\lim}_{k\to\infty}J(\vartheta_k)=\lim_{k\to\infty}J(\vartheta_k)=\lim_{k\to\infty}\int_{Q_T}f_0\left(x,t,u_k(x,t),\vartheta_k(x,t)\right)dxdt=\gamma.$$

Hence we get

$$\underline{\lim}_{k\to\infty}\sum_{l=1}^{k}\alpha_{ls}J\left(\vartheta_{n_s+l}\right)=\underline{\lim}_{k\to\infty}\sum_{l=1}^{k}\alpha_{ls}\int_{Q_T}f_0\left(x,t,u_{n_s+l}(x,t),\vartheta_{n_s+l}(x,t)\right)dxdt=\gamma.$$

So,

$$\int_{Q_T}Z_0(x,t)\,dxdt\leq\gamma. \tag{2.7}$$

Now, show that $(Z_0(x,t),Z(x,t))\in R^+(x,t,u_0(x,t))$. Denote by Q_1 the set of such points $(x,t)\in Q_T$ for which $Z_0(x,t)$ is finite as $s\to\infty$ $\psi_s(x,t)\to Z(x,t)$ and as $k\to\infty$ $u_k(x,t)\to u_0(x,t)$.

It is clear that $mesQ_1=mesQ_T$. For each k determine the set $E_k=\{(x,t)\,|\,(x,t)\in Q_T,\vartheta_k(x,t)\,\overline{\in}V\}$, $k=1,2,...$. By definition of admissible controls $mesE_k=0$, $k=1,2,...$. Let $E=\overset{\infty}{\underset{k=1}{U}}E_k$, $Q_2=\{(x,t)\,|\,(x,t)\in Q_T,(x,t)\,\overline{\in}E\}$, $Q_0=Q_1\cap Q_2$. It is clear that $mesQ_0=mesQ_T$.

Suppose that $(x,t)\in Q_0$. Since $\underline{\lim}_{s\to\infty}\lambda_s(x,t)=Z_0(x,t)$, then there is such a subsequence (denote it also by $\{\lambda_s(x,t)\}$), that for $\{\lambda_s(x,t)\}$ and appropriate sequence

$\{\psi_s(x,t)\}$, $\psi_s(x,t) \to Z(x,t)$ as $s \to \infty$. From $u_k(x,t) \to u_0(x,t)$ it follows that for any $\delta > 0$ there exists such $k_0(\delta) > 0$ that $k > k_0$ $|u_k(x,t) - u_0(x,t)| < \delta$.

Then for $k > k_0$ $(x,t,u_k(x,t)) \in N(x,t,u_0(x,t),\delta)$, where by $N(x,t,u_0,\delta)$ we denote a points set (x,t,u) for which $|u - u_0| \le \delta$. Therefore, for all $n_s + l > k_0$

$$(f_0(x,t,u_{n_s+l}(x,t),\vartheta_{n_s+l}(x,t)), f(x,t,u_{n_s+l}(x,t),\vartheta_{n_s+l}(x,t))) \in$$

$$\in R^+(N(x,t,u_0(x,t),\delta)).$$

Here

$$R^+(N(x,t,u_0(x,t),\delta)) = \underset{|u-u_0|<\delta}{U} \left\{R^+(x,t,u) \,|\, (x,t,u) \in N(x,t,u_0,\delta)\right\}.$$

From (2.5) and (2.6) it follows that

$$(\lambda_s(x,t), \psi_s(x,t)) \in coR^+(N(x,t,u_0(x,t),\delta)).$$

Since as $s \to \infty$ $\lambda_s(x,t) \to Z_0(x,t)$, $\psi_s(x,t) \to Z(x,t)$ then $(Z_0(x,t), Z(x,t)) \in clcoR^+(N(x,t,u_0(x,t),\delta))$, $\forall\delta > 0$, where $clcoB$ denotes a convex closed hull of the set B. Under the conditions imposed on the problem data, the Chesari condition [4] is fulfilled, i.e. in the given case

$$R^+(x,t,u_0(x,t)) \underset{\delta>0}{\cap} \underset{|u-u_0|<\delta}{U} \left\{R^+(x,t,u) \,|\, (x,t,u) \in N(x,t,u_0,\delta)\right\}.$$

Then hence it follows that

$$(Z_0(x,t), Z(x,t)) \in R^+(x,t,u_0(x,t)).$$

By definition of the set $R(x,t,u)$ there exists a function $\vartheta(x,t)$ such that it accepts the values from V, and

$$Z_0(x,t) \ge f_0(x,t,u_0(x,t),\vartheta(x,t)),$$

$$Z(x,t) = f(x,t,u_0(x,t),\vartheta(x,t)).$$

Then, by the Filippov's generalized lemma [5,6], there is a measurable function $\vartheta(x,t)$ such that

$$\vartheta_0(x,t) \in V,$$
$$Z_0(x,t) \ge f_0(x,t,u_0(x,t),\vartheta_0(x,t)),$$
$$Z(x,t) = f(x,t,u_0(x,t),\vartheta_0(x,t)). \tag{2.8}$$

Show that the function $u_0(x,t)$ is a solution of problem (1.1)-(1.3) corresponding to the control $\vartheta_0(x,t)$. By definition of the generalized solution of problem (1.1)-(1.3), for any function $\Phi(x,t) \in W_2^1(Q_T)$ such that $\Phi(x,T) = 0$ the following integral identity is fulfilled:

$$\int\limits_{Q_T}\sum_{l=1}^{k}\alpha_{ls}\left(-\frac{\partial u_{n_s+l}}{\partial t}\frac{\partial\Phi}{\partial t} + \sum_{i,j=1}^{m} a_{ij}\frac{\partial u_{n_s+l}}{\partial x_j}\frac{\partial\Phi}{\partial x_i}\right)dxdt-$$

$$-\int_0^T\int_{\partial\Omega}\Phi(x,t)\int_{\Omega}\sum_{l=1}^{k}\alpha_{ls}K(x,y,t,u_{n_s+l}(y,t))\,dydsdt-$$

$$-\int_{\Omega}\varphi_1(x)\Phi(x,0)\,dx=\int_{Q_T}\sum_{l=1}^{k}\alpha_{ls}f(x,t,u_{n_s+l}(x,t),\vartheta_{n_s+l}(x,t))\Phi(x,t)\,dxdt. \quad (2.9)$$

Passing to limit in (2.9) as $s\to\infty$, and taking into account (2.2),(2.3),(2.4),(2.5), we have

$$\int_{Q_T}\left(-\frac{\partial u(x,t)}{\partial t}\frac{\partial\Phi(x,t)}{\partial t}+\sum_{i,j=1}^{m}a_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_j}\frac{\partial\Phi(x,t)}{\partial x_i}\right)dxdt-$$

$$-\int_0^T\int_{\partial\Omega}\Phi(x,t)\int_{\Omega}K(x,y,t,u(y,t))\,dydsdt-$$

$$-\int_{\Omega}\varphi_1(x)\Phi(x,0)\,dx=\int_{Q_T}Z(x,t)\Phi(x,t)\,dxdt.$$

If here we take into account the third one from relations (2.8), we get that $u_0(x,t)$ is a generalized solution of problem (1.1)-(1.3) corresponding to $\vartheta_0(x,t)$.

Therefore

$$J(\vartheta_0)\geq\gamma. \quad (2.10)$$

Above we showed that $f_0(x,t,u_0(x,t),\vartheta_0(x,t))\leq Z_0(x,t)$. Then taking into account the last relation and (2.7), we have:

$$J(\vartheta_0)=\int_{Q_T}f_0(x,t,u_0(x,t),\vartheta_0(x,t))\,dxdt\leq\int_{Q_T}Z_0(x,t)\,dxdt\leq\gamma. \quad (2.11)$$

From (2.10) and (2.11) it follows that $J(\vartheta_0)=\gamma$, i.e. $(u_0(x,t),\vartheta_0(x,t))$ is an optimal pair, $\vartheta_0(x,t)$ is an optimal control. The theorem is proved.

References

[1]. Pontryagin L.S. *Ordinary differential Equations.* M. Nauka, 1974, 331 p. (Russian).

[2]. Ladyzhenskaya O.A. *Boundary value problems of mathematical physics.* M. Nauka, 1973, 408 p. (Russian).

[3]. Hille E., Fiblips R. *Functional analysis and semigroup.* M. IL, 1962, 826 p. (Russian).

[4]. Cesari L. *Existence theorems for abstract multidimentional control problems.*-Your. Opt. theory and Appl., 1970, 6, No 3, pp. 210-236

[5]. Filippov A.F. *On some problems of optimal control problem.* Vestnik MGU, ser. mat. mekh. Astr., 1959, 2, No 1, pp. 25-32 (Rusian).

[6]. Mc. Shane E.J. and Warfield R.B. *On Filippov implicit functions lemma.* Proc. Amer. Math. Soc., 1967, 18, No 1, pp. 41-47.

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