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A SOLUTION METHOD FOR A SYSTEM OF NONLINEAR DIFFERENTIAL EQUATIONS

Abstract

The system of nonlinear differential equations being a generalization of Toda and Volterra chains is considered. Lax representation for this system is found. The solution of the Cauchy problem for the mentioned system is found by the method of the inverse spectral problem.

Applications of the methods of the inverse spectral problem to integration of nonlinear dynamical systems such as Toda chain and Volterra chain (see [1]-[5] and their references) are known well. Initial boundary value problems for the systems of differential equations of Toda chain and Volterra chain type are undoubtedly of applied interest and therefore they are the subject of active study already in the course of a number of years (see [3], [4]).

For the real-valued functions $a_n = a_n(t) \in C^{(1)}[0,\infty)$, $b_n = b_n(t) \in$ $\in C^{(1)}[0,\infty)$ consider the Cauchy problem for the following system of equations

$$\begin{cases} a_n = \dfrac{\alpha}{2} a_n (b_n - b_{n+1}) + \dfrac{\beta}{2} a_n (a_{n-1}^2 - a_{n+1}^2 + b_n^2 - b_{n+1}^2), \\ b_n = \alpha (a_{n-1}^2 - a_n^2) + \beta [a_{n-1}^2 (b_{n-1} + b_n) - a_n^2 (b_n + b_{n+1}), \ n = 0, ..., N, \\ a_{-1} = a_N = 0, \quad \cdot = \dfrac{d}{dt}, \end{cases} \tag{1}$$

$$a_n(0) = a_n^0 > 0, \quad b_n(0) = b_n^0, \qquad n = 0, ..., N, \tag{2}$$

where α and β are real numbers. Note that the system of equations (1) becomes Toda's chain for $\alpha = 1, \ \beta = 0$ and Volterra's chain for $\alpha = 0, \ \beta = 1, \ b_n \equiv 0$.

In the present paper, by the method of the inverse spectral problem we obtain formulas for finding the solution of problem (1)-(2) at any time t. Global solvability of problem (1)-(2) is proved.

1. Preliminary information

In this item we formulate some known facts related to the inverse spectral problem for finite Jacobi matrices a lot of which are contained with their proofs in [2], [6]. [7].

Consider $(N+1)$-dimensional Jacobi matrix

$$L = \begin{pmatrix} b_0 & a_0 & 0 & & 0 & 0 \\ a_0 & b_1 & a_1 & & 0 & 0 \\ & & \cdots\cdots\cdots & & & \\ 0 & 0 & 0 & \ldots & b_{N-1} & b_{N-1} \\ 0 & 0 & 0 & \ldots & b_{N-1} & b_{N-1} \end{pmatrix}.$$

Introduce the difference equation

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \qquad n = 0, 1, ..., N,$$

$$a_{-1} = a_N = 0, \quad a_n > 0, \quad n = 0, ..., N-1. \tag{3}$$

where λ is a spectral parameter. Denote by $p_n = p_n(\lambda)$ the solution of equation (3) with initial conditions $p_{-1} = 0,\ p_0 = 1$. It is known that the eigen values, of the matrix L are real, prime and coincide with the zeros of the polynomial $p_{N+1}(\lambda)$.

Let $\lambda_0, \lambda_1, ..., \lambda_N$ be eigen values of the matrix L. Introduce the normalizing coefficients, assuming

$$\alpha_k = \sum_{n=0}^{\infty} p_n^2(\lambda_k), \quad k = 0, 1, ..., N. \tag{4}$$

The totality $\{\lambda_n, \alpha_n > 0\}_{n=0}^{N}$ is said to be spectral data of the matrix. The inverse spectral problem for the Jacobi matrix L consists of renewal of the elements $a_n, \quad n = 0, 1, ..., N$ and $b_n, \quad n = 0, 1, ..., N$ by spectral data $\{\lambda_n, \alpha_n\}_{n=0}^{N}$ for which equality (4) is valid.

The matrix L is uniquely determined by spectral data and may be found in the following way. Construct the moments $S_n,\ n = 0, 1, .., 2N+1$ by the formula

$$S_n = \sum_{k=0}^{N} \lambda_k^n \alpha_k^{-1} \tag{5}$$

and the Hankel determinants $D_n, -1 \le n \le N$ by the formulae

$$D_{-1} = 1,\ D_n = \begin{vmatrix} S_0 & S_1 & \ldots & S_n \\ S_1 & S_2 & \ldots & S_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ S_n & S_{n+1} & \ldots & S_{2n} \end{vmatrix}. \tag{6}$$

Introduce also the determinants $\Delta_{-1} = 0,\ \Delta_0 = S_0,\ \Delta_n,\ 1 \le n \le N$

$$\Delta_n = \begin{vmatrix} S_0 & S_1 & \ldots & S_{n-1} & S_{n+1} \\ S_1 & S_2 & \ldots & S_n & S_{n+2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ S_n & S_{n+1} & \ldots & S_{2n-1} & S_{2n+1} \end{vmatrix}.$$

Calculate a_n and b_n by the formulae

$$a_n = (D_{n-1} D_{n+1})^{\frac{1}{2}} D_n^{-1}, \quad b_n = \triangle_n D_n^{-1} - \triangle_{n-1} D_{n-1}^{-1}. \tag{7}$$

It should be noted that the Hankel determinants D_n are positive.Indeed, for any n the quadratic form $\sum\limits_{j,k=0}^{n} S_{j+k} x_j x_k$ is representable in the form

$$\sum_{j,k=0}^{n} S_{j+k} x_j x_k = \sum_{k=0}^{N} \alpha_k^{-1} \left(\sum_{j,k=0}^{n} x_j \lambda_k^j \right)^2,$$

whence it follows that

$$\sum_{j,k=0}^{n} S_{j+k} x_j x_k \ge 0,$$

and the sign of equality may be attained only when $x_0 = ... = x_n = 0$. Consequently, the quadratic form $\sum\limits_{j,k=0}^{n} S_{j+k}x_jx_k$ and the determinants D_n are positive.

2. Formulae for solving problem (1)-(2)

Now let the elements $a_n,\ b_n$ of the matrix L depend on t:$L = L(t)$ and satisfy system (1). The following lemma describs the evolution of spectral data $\{\lambda_k(t), \alpha_k(t)\}_{k=0}^{N}$ of the matrix $L = L(t)$.

Theorem 1. *Evolution of spectral data of the matrix $L = L(t)$ is described by the formulae*

$$\lambda_k(t) = \lambda_k(0) = \lambda_k,$$

$$\alpha_k^{-1}(t) = \alpha_k^{-1}(0)e^{-\frac{\alpha\lambda_k+\beta\lambda_k^2}{2}t}\left(\sum_{k=0}^{N}\alpha_k^{-1}(0)e^{-\frac{\alpha\lambda_k+\beta\lambda_k^2}{2}t}\right)^{-1}, \quad k = 0, ..., N. \tag{9}$$

Proof. Introduce the $(N+1)$-dimensional matrix $A = A(t)$ that acts on the vector $y = (y_0, ..., y_n)^T$ by the formula

$$(Ay)_n = \frac{\alpha}{2}\left(a_n y_{n+1} - a_{n-1}y_{n-1}\right) +$$

$$+\frac{\beta}{2}\left[a_n a_{n+1}y_{n+2} + a_n(b_n + b_{n+1})y_{n+1} - a_{n-1}(b_{n-1} + b_n)y_{n-1} - a_{n-2}a_{n-1}y_{n-2}\right].$$

where by calculating $(Ay)_j,\ j = 0, 1, N-1, N$ it should be taken into account that $y_k = 0,\ k = -2,\ -1,\ N+1,\ N+2$. It is easy to verify that the matrix A is skew-symmetric: $A^* = -A$. Furthermore, the matrices L and A form Lax pair, i.e. a system of equations (1) is equivalent to the matrix equation (10)

$$\dot{L} = LA - AL. \tag{10}$$

Since equality (10) implies (see [3], [4]) unitary equivalence of the family of matrices $L = L(t)$, then the eigen values $\lambda_k(t),\ k = 0, 1, ..., N$ of the matrix $L = L(t)$ are independent of t :

$$\lambda_k(t) = \lambda_k(0) = \lambda_k, \quad k = 0, 1, ..., N.$$

Consider the equation

$$Ly = \lambda y, \tag{11}$$

where the parameter λ is independent of t. Differentiating equality (11) with respect to t and using (10), we find that the matrix B acting by the formula

$$(By)_n = y_n + (Ay)_n, \qquad n = 0, 1, ..., N,$$

transforms the solution of equation (11) to the solution of just the same equation.

Now let $p(\lambda_k, t) = \{p_n(\lambda_k, t)\}_{n=0}^{N}$ be an eigen vector of the matrix $L = L(t)$ responding to the eigen value $\lambda_k,\ k = 0, 1, ..., N$, moreover $p_n(\lambda_k, t) = 1$. Then $Bp_n(\lambda_k, t)$ is also an eigen vector of the matrix L corresponding to the eigen value of the operator λ_k. Directly it is verified that $(Bp_n(\lambda_k, t))_0 =$

$=\beta\dfrac{\lambda_k^2-b_0^2-a_0^2}{2}+\alpha\dfrac{\lambda_k-b_0}{2}$. Since the eigen values of the operator $L=L(t)$ are prime, then the following equalities are true

$$Bp_n(\lambda_k,t)=\left[\alpha\frac{\lambda_k-b_0}{2}+\beta\frac{\lambda_k^2-b_0^2-a_0^2}{2}\right]p(\lambda_k,t)\,,\quad k=0,1,...,N. \tag{12}$$

Now find the dynamics of the normalizing coefficient

$$\alpha_k(t)=\sum_{n=0}^{N}p_n^2(\lambda_k,t)\,,\quad (k=0,1,...,N).$$

Taking into attention that

$$\dot{\alpha}_k(t)=2\sum_{n=0}^{\infty}p_n(\lambda_k,t)\,p_n(\lambda_k,t)\,,$$

allowing for formula (12) we find

$$\dot{\alpha}_k(t)=\left[\alpha\frac{\lambda_k-b_0}{2}+\beta\frac{\lambda_k^2-b_0^2-a_0^2}{2}\right]\alpha_k(t)+$$

$$+\sum_{n=0}^{N}(Ap_n^2(\lambda_k,t))_n p_n(\lambda_k,t)\,.$$

On the other hand, from the skew-symmetry of the matrix A it follows that the sum

$$\sum_{n=0}^{N}(Ap_n^2(\lambda_k,t))_n p_n(\lambda_k,t)$$

vanishes. Then we get the equation

$$\dot{\alpha}_k(t)=\left[\alpha\frac{\lambda_k-b_0}{2}+\beta\frac{\lambda_k^2-b_0^2-a_0^2}{2}\right]\alpha_k(t),$$

from which it follows that

$$\alpha_k^{-1}(t)=\alpha_k^{-1}(0)\exp\left(-\frac{\alpha\lambda_k+\beta\lambda_k^2}{2}\right)\times$$

$$\times\exp\left(\int_0^{\tau}\frac{\beta b_0^2(\tau)+\beta a_0^2(\tau)-ab_0(\tau)}{2}d\tau\right), \tag{13}$$

In (13), the functions $a_0(t)$, $b_0(t)$ may not be arbitrary, they should provide fulfilment of the equality

$$\sum_{n=0}^{N}\alpha_k^{-1}(t)=1.$$

Then we'll have

$$\exp\left(\int_0^{\tau}\frac{\beta b_0^2(\tau)+\beta a_0^2(\tau)-ab_0(\tau)}{2}d\tau\right)=$$

$$= \left(\sum_{n=0}^{N} \alpha_k^{-1}(0) \exp\left(-\frac{\alpha\lambda_k + \beta\lambda_k^2}{2} t \right) \right)^{-1}, \quad k = 0, 1, ..., N.$$

Together with (13) the last equality leads us to relation (9).

The theorem is proved.

The results of this item allow to find the solution of problem (1)-(2). More exactly, by initial data $a_n(0) = \widehat{a}_n, \quad b_n(0) = \widehat{b}_n$ we construct the spectral data $\{\lambda_k, \alpha_k(0)\}_{k=0}^N$. Calculate the spectral data $\{\lambda_k, \alpha_k(t)\}_{k=0}^N$ by formula (9). Solving in totality $\{\lambda_k, \alpha_k(t)\}_{k=0}^N$ the inverse spectral problem by means of formula (5)-(8), where instead of $\alpha_k^{-1}(0)$ we should substitute (9), we construct the solution.

3. Existence of the solution

The method for constructing the solution of the problem, suggested in the previous item requires a unique solvability of problem (1)-(2). In this item we'll be engaged in this matter.

Theorem 2. *For any initial data* $a_n^0 > 0, \quad b_n^0$ *problem (1)-(2) has a unique solution determined on the semi-axis* $[0, \infty)$.

Proof. First of all note that the right sides of the system of equations (1) are continuously differentiable functions of variables $a_0, ..., a_{N-1}, \; b_0, ..., b_{N-1}, b_N$.

Then passing to the integral equation by the standard method, and using the sequential approximations method, we find that problem (1)-(2) has at some interval $[0, t_0)$ a unique solution $a_n(t), \; b_n(t)$. Therewith, from (1) we have

$$a_0(t) = a_n(0) \exp\left(\int_0^t \left[\frac{\alpha}{2}(b_n(\tau) - b_{n+1}(\tau)) + \right.\right.$$

$$\left.\left. + \frac{\beta}{2}\left(a_{n-1}^2(\tau) - a_{n+1}^2(\tau) + b_n^2(\tau) - b_{n+1}^2(\tau)\right) \right] d\tau \right),$$

from which it follows that $a_n(t) > 0$.

Further, since for $t \in [0, t_0)$ equality (10) is true, then the family of the matrices $L = L(t)$ is unitary equivalent, i.e. there exists an $(N+1)$ dimensional unitary matrix $U(t)$ such that

$$U(0) = E, \quad L(t) = U^*(t) L(0) U(t),$$

where E is a $(N+1)-$dimensional matrix. From the last relations we get

$$\|L(t)\| = \|L(0)\|, \quad t \in [0, t_0), \tag{14}$$

where $\|\cdot\|$ is the norm of the matrix $L(t)$ in $(N+1)-$dimensional space of vectors $y = (y_0, ..., y_N)$ with the norm $\|y\| = \left(\sum_{n=0}^{N} y_n^2 \right)^{\frac{1}{2}}$. Now, from formula (14) allowing for very obvious inequalities

$$|a_n(t)| \le \|L(t)\|, \quad |b_n(t)| \le \|L(t)\|$$

it follows that the solution $a_n(t), \; b_n(t)$ is continuable on all the positive semi-axis.

The theorem is proved.

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