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ON CLOSE b-BASES

Abstract

In the paper the basis properties of close systems are studied. The results obtained in the paper generalize some known statements on close systems in Hilbert and Banach spaces. In particular, the results generalizing the Paley – Wiener and Krein – Rootman – Milman theorems are proved. The conditions of isomorphic property of b - bases are found.

Definition of the space of coefficients generated by the basis of the space is important while studying the properties of this or another basis space. Therewith, as is known, the identical spaces of coefficients correspond to isomorphic bases. From this point of view, study of isomorphic bases is of interest. In this direction, the results in Hilbert and Banach spaces are known. Concerning this theory one can see the papers, for example [1-7]. Stability of perturbation of the basis sometimes leads to obtaining asymptotic estimate. Under perturbation of the system of exponents in the space $L_2(-\pi,\pi)$ R. Paley and N. Wiener have received an asymptotic estimate. The "$\frac{1}{4}$ Kadets" theorem gives the exact estimate of this result.

The goal of the paper is to establish generalizations of some results on isomorphic bases in Hilbert and Banach spaces by means of bilinear mapping. At first some notion that are natural generalizations of classic notion, are obtained. Then, the generalizations of Paley-Wiener, Krein-Rootman-Milman theorems and also some results concerning the close systems are obtained.

1. Some notion and statements

Let X, Y and Z be $B-$spaces with corresponding norms $\|\cdot\|_X$, $\|\cdot\|_Y$, and $\|\cdot\|_Z$ be a bilinear mappind satisfying the condition:

$$\exists m, M\rangle 0 : m\|x\|_X\|y\|_Y \le \|b(x,y)\|_Z \le M\|x\|_X\|y\|_Y \quad \forall x\in X,\ y\in Y. \qquad (1)$$

We'll assume $xy \equiv b(x,y)$ $\forall x\in X$, $y\in Y$.

Let $M\subset Y$ be some set, $L_b(M)$ denote the aggregate of all possible final sums $\sum x_i m_i$, where $x_i\in X$, $m_i\in M$.

The system $\{y_n\}_{n\in N}\subset Y$ is called b-complete if $\overline{L_b(\{y_n\}_{n\in N})} = Z$.

The systems $\{y_n\}_{n\in N}\subset Y$ and $\{y_n^*\}_{n\in N}\subset L(Z,X)$ are said to be b-biorthogonal if

$$\forall n,k\in N,\ x\in X \quad y_n^*(xy_k) = \delta_{nk}x,$$

where δ_{nk} is Kronecker's symbol. Therewith the system $\{y_n^*\}_{n\in N}$ is called b-biorthogonal to the system $\{y_n\}_{n\in N}$.

The system $\{y_n\}_{n\in N} \subset Y$ is said to be b-minimal if

$$\forall x \in X \quad (x \neq 0)\,, \quad k \in K \quad xy_k \notin \overline{L_b\left(\{y_n\}_{n\in N, n\neq k}\right)}.$$

The system $\{y_n\}_{n\in N} \subset Y$ is called b-basis in Z if

$$\forall z \in Z \quad \exists!\, \{x_n\}_{n\in N} \subset X : z = \sum_{n=1}^{\infty} x_n y_n.$$

The system $\{y_n\}_{n\in N} \subset Y$ is said to be ω_b-linearly independent in Z if from $\sum_{n=1}^{\infty} x_n y_n = 0$ it follows that $x_n = 0$ for any $n \in N$.

Let $\{y_n\}_{n\in N} \subset Y$ form a b- basis in Z, and $\widetilde{X}$ be a space of coefficients that corresponds to the b - basis $\{y_n\}_{n\in N}$. The space $\widetilde{X}$ becomes a B - space by the norm $\|\widetilde{x}\|_{\widetilde{X}} = \sup\limits_n \left\|\sum_{k=1}^{n} x_k y_k\right\|_Z$ (see [8]). Furthermore, the operator T determined by the expression $T\widetilde{x} = z$, where $\widetilde{x} = \{x_n\}_{n\in N}$, $z = \sum_{n=1}^{\infty} x_n y_n$ is a linear bounded invertible operator from $\widetilde{X}$ to Z . The operator T is called a natural isomorphism between $\widetilde{X}$ and Z.

Statement 1. Let X, Y and Z be B - spaces, the system $\{y_n\}_{n\in N} \subset Y$ form a b - basis with b- biorthogonal system $\{y_n^*\}_{n\in N}$. Then there exist the constants a and c such that $a \leq \|y_n^*\|_{L(Z,X)} \|y_n\|_Y \leq c \; \forall n \in N$.

Proof. For $\forall x \in X$, $n \in N$ $\|x\|_X = \|y_n^*(xy_n)\|_X \leq \|y_n^*\|_{L(Z,X)} \|xy_n\|_Y \leq M \|y_n^*\|_{L(Z,X)} \|y_n\|_Y \|x\|_X$. Hence $a \leq \|y_n^*\|_{L(Z,X)} \|y_n\|_Y$, where $a = \frac{1}{M}$. Let $\widetilde{X}$ be a space of coefficients corresponding to the b - basis $\{y_n\}_{n\in N}$, T be an isomorphism between $\widetilde{X}$ and Z. We have:

$$\|y_n^*(z)\|_X \leq \frac{1}{m} \frac{\|y_n^*(z)\, y_n\|_Z}{\|y_n\|_Y} = \frac{1}{m} \frac{\left\|\sum_{k=1}^{n} y_k^*(z)\, y_k - \sum_{k=1}^{n-1} y_k^*(z)\, y_k\right\|_Z}{\|y_n\|_Y} \leq$$

$$\leq \frac{2}{m} \frac{\left\|T^{-1}z\right\|_{\widetilde{X}}}{\|y_n\|_Y} \leq \frac{2\left\|T^{-1}\right\|_{L(Z,\widetilde{X})}}{m\,\|y_n\|_Y} \|z\|_Z$$

$\forall z \in Z$, $n \in N$. From the last estimation we get $\|y_n^*\|_{L(Z,X)} \|y_n\|_Y \leq c$, where $c = \dfrac{2\left\|T^{-1}\right\|_{L(Z,\widetilde{X})}}{m}$. The theorem is proved.

2. b-isomorphism of b-bases

Let Y be a B -space, X and Z be H - spaces with appropriate scalar products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Z$. Fixing $z \in Z$ and $y \in Y$ in $(z, xy)_Z$ for an arbitrary value $x \in X$, consider the functional $f_{z,y}(x) = (z, xy)_Z$ in the space X. By the Riesz theorem there exists a uniquely determined element $< z, y > \in X$, such that $f_{z,y}(x) = (< z, y >, x)_X$ and $\|f_{z,y}\| = \|< z, y >\|$. So, $(z, xy)_Z = (< z, y >, x)_X$. Therewith it is easy to show that the element $< z, y >$ is linear in the argument $z \in Z$ and satisfies the inequality:

$$\|< z, y >\|_X \le M \|z\|_Z \|y\|_Y \quad \forall z \in Z, \quad y \in Y.$$

The systems $\{y_n\}_{n\in N}$ and $\{y_n^*\}_{n\in N} \subset Y$ are called b_Y - orthogonal if

$$\forall n, k \in N, \quad x \in X \qquad < xy_n, y_k^* >= \delta_{nk} x.$$

The system $\{y_n^*\}_{n\in N}$ is said to be b_Y- orthogonal to the system $\{y_n\}_{n\in N}$.

b - basis $\{y_n\}_{n\in N} \subset Y$ in Z is called b_Y - orthonormed if its b_Y -biorthogonal system coincides with $\{y_n\}_{n\in N}$.

b - basis $\{\varphi_n\}_{n\in N}$ is called the Riesz b - basis if there exists a bounded invertible operator $T \in L(Z) : T(x\varphi_n) = xy_n$ for any $x \in X$ and $n \in N$, where $\{y_n\}_{n\in N}$ is a b_Y – orthonormed b - basis.

The systems $\{y_n\}_{n\in N}$ and $\{y_n^*\}_{n\in N} \subset Y$ are said to be b - isomorphic if there exists a bounded invertible operator $T \in L(Z) : T(xy_n) = xy_n^*$ for any $x \in X$ and $n \in N$.

Theorem 1. *Let Y be a B - space, X and Z be H -spaces, the system $\{\varphi_n\}_{n\in N} \subset Y$ form a b_Y - orthonormed b-basis in Z , the system $\{\psi_n\}_{n\in N} \subset Y$ be such that $\exists \theta \in [0, 1)$, $\forall \{x_n\} \subset X$ (finite) the following relation be valid*

$$\left\| \sum_n x_n (\varphi_n - \psi_n) \right\|_Z^2 \le \theta^2 \sum_n \|x_n\|_X^2 . \qquad (2)$$

Then the system $\{\psi_n\}_{n\in N}$ forms a b basis b - isomorphic to $\{\varphi_n\}_{n\in N}$ i.e. the Riesz b - basis.

Proof. It is clear that in the case $\theta = 0$, we get $\varphi_n = \psi_n$ for every $\forall n \in N$. Let now $\theta \ne 0$. As $\{\varphi_n\}_{n\in N}$ is an b_Y -orthonormed b -basis, it is clear that $\forall z \in Z$ $\sum_{n=1}^{\infty} \|< z, \varphi_n >\|_X^2 = \|z\|_Z^2$. Consider the sequence $z_m = \sum_{n=1}^{m} < z, \varphi_n > (\varphi_n - \psi_n)$. Show the convergence of the sequence z_m. From the convergence of the series $\sum_{n=1}^{\infty} \|< z, \varphi_n >\|_X^2$ it follows that

$$\forall \varepsilon > 0 \;\; \exists m_0 \;\; \forall m \in N \;\; m \ge m_0 \;\; \forall p \in N \quad \sum_{n=m+1}^{m+p} \|< z, \varphi_n >\|_X^2 < \frac{\varepsilon^2}{\theta^2}$$

Therefore, applying (2), we get that $\forall \varepsilon > 0$ $\exists m_0$ $\forall m \in N$ $m \ge m_0$ $\forall p \in N$

$$\left\| \sum_{n=m+1}^{m+p} < z, \varphi_n > (\varphi_n - \psi_n) \right\|_Z^2 \le \theta^2 \sum_{n=m+1}^{m+p} \|< z, \varphi_n >\|_X^2 < \varepsilon^2$$

i.e. the sequence z_m is fundamental, and so by the completeness of Z it converges. Consequently, the operator K determined by the formula $Kz = \sum\limits_{n=1}^{\infty} < z, \varphi_n > (\varphi_n - \psi_n)$, $z \in Z$ is well-defined. The linearity of the operator K is obvious. From (2) we get $\left\| \sum\limits_{n=1}^{m} < z, \varphi_n > (\varphi_n - \psi_n) \right\|_Z^2 \leq \theta^2 \sum\limits_{n=1}^{m} \| < z, \varphi_n > \|_X^2$. Hence, passing to limit as $m \to \infty$, we have

$$\left\| \sum_{n=1}^{\infty} < z, \varphi_n > (\varphi_n - \psi_n) \right\|_Z \leq \theta \|z\|_Z .$$

Thus, $\|Kz\|_Z \leq \theta \|z\|_Z$, by the same token $\|K\|_{L(Z)} < 1$. Then the operator $I - K$ is boundedly invertible, where I is an identity operator in Z. On the other hand, $(I - K) z = \sum\limits_{n=1}^{\infty} < z, \varphi_n > \psi_n$, $z \in Z$, and $(I - K)(x\varphi_n) = x\psi_n \ \forall n \in N$. The theorem is proved.

Now consider the case of Banach spaces.

Theorem 2. *Let X, Y and Z be B -spaces, the system $\{\varphi_n\}_{n\in N} \subset Y$ form a b basis in Z with b - biorthogonal system $\{\varphi_n^*\}_{n\in N}$, the system $\{\psi_n\}_{n\in N} \subset Y$ be such that $\exists \theta \in [0, 1)$, for any finite $\{x_n\} \subset Y$ the following relation be valid*

$$\left\| \sum_n x_n (\varphi_n - \psi_n) \right\|_Z \leq \theta \left\| \sum_n x_n \varphi_n \right\|_Z . \tag{3}$$

Then the system $\{\psi_n\}_{n\in N}$ forms a b- basis b - isomorphic to $\{\varphi_n\}_{n\in N}$.

Proof. Consider the operator K determined by the expression $Kz = \sum\limits_{n=1}^{\infty} \varphi_n^* (z) (\varphi_n - \psi_n)$, $z \in Z$. Similar to the above mentioned one, it is easy to show that the series $\sum\limits_{n=1}^{\infty} \varphi_n^* (z) (\varphi_n - \psi_n)$ converges $\forall z \in Z$. Consequently, the operator K is well-defined. It is clear that the operator K is linear. From (3) we get that $\forall m \in N$ $\left\| \sum\limits_{n=1}^{m} \varphi_n^* (z) (\varphi_n - \psi_n) \right\|_Z \leq \theta \left\| \sum\limits_{n=1}^{m} \varphi_n^* (z) \varphi_n \right\|_Z$, $z \in Z$. So, $\|K\|_{L(Z)} < 1$. Consequently, $I - K$ is boundedly invertible, and $(I - K)(x\varphi_n) = x\psi_n \ \forall n \in N$, $x \in X$.

The theorem is proved.

Theorem 3. *Let X, Y and Z be B -spaces, the system $\omega = \{\omega_n\}_{n\in N} \subset Y$ form a b - basis in Z with b -biorthogonal system $\{\omega_n^*\}_{n\in N}$, the systems $\{\varphi_n\}_{n\in N}$ and $\{\psi_n\}_{n\in N} \subset Y$ be close in the sense that $\exists \theta \in [0, 1)$ such that for any finite $\{x_n\} \subset Y$*

$$\left\| \sum_n x_n (\varphi_n - \psi_n) \right\|_Z \leq \theta \left\| \sum_n x_n \omega_n \right\|_Z . \tag{4}$$

Then the system $\{h_n = \varphi_n - \psi_n + \omega_n\}_{n\in N}$ forms a b - basis b - isomorphic to $\{\omega_n\}_{n\in N}$.

Proof. Let K be an operator determined by the expression
$Kz = \sum\limits_{n=1}^{\infty} \omega_n^* (z) (\varphi_n - \psi_n)$, $z \in Z$. By (4) the operator K is determined. The linearity of the operator K is obvious. From (4) we get

$$\|Kz\|_Z = \left\| \sum_{n=1}^{\infty} \omega_n^* (z) (\varphi_n - \psi_n) \right\|_Z \leq \theta \left\| \sum_{n=1}^{\infty} \omega_n^* (z) \omega_n \right\|_Z, \quad z \in Z.$$

Hence, $\|K\|_{L(Z)} < 1$. Consequently, $I + K$ is boundedly invertible and $(I + K)(x\omega_n) = xh_n$ for any $n \in N$ and $x \in X$. The theorem is proved.

We give the closeness condition when the systems have the same b basis properties.

Theorem 4. *Let X, Y and Z be B -spaces, the systems $\{\varphi_n\}_{n \in N}$ and $\{\psi_n\}_{n \in N} \subset Y$ be Y - close in the sense that $\exists \theta \in [0, 1)$ such that for any finite $\{x_n\} \subset X$*

$$\left\| \sum_n x_n (\varphi_n - \psi_n) \right\|_Z \leq \theta \left(\left\| \sum_n x_n \varphi_n \right\|_Z + \left\| \sum_n x_n \psi_n \right\|_Z \right). \tag{5}$$

Then the systems $\{\varphi_n\}_{n \in N}$ and $\{\psi_n\}_{n \in N}$ are simultaneously:

a) b - complete in Z;

b) b - minimal in Z;

c) ω_b - linearly independent in Z ;

d) form a b - basis in Z.

Proof.

$$\frac{1 - \theta}{1 + \theta} \left\| \sum_n x_n \varphi_n \right\|_Z \leq \left\| \sum_n x_n \psi_n \right\|_Z \leq \frac{1 + \theta}{1 - \theta} \left\| \sum_n x_n \varphi_n \right\|_Z \tag{6}$$

is valid for any finite collection $\{x_n\}$. Indeed, from (5) we get

$$\left\| \sum_n x_n \psi_n \right\|_z = \left\| \sum_n x_n (\psi_n - \varphi_n) + \sum_n x_n \varphi_n \right\|_z \leq$$

$$\leq \left\| \sum_n x_n (\psi_n - \varphi_n) \right\|_Z + \left\| \sum_n x_n \varphi_n \right\|_z \leq (1 + \theta) \left\| \sum_n x_n \varphi_n \right\|_z + \theta \left\| \sum_n x_n \psi_n \right\|_z.$$

Therefore,

$$\left\| \sum_n x_n \psi_n \right\|_z \leq \frac{1 + \theta}{1 - \theta} \left\| \sum_n x_n \varphi_n \right\|_z.$$

Similarly

$$\left\| \sum_n x_n \varphi_n \right\|_z \leq \frac{1 + \theta}{1 - \theta} \left\| \sum_n x_n \psi_n \right\|_z.$$

Validity of (6) follows from the last two inequalities.

Statements b) and c) follow from (6). Show the validity of the statements a) and d). let the system $\{\varphi_n\}_{n\in N}$ be b - complete in Z. Consider the operator K given on $L_b(\{\varphi_n\})$ by the expression $K\left(\sum\limits_n x_n\varphi_n\right)=\sum\limits_n x_n\psi_n$. Obviously, the operator K is linear and by (6) is well-defined. By b - completeness of the system $\{\varphi_n\}_{n\in N}$, continueing the operator K by the continuity on the Z, from (6) we get

$$m\left\|z\right\|_Z\leq\left\|Tz\right\|_Z\leq M\left\|z\right\|_Z,\quad z\in Z, \tag{7}$$

where m,M are some constants, T is the continuation of the operator K. Hence $\ker T=\{0\}$, and $\operatorname{Im}T$ is closed. Following the proof theorem 9.2 (see [4], p.87) we get exists a constant $c>0$ such that $\left\|T^*f\right\|_{Z^*}\geq c\left\|f\right\|_{Z^*}$ for any $f\in Z^*$. Consequently, $\operatorname{Im}T=Z$ by the same taken, the operator T is boundedly invertible and from $T(x\varphi_n)=x\psi_n$ $\forall x\in X$, $n\in N$ we get the b - completeness of the system $\{\psi_n\}_{n\in N}$.

The validity of the statement d) is shown by the similar reasoning.

Theorem 5. *Let X, Y and Z be B - spaces, the system $\{\varphi_n\}_{n\in N}\subset Y$ be normalized and form a b - basis in Z with the b - biorthogonal system $\{\varphi_n^*\}_{n\in N}$, the system $\{\psi_n\}_{n\in N}\subset Y$ be such that*

$$\sum_{n=1}^{\infty}\left\|\psi_n-\varphi_n\right\|_Y<\frac{1}{\gamma},$$

where $\gamma=\sup\limits_n\left\|\varphi_n^*\right\|_{L(Z,X)}<+\infty$. Then, $\{\psi_n\}_{n\in N}$ forms a b - basis b- isomorphic to $\{\varphi_n\}_{n\in N}$.

Proof. From statement 1 it follows that there exist the constants a and c such that $a\leq\left\|\varphi_n^*\right\|_{L(Z,X)}\left\|\varphi_n\right\|_Y\leq c$ $\forall n\in N$. Since $\{\varphi_n\}_{n\in N}$ is normalized, then $a\leq\left\|\varphi_n^*\right\|_{L(Z,X)}\leq c$ $\forall n\in N$. So, γ is some number. Consider the operator T determined from the formula $Tz=\sum\limits_{n=1}^{\infty}\varphi_n^*(z)(\psi_n-\varphi_n)$. Following the proof of theorem 1, it is obvious that the operator T is well-defined and linear. Further, from

$$\left\|Tz\right\|_Z\leq\sum_{n=1}^{\infty}\left\|\psi_n-\varphi_n\right\|_Y\cdot\left\|\varphi_n^*(z)\right\|_X\leq$$

$$\leq\left\|z\right\|_Z\sum_{n=1}^{\infty}\left\|\psi_n-\varphi_n\right\|_Y\cdot\left\|\varphi_n^*\right\|_{L(Z,X)}<\left\|z\right\|_Z,$$

it follows that $\left\|T\right\|_{L(Z)}<1$. Therefore, the operator $F=I-T$ is boundedly invertible. On the other hand, we have $F(x\varphi_n)=x\psi_n$ $\forall n\in N$. Q. E. D. The theorem is proved.

Example 1. Let $X=L_p(0,T)$, $0<T<+\infty$, $Y=L_p(-\pi,\pi)$ $(p\geq 2)$. Take an arbitrary orthogonal normalized basis $\{\varphi_n(x)\}_{n\in N}$ in $L_p(-\pi,\pi)$. Denote by $Z=L_{p,p-2}(D_T)$, $D_T=(0,T)\times(-\pi,\pi)$ a closed subspace $L_p(D_T)$ of the functions

$f(t,x)$ for which

$$\sum_{n=1}^{\infty} n^{p-2}\int_0^T |f_n(t)|^p\,dt<+\infty,$$

where $f_n(t)=\int_{-\pi}^{\pi} f(t,x)\,\varphi_n(x)\,dx$. If $b(f,y)=f(t)\,y(x)$ then $\{\varphi_n(x)\}_{n\in N}$ forms a b - basis in $L_{p,p-2}(D_T)$. Really, by the Paley theorem (see [1,2]) $\forall f\in Z$

$$\int_0^T\int_{-\pi}^{\pi}\left|f(t,x)-\sum_{n=1}^{m} f_n(t)\,\varphi_n(x)\right|^p dxdt\le$$

$$\le N_p\sum_{n=m+1}^{\infty} n^{p-2}\int_0^T |f_n(t)|^p\,dt\to 0\quad as\quad m\to\infty.$$

Example 2. Let $X=L_2(a,b)$, $Y=L_2(c,d)$, $Z=L_2(\Pi)$, $\Pi=(a,b)\times(c,d)$ and $b(f,y)=f(t)\,y(x)$, $f(t)\in L_2(a,b)$, $y(x)\in L_2(c,d)$. Suppose that $\{\varphi_n(x)\}_{n\in N}$ is an arbitrary orthonormed basis in $L_2(c,d)$. If $\{\psi_n(x)\}_{n\in N}$ from $L_2(c,d)$ is such that

$$\int_a^b\int_c^d\left|\sum_{n=1}^{m} f_n(t)\left(\varphi_n(x)-\psi_n(x)\right)\right|^2 dxdt\le\theta^2\int_a^b\int_c^d\left|\sum_{n=1}^{m} f_n(t)\,\varphi_n(x)\right|^2 dxdt,$$

where $\theta\in[0,1)$, then by theorem 2 the system $\{\psi_n(x)\}_{n\in N}$ forms a b - basis in $L_2(\Pi)$.

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