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PARABOLIC FRACTIONAL MAXIMAL OPERATOR IN PARABOLIC LOCAL MORREY-TYPE SPACES

Abstract

In this paper, we study the boundedness of the parabolic fractional maximal operator in parabolic local Morrey-type spaces. We reduce the problem of boundedness of the parabolic fractional maximal operator M_α, $0 \le \alpha < \gamma$ in general parabolic local Morrey-type spaces to the problem of boundedness of the supremal operator in weighted L_p-spaces on the cone of non-negative non-decreasing functions.

1. Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r and ${}^{\complement}B(x,r)$ denote the set $\mathbb{R}^n \backslash B(x,r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have positive real part. Let $A_t = t^P \quad (t > 0)$, and set $\gamma = trP$. Then, there exists a quasi-distance ρ associated with P such that (see, for example, [4, 5])

$$
\begin{aligned}
&(a) \quad \rho(A_t x) = t\rho(x), \quad t > 0, \quad \text{for every} \quad x \in \mathbb{R}^n; \\
&(b) \quad \rho(0) = 0, \quad \rho(x-y) = \rho(y-x) \ge 0 \\
&\qquad \text{and} \quad \rho(x-y) \le k(\rho(x-z) + \rho(y-z)); \\
&(c) \quad dx = \rho^{\gamma-1} d\sigma(w) d\rho, \quad \text{where} \quad \rho = \rho(x), w = A_{\rho^{-1}} x \\
&\qquad \text{and } d\sigma(w) \text{is a } C^\infty \text{measure on the ellipsoid } \{w : \rho(w) = 1\}.
\end{aligned}
$$

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Moreover, we always assume the following properties on ρ:

(d) For every x,

$$
\begin{aligned}
c_1 |x|^{\alpha_1} \le \rho(x) \le c_2 |x|^{\alpha_2} \quad \text{if} \quad \rho(x) \ge 1 \\
c_3 |x|^{\alpha_3} \le \rho(x) \le c_4 |x|^{\alpha_4} \quad \text{if} \quad \rho(x) \le 1
\end{aligned}
$$

and

$$\rho(\theta x) \le \rho(x) \quad \text{for} \quad 0 < \theta < 1.$$

Here α_i and $c_i \quad (i = 1, \ldots, 4)$ are some positive constants. Similar properties hold for ρ^* which is associated with the matrix P^*.

There are some important examples for the above spaces:

1. Let $(Px, x) \ge (x, x) \quad (x \in \mathbb{R}^n)$. In this case, $\rho(x)$ is defined by the unique solution of $|A_{t^{-1}} x| = 1$, and $k = 1$. This space is just the one studied by Calderon and Torchinsky in [4].

2. Let P be a diagonal matrix with positive diagonal entries, and let $\rho(x)$ be the unique solution of $|A_{t^{-1}} x| = 1$.

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2_a) If all diagonal entries are greater than or equal to 1, this space was studied by E.B. Fabes and N.M. Riviere [5]. More precisely they studied the weak $(1,1)$ and L^p estimates of the singular integral operators on this space in 1966.

2_b) If there are diagonal entries smaller than 1, then ρ satisfies the above $(a)-(d)$ with $k \geq 1$.

Let $f \in L_1^{\rm loc}$. The parabolic fractional maximal function $M_\alpha^P f$ is defined by

$$M_\alpha^P f(x) = \sup_{t>0} |\mathcal{E}_P(x,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}_P(x,t)} |f(y)|dy, \quad 0 \leq \alpha < \gamma.$$

If $\alpha = 0$, then $M^P \equiv M_0^P$ is the parabolic maximal operator. If $P = I$, then $M_\alpha \equiv M_\alpha^{\mathbf{1}}$ is the fractional maximal operator and $M \equiv M_0^I$ is the Hardy-Littlewood maximal operator.

In the theory of partial differential equations, together with weighted $L_{p,w}$ spaces, Morrey spaces $\mathcal{M}_{p,\lambda}$ play an important role. They were introduced by C. Morrey in 1938 [9]. These spaces appeared to be quite useful in the study of a number of problems in the theory of partial differential equations, in particular in the study of local behavior of solutions of parabolic or quasi-elliptic differential equations. The parabolic Morrey space is defined as follows: for $1 \leq p \leq \infty$, $0 \leq \lambda \leq \gamma$, a function $f \in \mathcal{M}_{p,\lambda,P}$ if $f \in L_p^{\rm loc}$ and

$$\|f\|_{\mathcal{M}_{p,\lambda,P}} \equiv \|f\|_{\mathcal{M}_{p,\lambda,P}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\lambda/p} \|f\|_{L_p(\mathcal{E}_P(x,r))} < \infty.$$

Note that $\mathcal{M}_{p,\lambda} \equiv \mathcal{M}_{p,\lambda,\mathbf{1}}$. (If $\lambda = 0$, then $\mathcal{M}_{p,0,P} = L_p$; if $\lambda = \gamma$, then $\mathcal{M}_{p,\gamma,P} = L_\infty$; if $\lambda < 0$ or $\lambda > \gamma$, then $\mathcal{M}_{p,\lambda,P} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.)

Also, by $W\mathcal{M}_{p,\lambda,P}$ we denote the weak Morrey space of all functions $f \in WL_p^{\rm loc}$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda,P}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda,P}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\lambda/p} \|f\|_{WL_p(\mathcal{E}_P(x,r))} < \infty,$$

where $WL_p(\mathcal{E}_P(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(\mathcal{E}_P(x,r))} \equiv \|f\chi_{\mathcal{E}_P(x,r)}\|_{WL_p(\mathbb{R}^n)} = \sup_{t>0} t\,|\{y \in \mathcal{E}_P(x,r) : \,|f(y)| > t\}|^{1/p}. \quad (1)$$

If in the place of the power function $r^{-\lambda/p}$ in the definition of $\mathcal{M}_{p,\lambda,P}$ we consider any positive measurable weight function w defined on $(0,\infty)$, then it becomes the Morrey-type space $\mathcal{M}_{p,w,P}$.

The following statement, containing the results in [6] was proved in [7] (see also [8]).

Theorem 1.1. *Let $1 \leq p_1 \leq p_2 < \infty$ and $\alpha = \gamma\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$. Moreover, let w_1 and w_2 be positive measurable functions satisfying the following condition:*

$$\left\|w_1^{-1}(r)\, r^{\alpha-\frac{\gamma}{p_1}-1}\right\|_{L_1(t,\infty)} \leq c w_2^{-1}(t)\, t^{\alpha-\frac{\gamma}{p_1}}. \quad (2)$$

Then for $p_1 > 1$ M_α^P is bounded from $\mathcal{M}_{p_1,w_1,P}$ to $\mathcal{M}_{p_2,w_2,P}$, and for $p_1 = 1$ M_α^P is bounded from $\mathcal{M}_{1,w_1,P}$ to $W\mathcal{M}_{p_2,w,P}$.

Earlier, in [6] a weaker version of Theorem 1.1 was proved: it was assumed that $w_1 = w_2 = w$ and that w is a positive non-increasing function satisfying the pointwise doubling condition, namely that for some $c > 0$

$$c^{-1}w(r) \le w(t) \le cw(r)$$

for all t, $r > 0$ such that $0 < r \le t \le 2r$.

2. Definitions and basic properties of parabolic local Morrey-type spaces

Definition 2.1. *Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0, \infty)$. We denote by $LM_{p\theta,w,P}$, $GM_{p\theta,w,P}$, the parabolic local Morrey-type spaces, the global Morrey-type spaces respectively, the spaces of all functions $f \in L_p^{\rm loc}$ with finite quasinorms*

$$\|f\|_{LM_{p\theta,w,P}} \equiv \|f\|_{LM_{p\theta,w,P}(\mathbb{R}^n)} = \left\|w(r)\|f\|_{L_p(\mathcal{E}_P(0,r))}\right\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GM_{p\theta,w,P}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w,P}}$$

respectively.

We denote by the (isotropic) local Morrey-type spaces, the global Morrey-type spaces respectively $LM_{p\theta,w} \equiv LM_{p\theta,w,I}$, $GM_{p\theta,w} \equiv GM_{p\theta,w,I}$, where I be a $n \times n$ identity matrix and

$$\|f\|_{LM_{p\infty,1,P}} = \|f\|_{GM_{p\infty,1,P}} = \|f\|_{L_p}.$$

Furthermore, $GM_{p\infty,r^{-\lambda/p},P} \equiv \mathcal{M}_{p,\lambda,P}$, $0 \le \lambda \le \gamma$.

Lemma 2.2. *Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0, \infty)$.*

1.If for all $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} = \infty, \tag{3}$$

then $LM_{p\theta,w,P} = GM_{p\theta,w,P} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

2. If for all $t > 0$

$$\|w(r)r^{\gamma/p}\|_{L_\theta(0,t)} = \infty, \tag{4}$$

then for all functions $f \in LM_{p\theta,w,P}$, continuous at 0, $f(0) = 0$, and for $0 < p < \infty$ $GM_{p\theta,w,P} = \Theta$.

Proof. 1. Let (3) be satisfied and f be not equivalent to zero. Then for some $t_0 > 0$

$$A = \|f\|_{L_p(\mathcal{E}_P(0,t_0))} > 0.$$

Hence

$$\|f\|_{GM_{p\theta,w,P}} \ge \|f\|_{LM_{p\theta,w,P}} \ge \left\|w(r)\|f\|_{L_p(\mathcal{E}_P(0,r))}\right\|_{L_\theta(t_0,\infty)} \ge A\|w(r)\|_{L_\theta(t_0,\infty)}.$$

Therefore $\|f\|_{GM_{p\theta,w,P}} = \|f\|_{LM_{p\theta,w,P}} = \infty$.

2. Let (4) be satisfied. If $f \in LM_{p\theta,w,P}$ and there exists

$$\lim_{r\to 0} |\mathcal{E}_P(0,r)|^{-1/p}\|f\|_{L_p(\mathcal{E}_P(0,r))} = B, \tag{5}$$

then $B = 0$.
Indeed, if $B > 0$, then there exists $t_0 > 0$ such that

$$|\mathcal{E}_P(0,r)|^{-1/p}\|f\|_{L_p(\mathcal{E}_P(0,r))} \geq \frac{B}{2} \tag{6}$$

for all $0 < r \leq t_0$. Consequently,

$$\|f\|_{LM_{p\theta,w,P}} \geq \left\|w(r)\|f\|_{L_p(\mathcal{E}_P(0,r))}\right\|_{L_\theta(0,t_0)} \geq \frac{B}{2}v_n^{1/p}\left\|w(r)r^{\gamma/p}\right\|_{L_\theta(0,t_0)},$$

where v_n is the volume of the unit ellipsoid $\{x : \rho(x) = 1\}$. Hence $\|f\|_{LM_{p\theta,w,P}} = \infty$, $f \notin LM_{p\theta,w,P}$ and we have arrived at a contradiction.

If $f \in LM_{p\theta,w,P}$ and it is continuous at 0, then (5) holds with $B = |f(0)|$. Hence $f(0) = 0$.

Next let $0 < p < \infty$ and let $f \in GM_{p\theta,w,P}$, then by the generalized Lebesgue theorem on differentiation of integrals (see, for example, [10]) for almost all $x \in \mathbb{R}^n$

$$\lim_{r\to 0}|\mathcal{E}_P(x,r)|^{-1/p}\|f\|_{L_p(\mathcal{E}_P(x,r))} = |f(x)|.$$

By the above argument for all those x we have $f(x) = 0$. Hence f is equivalent to zero.

Definition 2.3. *Let $0 < p, \theta \leq \infty$. We denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t > 0$*

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

Moreover, we denote by $\Omega_{p,\theta,P}$ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t_1, t_2 > 0$

$$\|w(r)\|_{L_\theta(t_1,\infty)} < \infty, \qquad \|w(r)r^{\gamma/p}\|_{L_\theta(0,t_2)} < \infty.$$

Keeping in mind Lemma 2.2, when considering the spaces $LM_{p\theta,w,P}$ we always assume that $w \in \Omega_\theta$, and when considering the spaces $GM_{p\theta,w,P}$ we always assume that $w \in \Omega_{p,\theta,P}$.

Example 2.4. Defined the test function f_t, $t > 0$, by the following way

$$f_t(x) = \chi_{\mathcal{E}_P(0,2t)\setminus\mathcal{E}_P(0,t)}(x), \quad x \in \mathbb{R}^n, \quad t > 0.$$

Note that, for $0 < p < \infty$

$$\|f_t\|_{L_p(\mathcal{E}_P(0,r))} = 0, \;\; 0 < r \leq t, \quad \|f_t\|_{L_p(\mathcal{E}_P(0,r))} \leq Ct^{\frac{\gamma}{p}}, \;\; t < r < \infty, \tag{7}$$

where $C > 0$ depends only on n and p. Then

$$\|f_t\|_{LM_{p\theta,w,P}} = \left\|w(r)\|f_t\|_{L_p(\mathcal{E}_P(0,r))}\right\|_{L_\theta(t,\infty)} \leq Ct^{\frac{\gamma}{p}}\|w(r)\|_{L_\theta(t,\infty)}.$$

Then $f_t \in LM_{p\theta,w,P}$ for some $t > 0$ and $w \in \Omega_\theta$.

Lemma 2.5. *Let $1 < p_1 \leq \infty$, $0 < p_2 \leq \infty$, $0 \leq \alpha < \gamma$, $0 < \theta_1, \theta_2 \leq \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$. Then the condition*

$$\alpha \leq \frac{\gamma}{p_1}$$

is necessary for the boundedness of M_α^P *from* $LM_{p_1\theta_1,w_1,P}$ *to* $LM_{p_2\theta_2,w_2,P}$.

Proof. Assume that $\alpha > \gamma/p_1$ and M_α^P is bounded from $LM_{p_1\theta_1,w_1,P}$ to $LM_{p_2\theta_2,w_2,P}$. Since $w_1 \in \Omega_{\theta_1}$ for some $t > 0$ $\|w_1\|_{L_\theta(t,\infty)} < \infty$. Let $f(x) = \rho(x)^\beta \chi_{\complement_{\mathcal{E}_P(0,t)}}$, where $-\alpha < \beta < -\gamma/p$. Note that $f \in LM_{p_1\theta_1,w_1,P}$. On the other hand for all $x \in \mathbb{R}^n$

$$M_\alpha^P f(x) \geq \lim_{t\to\infty} |\mathcal{E}_P(x,t)|^{-1+\alpha/\gamma} \int\limits_{\mathcal{E}_P(x,t)\setminus\mathcal{E}_P(x,\rho(x)+2)} \rho(y)^\beta dy \geq c \lim_{t\to\infty} t^{\alpha+\beta} = \infty,$$

where c depends only on n, α and β, hence $f \notin LM_{p_2\theta_2,w_2,P}$.

For the isotropic case $P = I$, Lemma 2.2 was proved in [1] and Lemma 2.5 was proved in [2].

Throughout this paper $a \lesssim b$, ($b \gtrsim a$), means that $a \leq \lambda b$, where $\lambda > 0$ depends on unessential parameters. If $b \lesssim a \lesssim b$, then we write $a \approx b$.

3. L_p-estimates of parabolic fractional maximal function over ellipsoids

We consider the following "partial" parabolic fractional maximal functions

$$\underline{M}^P_{\alpha,r} f(x) = \sup_{0<t\leq r} |\mathcal{E}_P(x,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}_P(x,t)} |f(y)|dy,$$

$$\overline{M}^P_{\alpha,r} f(x) = \sup_{t>r} |\mathcal{E}_P(x,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}_P(x,t)} |f(y)|dy.$$

Lemma 3.1. *Let* $0 < p \leq \infty$, $0 \leq \alpha < \gamma$ *and* $f \in L_1^{\rm loc}$. *Then for any ellipsoid* $\mathcal{E}_P(x,r)$ *in* $\mathbb{R}^n$

$$\|M_\alpha^P f\|_{WL_p(\mathcal{E}_P(x,r))} \gtrsim r^{\frac{\gamma}{p}}\, \overline{M}^P_{\alpha,r} f(x). \tag{8}$$

Proof. If $y \in \mathcal{E}_P(x,r)$ and $t > 2kr$, then $\mathcal{E}_P(x,\frac{t}{2k}) \subset \mathcal{E}_P(y,t)$ and

$$M_\alpha^P f(y) \geq 2^{\alpha-\gamma} \sup_{t>2r} \frac{1}{|\mathcal{E}_P(x,\frac{t}{2k})|^{1-\frac{\alpha}{\gamma}}} \int_{\mathcal{E}_P(x,\frac{t}{2k})} |f(z)|dz = 2^{\alpha-\gamma} \overline{M}^P_{\alpha,r} f(x).$$

Hence, if f is not equivalent to 0 on $\mathbb{R}^n$, then

$$\|M_\alpha^P f\|_{WL_p(\mathcal{E}_P(x,r))} \geq \sup_{0<t<2^{\alpha-\gamma}\overline{M}^P_{\alpha,r} f(x)} t \left|\{y \in \mathcal{E}_P(x,r) : M_\alpha^P f(y) > t\}\right|^{\frac{1}{p}} \geq$$

$$\geq \sup_{0<t<2^{\alpha-\gamma}\overline{M}^P_{\alpha,r} f(x)} t(v_n r^\gamma)^{\frac{1}{p}} = 2^{\alpha-\gamma} v_n^{\frac{1}{p}}\, r^{\frac{\gamma}{p}}\, \overline{M}^P_{\alpha,r} f(x).$$

(If f is equivalent to 0 inequality (8) is trivial.)

Lemma 3.2. *Let* $0 < p \leq \infty$, $0 \leq \alpha < \gamma$ *and* $f \in L_1^{\rm loc}$. *Then for any ellipsoid* $\mathcal{E}_P(x,r)$ *in* $\mathbb{R}^n$

$$\|M_\alpha^P f\|_{L_p(\mathcal{E}_P(x,r))} \approx \|M_\alpha^P (f\chi_{\mathcal{E}_P(x,2kr)})\|_{L_p(\mathcal{E}_P(x,r))} + r^{\frac{\gamma}{p}}\, \overline{M}^P_{\alpha,2kr} f(x). \tag{9}$$

Proof. It is obvious that for any ellipsoid $\mathcal{E}_P(x,r)$

$$\|M_\alpha^P f\|_{L_p(\mathcal{E}_P(x,r))} \lesssim \|M_\alpha^P (f\chi_{\mathcal{E}_P(x,2kr)})\|_{L_p(\mathcal{E}_P(x,r))} + \|M_\alpha^P (f\chi_{\complement_{\mathcal{E}_P(x,2kr)}})\|_{L_p(\mathcal{E}_P(x,r))}.$$

Let y be an arbitrary point in $\mathcal{E}_P(x,r)$. If $\mathcal{E}_P(y,t)\cap {}^{\complement}\mathcal{E}_P(x,2kr)\neq\emptyset$, then $t>r$. Indeed, if $z\in\mathcal{E}_P(y,t)\cap {}^{\complement}\mathcal{E}_P(x,2kr)$, then $t>\rho(z-y)\geq\frac{1}{k}\rho(z-x)-\rho(x-y)>2r-r=r$.

On the other hand $\mathcal{E}_P(y,t)\cap {}^{\complement}\mathcal{E}_P(x,2kr)\subset\mathcal{E}_P(x,2kt)$. Indeed, if $z\in\mathcal{E}_P(y,t)\cap {}^{\complement}\mathcal{E}_P(x,2kr)$, then we get $\rho(z-x)\leq k\rho(z-y)+k\rho(y-x)<kt+kr<2kt$.

Hence

$$M^P_\alpha(f\chi_{{}^{\complement}\mathcal{E}_P(x,2kr)})(y)=\sup_{t>0}\frac{1}{|\mathcal{E}_P(y,t)|^{1-\frac{\alpha}{\gamma}}}\int_{\mathcal{E}_P(y,t)\cap{}^{\complement}\mathcal{E}_P(x,2kr)}|f(z)|dz\leq$$

$$\lesssim\sup_{t\geq r}\frac{1}{|\mathcal{E}_P(x,2kt)|^{1-\frac{\alpha}{\gamma}}}\int_{\mathcal{E}_P(x,2kt)}|f(y)|dy=\overline{M}^P_{\alpha,2kr}f(x)$$

and the right-hand side inequality in (9) follows.

The left-hand side inequality in (9) follows by Lemma 3.1 and obvious inequality

$$\|M^P_\alpha f\|_{L_p(\mathcal{E}_P(x,r))}\geq\|M^P_\alpha(f\chi_{\mathcal{E}_P(x,2kr)})\|_{L_p(\mathcal{E}_P(x,r))}.$$

Lemma 3.3. *Let $1\leq p_1\leq p_2\leq\infty$ and $0\leq\alpha<\gamma$. The inequality*

$$\|M^P_\alpha(f\chi_{\mathcal{E}_P(x,2kr)})\|_{L_{p_2}(\mathcal{E}_P(x,r))}\lesssim r^{\alpha-\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L_{p_1}(\mathcal{E}_P(x,2kr))}\tag{10}$$

holds for all $f\in L^{\rm loc}_{p_1}$ if and only if in the case $p_1>1$

$$\alpha\geq\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right),\tag{11}$$

and in the case $p_1=1$

$$p_2<\infty\qquad and\qquad\alpha>\gamma\left(1-\frac{1}{p_2}\right).\tag{12}$$

Moreover for $1\leq p_2<\infty$ and $\alpha=\gamma\left(1-\frac{1}{p_2}\right)$ the inequality

$$\|M^P_\alpha(f\chi_{\mathcal{E}_P(x,2kr)})\|_{WL_{p_2}(\mathcal{E}_P(x,r))}\lesssim\|f\|_{L_1(\mathcal{E}_P(x,2kr))}\tag{13}$$

holds for all $f\in L^{\rm loc}_1$.

Proof. Recall the well-known inequalities for the fractional maximal operator [10]. If $1<p_1\leq p_2\leq\infty$, then

$$\|M^P_{\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}f\|_{L_{p_2}(\mathbb{R}^n)}\lesssim\|f\|_{L_{p_1}(\mathbb{R}^n)}.\tag{14}$$

Also if $1\leq p_2<\infty$, then

$$\|M^P_{\gamma\left(1-\frac{1}{p_2}\right)}f\|_{WL_{p_2}(\mathbb{R}^n)}\lesssim\|f\|_{L_1(\mathbb{R}^n)}.\tag{15}$$

If $1<p_1\leq p_2\leq\infty$, inequality (11) holds and $z\in\mathcal{E}_P(x,r)$, then

$$M^P_\alpha(f\chi_{\mathcal{E}_P(x,2kr)})(z)=\sup_{0<t\leq 3kr}|\mathcal{E}_P(z,t)|^{\frac{\alpha}{\gamma}-1}\int_{\mathcal{E}_P(z,t)}\left|f(y)\chi_{\mathcal{E}_P(x,2kr)}(y)\right|dy,$$

because for $t > 3kr$ $\mathcal{E}_P(z,t) \supset \mathcal{E}_P(x,2kr)$ hence

$$|\mathcal{E}_P(z,t)|^{\frac{\alpha}{\gamma}-1} \int_{\mathcal{E}_P(z,t)} \left| f(y)\chi_{\mathcal{E}_P(x,2kr)}(y) \right| dy \le$$

$$\le |\mathcal{E}_P(z,3kr)|^{\frac{\alpha}{\gamma}-1} \int_{\mathcal{E}_P(z,3kr)} \left| f(y)\chi_{\mathcal{E}_P(x,2kr)}(y) \right| dy.$$

Therefore

$$M^P_\alpha (f\chi_{\mathcal{E}_P(x,2kr)})(z) \lesssim r^{\alpha-\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} M_{\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \left(f\chi_{\mathcal{E}_P(x,2kr)}\right)(z)$$

and by (14)

$$\|M^P_\alpha (f\chi_{\mathcal{E}_P(x,2kr)})\|_{L_{p_2}(\mathcal{E}_P(x,r))} \lesssim r^{\alpha-\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \left\| M_{\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \left(f\chi_{\mathcal{E}_P(x,2kr)}\right)\right\|_{L_{p_2}(\mathbb{R}^n)} \lesssim$$

$$\lesssim r^{\alpha-\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(\mathcal{E}_P(x,2kr))}.$$

If $1 \le p_2 < \infty$ and inequality (13) holds, then by (15) and (1)

$$\|M^P_\alpha \left(f\chi_{\mathcal{E}_P(x,2kr)}\right)\|_{L_{p_2}(\mathcal{E}_P(x,r))} \le \| \left(M^P_\alpha \left(f\chi_{\mathcal{E}_P(x,2kr)}\right)\right)^* \|_{L_{p_2}(0,|\mathcal{E}_P(x,r)|)} \le$$

$$\le \sup_{0<t\le|\mathcal{E}_P(x,r)|} t^{1-\frac{\alpha}{\gamma}} \left(M^P_\alpha \left(f\chi_{\mathcal{E}_P(x,2kr)}\right)\right)^*(t) \|t^{\frac{\alpha}{\gamma}-1}\|_{L_{p_2}(0,|\mathcal{E}_P(x,r)|)} \lesssim$$

$$\lesssim r^{\alpha-\gamma\left(1-\frac{1}{p_2}\right)} \left\|M^P_\alpha \left(f\chi_{\mathcal{E}_P(x,2kr)}\right)\right\|_{WL_{\frac{\gamma}{\gamma-\alpha}}(\mathbb{R}^n)} \lesssim r^{\alpha-\gamma\left(1-\frac{1}{p_2}\right)} \|f\|_{L_1(\mathcal{E}_P(x,2kr))}.$$

Inequality (13) follows directly from (15).

If $p_1 > 1$ and $\alpha < \gamma\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, then inequality (14) cannot hold for all $f \in L^{\rm loc}_{p_1}$. Indeed if $f \in L_{p_1}(\mathbb{R}^n)$ and $f \nsim 0$ then by passing in (10) to the limit as $r \to \infty$ we arrive at a contradiction.

Assume that $p_1 = 1$, $1 \le p_2 < \infty$ and $\alpha = \gamma\left(1 - \frac{1}{p_2}\right)$. Then by passing to the limit in (10) we get

$$\|M^P_\alpha f\|_{L_{p_2}(\mathbb{R}^n)} \lesssim \|f\|_{L_1(\mathbb{R}^n)}$$

which, according to known results [10], is not possible.

Corollary 3.4. *Let* $1 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $\gamma\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \le \alpha < \gamma$ *if* $p_1 > 1$, *and* $\gamma\left(1 - \frac{1}{p_2}\right)_+ < \alpha < \gamma$ *if* $p_1 = 1$. *Then the inequality*

$$\|M^P_\alpha (f\chi_{\mathcal{E}_P(x,2kr)})\|_{L_{p_2}(\mathcal{E}_P(x,r))} \lesssim r^{\alpha-\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(\mathcal{E}_P(x,2kr))}$$

holds for all $f \in L^{\rm loc}_{p_1}$.

Moreover for $0 < p_2 < \infty$ *and* $\alpha = \gamma\left(1 - \frac{1}{p_2}\right)_+$ *the inequality*

$$\|M^P_\alpha (f\chi_{\mathcal{E}_P(x,2kr)})\|_{WL_{p_2}(\mathcal{E}_P(x,r))} \lesssim r^{\alpha-\gamma\left(1-\frac{1}{p_2}\right)} \|f\|_{L_1(\mathcal{E}_P(x,2kr))} \qquad (16)$$

holds for all $f \in L_1^{\rm loc}$.

Proof. If $p_2 \geq p_1$, the statement follows by Lemma 3.3. If $p_2 < p_1$, then by applying Hölder's inequality and statement of Lemma 3.3 we have

$$\|M_\alpha^P(f\chi_{\mathcal{E}_P(x,2kr)})\|_{L_{p_2}(\mathcal{E}_P(x,r))} \lesssim r^{\frac{\gamma}{p_2}-\frac{\gamma}{p_1}}\|M_\alpha^P(f\chi_{\mathcal{E}_P(x,2kr)})\|_{L_{p_1}(\mathcal{E}_P(x,r))} \leq$$
$$\lesssim r^{\alpha-\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L_{p_1}(\mathcal{E}_P(x,2kr))}.$$

Inequality (16) similarly follows by Hölder's inequality for weak L_p-spaces.

Lemmas 3.2, 3.3 and Corollary 3.4 imply the following statement.

Lemma 3.5. *Let* $1 \leq p_1 \leq \infty$, $0 < p_2 \leq \infty$, $\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \leq \alpha < \gamma$ *if* $p_1 > 1$, *and* $\gamma\left(1-\frac{1}{p_2}\right)_+ < \alpha < \gamma$ *if* $p_1 = 1$. *Then for any ellipsoid* $\mathcal{E}_P(x,r) \subset \mathbb{R}^n$ *the inequality*

$$\|M_\alpha^P f\|_{L_{p_2}(\mathcal{E}_P(x,r))} \lesssim r^{\alpha-\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L_{p_1}(\mathcal{E}_P(x,2kr))} + r^{\frac{\gamma}{p_2}}\,\overline{M}_{\alpha,2kr}^P f(x) \tag{17}$$

holds for all $f \in L_{p_1}^{\rm loc}$.

Moreover for $0 < p_2 < \infty$ *and* $\alpha = \gamma\left(1-\frac{1}{p_2}\right)_+$ *the inequality*

$$\|M_\alpha^P f\|_{WL_{p_2}(\mathcal{E}_P(x,r))} \lesssim r^{\alpha-\gamma\left(1-\frac{1}{p_2}\right)}\|f\|_{L_1(\mathcal{E}_P(x,2kr))} + r^{\frac{\gamma}{p_2}}\,\overline{M}_{\alpha,2kr}^P f(x) \tag{18}$$

holds for all $f \in L_1^{\rm loc}$.

Lemma 3.6. *Let* $0 < p < \infty$.

1. *If* $\gamma\left(1-\frac{1}{p}\right)_+ < \alpha < \gamma$, *then for any ellipsoid* $\mathcal{E}_P(x,r) \subset \mathbb{R}^n$ *the equivalences*

$$\|M_\alpha^P f\|_{L_p(\mathcal{E}_P(x,r))} \approx \|M_\alpha^P f\|_{WL_p(\mathcal{E}_P(x,r))} \approx r^{\frac{\gamma}{p}}\,\overline{M}_{\alpha,r}^P f(x) \tag{19}$$

hold for all $f \in L_1^{\rm loc}$.

2. *If* $\alpha = \gamma\left(1-\frac{1}{p}\right)_+$, *then for any ellipsoid* $\mathcal{E}_P(x,r) \subset \mathbb{R}^n$ *the equivalence*

$$\|M_\alpha^P f\|_{WL_p(\mathcal{E}_P(x,r))} \approx r^{\frac{\gamma}{p}}\,\overline{M}_{\alpha,r}^P f(x) \tag{20}$$

holds for all $f \in L_1^{\rm loc}$.

3. *If* $1 < p_1 < \infty$, $\gamma\left(\frac{1}{p_1}-\frac{1}{p}\right)_+ \leq \alpha < \frac{\gamma}{p_1}$, *then for any ellipsoid* $\mathcal{E}_P(x,r) \subset \mathbb{R}^n$ *the inequalities*

$$r^{\frac{\gamma}{p}}\,\overline{M}_{\alpha,r}^P f(x) \lesssim \|M_\alpha^P f\|_{L_p(\mathcal{E}_P(x,r))} \lesssim r^{\frac{\gamma}{p}}\left(\overline{M}_{\alpha p_1,r}^P(|f|^{p_1})(x)\right)^{\frac{1}{p_1}} \tag{21}$$

hold for all $f \in L_1^{\rm loc}$.

Proof. Denote

$$A_1 := r^{\frac{\gamma}{p}}\sup_{t\geq 2kr}\frac{1}{|\mathcal{E}_P(x,t)|^{1-\frac{\alpha}{\gamma}}}\int_{\mathcal{E}_P(x,t)}|f(y)|dy,$$
$$A_2 := r^{\alpha-\gamma\left(\frac{1}{p_1}-\frac{1}{p}\right)}\|f\|_{L_{p_1}(\mathcal{E}_P(x,2kr))}.$$

By Lemma 3.5

$$\|M^P_\alpha f\|_{L_p(\mathcal{E}_P(x,r))} \le A_1 + A_2.$$

By applying Hölder's inequality we get

$$A_1 \lesssim r^{\frac{\gamma}{p}} \sup_{t\ge 2kr} \frac{1}{|\mathcal{E}_P(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{\gamma}}} \left(\int_{\mathcal{E}_P(x,t)} |f(y)|^{p_1} dy\right)^{\frac{1}{p_1}} = r^{\frac{\gamma}{p}} \left(\overline{M}^P_{\alpha p_1,2kr}(|f|^{p_1})(x)\right)^{\frac{1}{p_1}}.$$

On the other hand, since $\alpha < \frac{\gamma}{p_1}$ it follows that

$$A_2 \approx r^{\frac{\gamma}{p}} \left(\sup_{t\ge 2kr} |\mathcal{E}_P(x,t)|^{\frac{\alpha}{\gamma}-\frac{1}{p_1}}\right) \|f\|_{L_{p_1}(\mathcal{E}_P(x,2kr))} \lesssim$$

$$\lesssim r^{\frac{\gamma}{p}} \left(\sup_{t\ge 2kr} \frac{1}{|\mathcal{E}_P(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{\gamma}}} \left(\int_{\mathcal{E}_P(x,t)} |f(y)|^{p_1} dy\right)^{\frac{1}{p_1}}\right) \lesssim$$

$$\lesssim r^{\frac{\gamma}{p}} \left(\overline{M}^P_{\alpha p_1,r}(|f|^{p_1})(x)\right)^{\frac{1}{p_1}}.$$

Estimates from below follow by Lemma 3.1.

Remark 3.7. We note that the right-hand side inequality in (21) implies the inequality

$$\|M^P_\alpha f\|_{L_p(\mathcal{E}_P(x,r))} \lesssim r^{\frac{\gamma}{p}} \left(\int_r^\infty \left(\int_{\mathcal{E}_P(x,t)} |f(y)|^{p_1} dy\right) \frac{dt}{t^{\gamma-\alpha p_1+1}}\right)^{\frac{1}{p_1}}.$$

This follows since

$$\left(\overline{M}^P_{\alpha p_1,r}(|f|^{p_1})(x)\right)^{\frac{1}{p_1}} \lesssim \left(\int_r^\infty \left(\int_{\mathcal{E}_P(x,t)} |f(y)|^{p_1} dy\right) \frac{dt}{t^{\gamma-\alpha p_1+1}}\right)^{\frac{1}{p_1}}.$$

In fact

$$\left(\overline{M}^P_{\alpha p_1,r}(|f|^{p_1})(x)\right)^{\frac{1}{p_1}} = \sup_{t\ge r} \frac{1}{|\mathcal{E}_P(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{\gamma}}} \left(\int_{\mathcal{E}_P(x,t)} |f(y)|^{p_1} dy\right)^{\frac{1}{p_1}} \le$$

$$\le \sup_{t\ge r} \frac{1}{|\mathcal{E}_P(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{\gamma}}} \left(\int_{\mathcal{E}_P(x,r)} |f(y)|^{p_1} dy\right)^{\frac{1}{p_1}} +$$

$$+ \sup_{t\ge r} \frac{1}{|\mathcal{E}_P(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{\gamma}}} \left(\int_{\mathcal{E}_P(x,t)\setminus\mathcal{E}_P(x,r)} |f|^{p_1} dy\right)^{\frac{1}{p_1}} \lesssim$$

$$\lesssim \frac{1}{|\mathcal{E}_P(x,r)|^{\frac{1}{p_1}-\frac{\alpha}{\gamma}}} \left(\int_{\mathcal{E}_P(x,r)} |f(y)|^{p_1} dy\right)^{\frac{1}{p_1}} + \sup_{t\ge r} \left(\int_{\mathcal{E}_P(x,t)\setminus\mathcal{E}_P(x,r)} \frac{|f(y)|^{p_1}}{\rho(y)^{\gamma-\alpha p_1}} dy\right)^{\frac{1}{p_1}}.$$

By using the equality

$$\frac{1}{\rho^{\gamma-\alpha p_1}} = \frac{1}{\gamma-\alpha p_1} \int_\rho^\infty \frac{d\tau}{\tau^{\gamma-\alpha p_1+1}}$$

with $\rho = r$ or $\rho = \rho(y)$ and the Fubini theorem we get

$$\left(\overline{M}^P_{\alpha p_1,r}(|f|^{p_1})(x)\right)^{\frac{1}{p_1}} \lesssim \left(\int_r^\infty \left(\int_{\mathcal{E}_P(x,r)} |f(y)|^{p_1} dy\right) \frac{d\tau}{\tau^{\gamma-\alpha p_1+1}}\right)^{\frac{1}{p_1}} +$$

$$+\sup_{t\geq r} \left(\int_r^t \left(\int_{\mathcal{E}_P(x,\tau)\setminus\mathcal{E}_P(x,r)} |f(y)|^{p_1} dy\right) \frac{d\tau}{\tau^{\gamma-\alpha p_1+1}}\right)^{\frac{1}{p_1}} \lesssim$$

$$\lesssim \left(\int_r^\infty \left(\int_{\mathcal{E}_P(x,\tau)} |f(y)|^{p_1} dy\right) \frac{d\tau}{\tau^{\gamma-\alpha p_1+1}}\right)^{\frac{1}{p_1}}.$$

Remark 3.8. Statement 3 of Lemma 3.6 also makes sense if $\alpha = \frac{\gamma}{p_1}$ in which case the right-hand side inequality in (21) takes the form

$$\|M^P_{\frac{\gamma}{p_1}} f\|_{L_p(\mathcal{E}_P(x,r))} \lesssim r^{\frac{\gamma}{p}} \, \|f\|_{L_{p_1}(\mathbb{R}^n)}.$$

This inequality easily follows directly by the definition of $M^P_{\frac{\gamma}{p_1}} f$ and Hölder's inequality.

Remark 3.9. All statements of this section in the isotropic case $P = I$ were proved in [3].

4. Parabolic fractional maximal operator and supremal operator

For a measurable set $E \subset \mathbb{R}^n$ and a function v non-negative and measurable on E, let $L_{p,v}(E)$ be the weighted L_p-space of all functions f measurable on E for which

$$\|f\|_{L_{p,v}(E)} = \|vf\|_{L_p(E)} < \infty.$$

Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all non-negative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and we set

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

Let u be continuous and non-negative on $(0,\infty)$. We define the supremal operators $\underline{S}_u$ and $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\underline{S}_u g)(t) := \|u\, g\|_{L_\infty(0,t)}, \quad t \in (0,\infty),$$
$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

In the case $u(r) = r^\beta$, $\beta \in \mathbb{R}$

$$(\underline{S}_\beta g)(t) := \|r^\beta\, g(r)\|_{L_\infty(0,t)}, \quad t \in (0,\infty),$$
$$(\overline{S}_\beta g)(t) := \|r^\beta\, g(r)\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

Also let $\underline{S} \equiv \underline{S}_0$ and $\overline{S} \equiv \overline{S}_0$.

If in Lemma 3.6 $x = 0$, then in the above notation it reduces to the following statement.

Lemma 4.1. *Let $0 < p < \infty$.*

1. *If $\gamma\left(1-\frac{1}{p}\right)_+ < \alpha < \gamma$, then for any $r > 0$ the inequalities*

$$\|M_\alpha^P f\|_{L_p(\mathcal{E}_P(0,r))} \approx \|M_\alpha^P f\|_{WL_p(\mathcal{E}_P(0,r))} \approx r^{\frac{\gamma}{p}}\, \overline{S}_{\alpha-\gamma}\left(\|f\|_{L_1(\mathcal{E}_P(0,\cdot))}\right)(r) \tag{22}$$

holds for all $f \in L_1^{\rm loc}$.

2. *If $\alpha = \gamma\left(1-\frac{1}{p}\right)_+$, then for any $r > 0$ the inequality*

$$\|M_\alpha^P f\|_{WL_p(\mathcal{E}_P(0,r))} \approx r^{\frac{\gamma}{p}}\, \overline{S}_{\alpha-\gamma}\left(\|f\|_{L_1(\mathcal{E}_P(0,\cdot))}\right)(r) \tag{23}$$

holds for all $f \in L_1^{\rm loc}$.

3. *If $1 < p_1 < \infty$, $\gamma\left(\frac{1}{p_1}-\frac{1}{p}\right)_+ \le \alpha < \frac{\gamma}{p_1}$, then for any $r > 0$ the inequality*

$$\begin{aligned} r^{\frac{\gamma}{p}}\, \overline{S}_{\alpha-\gamma}\left(\|f\|_{L_1(\mathcal{E}_P(0,\cdot))}\right)(r) \lesssim \|M_\alpha^P f\|_{L_p(\mathcal{E}_P(0,r))} \lesssim \\ \lesssim r^{\frac{\gamma}{p}}\, \overline{S}_{\alpha-\frac{\gamma}{p_1}}\left(\|f\|_{L_{p_1}(\mathcal{E}_P(0,\cdot))}\right)(r) \end{aligned} \tag{24}$$

holds for all $f \in L_1^{\rm loc}$.

4. *If $1 \le p_1 < \infty$, $\gamma\left(\frac{1}{p_1}-\frac{1}{p}\right)_+ \le \alpha < \frac{\gamma}{p_1}$, then for any $r > 0$ the inequality*

$$\begin{aligned} r^{\frac{\gamma}{p}}\, \overline{S}_{\alpha-\gamma}\left(\|f\|_{L_1(\mathcal{E}_P(0,\cdot))}\right)(r) \lesssim \|M_\alpha^P f\|_{WL_p(\mathcal{E}_P(0,r))} \lesssim \\ \lesssim r^{\frac{\gamma}{p}}\, \overline{S}_{\alpha-\frac{\gamma}{p_1}}\left(\|f\|_{L_{p_1}(\mathcal{E}_P(0,\cdot))}\right)(r) \end{aligned} \tag{25}$$

holds for all $f \in L_1^{\rm loc}$.

Lemma 4.2. *Let $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \le \alpha < \frac{\gamma}{p_1}$ if $p_1 > 1$, and $\gamma\left(1-\frac{1}{p_2}\right)_+ < \alpha < \gamma$ if $p_1 = 1$. Let also $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$.*

Then the operator M_α^P is bounded from $LM_{p_1\theta_1,w_1,P}$ to $LM_{p_2\theta_2,w_2,P}$ if, and in the case $p_1 = 1$ only if, the operator $\overline{S}_{\alpha-\frac{\gamma}{p_1}}$ is bounded from $L_{\theta_1,w_1(r)}(0,\infty)$ to $L_{\theta_2,w_2(r)r^{\frac{\gamma}{p_2}}}(0,\infty)$ on the cone $\mathbb{A}$.

Proof.

Sufficiency. Since $\overline{S}_{\alpha-\frac{\gamma}{p_1}}$ is bounded from $L_{\theta_1,w_1(r)}(0,\infty)$ to $L_{\theta_2,w_2(r)r^{\frac{\gamma}{p_2}}}(0,\infty)$ on the cone $\mathbb{A}$, by Lemma 4.1 we have

$$\begin{aligned} \|M_\alpha^P f\|_{LM_{p_2\theta_2,w_2,P}} \lesssim \|\overline{S}_{\alpha-\frac{\gamma}{p_1}}\left(\|f\|_{L_{p_1}(\mathcal{E}_P(0,\cdot))}\right)\|_{L_{\theta_2,w_2(r)r^{\frac{\gamma}{p_2}}}} \lesssim \\ \lesssim \|w_1(r)\|f\|_{L_{p_1}(\mathcal{E}_P(0,r))}\|_{L_{\theta_1}(0,\infty)} = \|f\|_{LM_{p_1\theta_1,w_1,P}}. \end{aligned} \tag{26}$$

Necessity. Let $p_1 = 1$ and the inequality

$$\|M_\alpha^P f\|_{LM_{p_2\theta_2,w_2,P}} \lesssim \|f\|_{LM_{1\theta_1,w_1,P}}$$

be satisfied. Then by (23)

$$\|\overline{S}_{\alpha-\gamma}\left(\|f\|_{L_1(\mathcal{E}_P(0,\cdot))}\right)\|_{L_{\theta_2,w_2(r)r^{\frac{\gamma}{p_2}}}} \lesssim \|\|f\|_{L_1(\mathcal{E}_P(0,\cdot))}\|_{L_{\theta_1,w_1}}. \qquad (27)$$

Let $g \in \mathbb{A}$. Then there exists a sequence of non-negative functions $f_n \in L_1^{\rm loc}$ such that

$$g_n(r) = \|f_n\|_{L_1(\mathcal{E}_P(0,r))} \nearrow g(r), \quad r \in (0,\infty).$$

By (27) and the Fatou lemma

$$\|\overline{S}_{\alpha-\gamma}\, g\|_{L_{\theta_2,w_2(r)r^{\frac{\gamma}{p_2}}}} \lesssim \|g\|_{L_{\theta_1,w_1}}.$$

5. Necessary and sufficient conditions

By Lemma 4.2 and Theorem 5.4 in [3] we get

Theorem 5.1. *Let $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \le \alpha < \frac{\gamma}{p_1}$ if $p_1 > 1$, and $\gamma\left(1-\frac{1}{p_2}\right)_+ < \alpha < \gamma$ if $p_1 = 1$. Let also $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$.*

Then the operator M_α^P is bounded from $LM_{p_1\theta_1,w_1,P}$ to $LM_{p_2\theta_2,w_2,P}$ if, and in the case $p_1 = 1$ only if,

(i) if $\theta_1 \le \theta_2$ and $\theta_1 < \infty$, then

$$\sup_{t>0}\left(t^{\alpha-\frac{\gamma}{p_1}}\|w_2(r)r^{\frac{\gamma}{p_2}}\|_{L_{\theta_2}(0,t)} + \|w_2(r)r^{\alpha-\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|_{L_{\theta_2}(t,\infty)}\right)\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty; \quad (28)$$

(ii) if $\theta_2 < \theta_1 < \infty$, then

$$\left\|w_2(t)t^{\alpha-\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|w_2(r)r^{\alpha-\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|^{\frac{\theta_2}{\theta_1-\theta_2}}_{L_{\theta_2}(t,\infty)}\|w_1\|^{-\frac{\theta_1}{\theta_1-\theta_2}}_{L_{\theta_1}(t,\infty)}\right\|_{L_{\theta_2}(0,\infty)} < \infty \qquad (29)$$

and

$$\left\|w_2(t)t^{\frac{\gamma}{p_2}}\|w_2(r)r^{\frac{\gamma}{p_2}}\|^{\frac{\theta_2}{\theta_1-\theta_2}}_{L_{\theta_2}(0,t)}\ \overline{S}\left(r^{\alpha-\frac{\gamma}{p_1}}\|w_1\|^{-1}_{L_{\theta_1}(r,\infty)}\right)(t)^{\frac{\theta_1}{\theta_1-\theta_2}}\right\|_{L_{\theta_2}(0,\infty)} < \infty; \qquad (30)$$

(iii) if $\theta_1 = \infty$, then

$$\left\|w_2(t)t^{\frac{\gamma}{p_2}}\,\overline{S}\left(r^{\alpha-\frac{\gamma}{p_1}}\|w_1\|^{-1}_{L_\infty(r,\infty)}\right)(t)\right\|_{L_{\theta_2}(0,\infty)} < \infty. \qquad (31)$$

Corollary 5.2. *Let $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $\gamma\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \le \alpha < \frac{\gamma}{p_1}$ if $p_1 > 1$, and $\gamma\left(1-\frac{1}{p_2}\right)_+ < \alpha < \gamma$ if $p_1 = 1$. Let also w_1, w_2 be non-negative measurable functions satisfying $w_1 \in \Omega_{p_1\infty}$, $w_2 \in \Omega_{p_2\infty}$ and*

$$\operatorname*{ess\,sup}_{t>0}\left(w_2(t)t^{\frac{\gamma}{p_2}}\operatorname*{ess\,sup}_{t<r<\infty}\frac{r^{\alpha-\frac{\gamma}{p_1}}}{\|w_1\|_{L_\infty(r,\infty)}}\right) < \infty, \qquad (32)$$

Then M_α^P is bounded from $\mathcal{M}_{p_1,w_1,P}$ to $\mathcal{M}_{p_2,w_2,P}$.

Proof. It is easy to see that boundedness of M_α^P from $LM_{p_1\infty,w_1,P}$ to $LM_{p_2\infty,w_2,P}$ implies boundedness of M_α^P from $GM_{p_1\infty,w_1,P} \equiv \mathcal{M}_{p_1,w_1,P}$ to $GM_{p_2\infty,w_2,P} \equiv \mathcal{M}_{p_2,w_2,P}$.

Remark 5.3. Note that condition (32) is weaker than condition (2) in Theorem 1.1. Indeed, if condition (2) holds, then for any r satisfying $t < r < \infty$ we get

$$\frac{1}{w_2(t)t^{\frac{\gamma}{p_2}}} \gtrsim \int_t^\infty \frac{ds}{w_1(s)s^{\frac{\gamma}{p_1}-\alpha+1}} \geq \int_r^\infty \frac{ds}{w_1(s)s^{\frac{\gamma}{p_1}-\alpha+1}} \geq$$

$$\geq \int_r^\infty \frac{ds}{\|w_1\|_{L_\infty(s,\infty)}s^{\frac{\gamma}{p_1}-\alpha+1}} \geq \frac{1}{\|w_1\|_{L_\infty(r,\infty)}} \int_r^\infty \frac{ds}{s^{\frac{\gamma}{p_1}-\alpha+1}} \approx$$

$$\approx \frac{1}{\|w_1\|_{L_\infty(r,\infty)} r^{\frac{\gamma}{p_1}-\alpha}}.$$

Thus

$$\operatorname*{ess\,sup}_{t<r<\infty} \frac{r^{\alpha-\frac{\gamma}{p_1}}}{\|w_1\|_{L_\infty(r,\infty)}} \lesssim \frac{1}{w_2(t)t^{\frac{\gamma}{p_2}}}, \quad t \in (0,\infty),$$

so condition (32) holds.

On the other hand the functions $w_1(t) = t^{\alpha-\frac{\gamma}{p_1}}$, $w_2(t) = t^{-\frac{\gamma}{p_2}}$ satisfy condition (32), but do not satisfy condition (2).

Theorem 5.1 contains necessary and sufficient conditions if $p_1 = 1$. If $p_1 > 1$ it contains sufficient conditions. However for $\theta_1 \leq \theta_2$ and the limiting case $\alpha = \gamma\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ Theorem 5.1 together with the appropriate necessity condition implies necessary and sufficient conditions.

Theorem 5.4. *Let $1 < p_1 \leq p_2 < \infty$, $0 < \theta_1 \leq \theta_2 \leq \infty$, $\alpha = \gamma\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then the condition*

$$\left\| w_2(r)\left(\frac{r}{t+r}\right)^{\frac{\gamma}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} \leq c\|w_1\|_{L_{\theta_1}(t,\infty)} \tag{33}$$

for all $t > 0$, where $c > 0$ is independent of t, is necessary and sufficient for the boundedness of M_α^P from $LM_{p_1\theta_1,w_1,P}$ to $LM_{p_2\theta_2,w_2,P}$.

Proof. Sufficiently follows by Theorem 5.1 because condition (33) is equivalent to condition (28) if $\theta_1 < \infty$ and to condition (31) if $\theta_1 = \theta_2 = \infty$. To prove necessity one should act like in paper [2].

Recall that, for $0 < p \leq \infty$

$$\|f\|_{LM_{pp,w}} = \|f\|_{L_{p,W}},$$

where for all $x \in \mathbb{R}^n$ $W(x) = \|w\|_{L_p(\rho(x),\infty)}$. For this reason Theorem 5.4 implies necessary and sufficient conditions for boundedness of M_α^P from one weighted Lebesgue spaces L_{p_1,W_1} to another one L_{p_2,W_2} for the case of radially non-increasing weights W_1 and W_2.

Corollary 5.5. *Let $1 < p_1 \leq p_2 < \infty$, $\alpha = \gamma\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, and W_1, W_2 be non-increasing radially symmetric functions with respect to the distance ρ. Then the* *condition*

$$\left\| w_2(r) \left(\frac{r}{t+r} \right)^{\gamma/p_2} \right\|_{L_{\theta_2}(0,\infty)} \leq c \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{34}$$

for all $t > 0$*, where functions* w_1 *and* w_2 *are defined by the equations*

$$W_1(x) = \|w_1\|_{L_{p_1}(\rho(x),\infty)}, \qquad W_2(x) = \|w_2\|_{L_{p_2}(\rho(x),\infty)}, \quad x \in \mathbb{R}^n, \tag{35}$$

$c > 0$ *is independent of* t*, is necessary and sufficient for the boundedness of* M^P_α *from* L_{p_1,W_1} *to* L_{p_2,W_2}.

In the isotropic case Corollary 5.5 was proved in [3].

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