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# INVESTIGATION OF A LINEAR BOUNDARY VALUE PROBLEM FOR A COMPOSITE TYPE TWO-DIMENSIONAL DIFFERENTIAL EQUATION OF THIRD ORDER WITH GENERAL BOUNDARY CONDITIONS

**Abstract**

*The boundary value problem is considered for the linear, two-dimensional, integro-differential, composite type loaded third order equation with non-local and global terms in the boundary conditions. The principal part of the equation is a derivative with respect to variable $x_2$ from two-dimensional Laplace equation. Taking into account ill-posedness of boundary value problems for hyperbolic differential equations, the principal parts of boundary conditions are chosen in the special form dictated by the obtained necessary conditions. These conditions are such that each solution of the considered equation determined in the considered domain satisfies these conditions.*

**Introduction.** As is known, in the case of ordinary differential operators for finding the solutions of boundary value problems, the Lagrange formula is the main tool [12]. But when the operator is generated by means of the boundary value problem for partial equations, the Green second formula [6], [15] becomes basic. For each concrete case [3], [11], [13], [15],some potentials (with unknown densities) that are the solutions of the stated problems are composed proceeding from boundary conditions.

The form of the kernel of the potential is determined by Green's formula mentioned above. The study of properties of constructed potentials enables to define an unknown density of some integral equations. Because of study of properties of simple and double layers (a step formula) it was possible to investigate the solution of Dirichlet and Neumann problems. In spite of the fact that the limit theorems both for normal derivatives of double layer potentials and tangential derivatives of simple and double layers are known, for some reason, they haven't applied enough to investigations of boundary value problems.

For solving the boundary value problems with oblique derivatives, a jump formula obtained in [4], [13] for a derivative of a simple layer potential when the derivative's direction given on the boundary of the considered domain is not tangential to the boundary is used.

**Problem statement.** Let $D$ be a bounded, convex in the direction of $x_2$ plane domain with Lyapunov type $\Gamma$-line boundary [15]. when the domain $D$ is orthogonal projected on the axis $x_1$ (parallel to $x_2$), the boundary $\Gamma$ is divided into the parts $\Gamma_1$ and $\Gamma_2$. The equations of these lines are denoted by $x_2 = \gamma_k(x_1)$, $k = 1, 2;$ $x_1 \in [a_1, b_1]$.

Consider the following boundary value problem

$$lu \equiv \frac{\partial^3 u(x)}{\partial x_2^3} + \frac{\partial^3 u(x)}{\partial x_1^2 \partial x_2} + \sum_{k=0}^{2} a_{2k}(x) \frac{\partial^2 u(x)}{\partial x_1^k \partial x_2^{2-k}} +$$

$$+\sum_{k=1}^{2} a_{1k}(x)\frac{\partial u(x)}{\partial x_k}+a_0(x)\,u(x)+$$

$$+\sum_{m=0}^{2}\sum_{n=1}^{2}\int_{a_1}^{b_1} K_{2mn}(x,\eta_1)\left.\frac{\partial^2 u(\eta)}{\partial\eta_1^m\partial\eta_2^{2-m}}\right|_{\eta_2=\gamma_n(\eta_1)} d\eta_1+$$

$$+\sum_{m=1}^{2}\sum_{n=1}^{2}\int_{a_1}^{b_1} K_{1mn}(x,\eta_1)\left.\frac{\partial u(\eta)}{\partial\eta_m}\right|_{\eta_2=\gamma_n(\eta_1)} d\eta_1+$$

$$+\sum_{n=1}^{2}\int_{a_1}^{b_1} K_{0n}(x,\eta_1)\;u(\eta_1,\gamma_n(\eta_1))\,d\eta_1 = f(x),\quad x\in D\subset\mathbb{R}^2, \tag{1}$$

$$l_k u\equiv \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_k(x_1)}-\sum_{p=1}^{2}\sum_{j=1}^{2}\alpha_{kjp}(x_1)\left.\frac{\partial u(x)}{\partial x_j}\right|_{x_2=\gamma_p(x_1)}-$$

$$-\sum_{p=1}^{2}\alpha_{kp}(x_1)\,u\left(x_1,\gamma_p(x_1)\right)-\sum_{p=1}^{2}\sum_{j=1}^{2}\int_{a_1}^{b_1}\alpha_{kjp}(x_1,\eta_1)\left.\frac{\partial u(\eta)}{\partial\eta_j}\right|_{\eta_2=\gamma_p(\eta_1)} d\eta_1-$$

$$-\sum_{p=1}^{2}\int_{a_1}^{b_1}\alpha_{kp}(x_1,\eta_1)\;u\left(\eta_1,\gamma_p(\eta_1)\right)d\eta_1 = f_k(x_1),\quad k=1,2;\quad x_1\in[a_1,b_1], \tag{2}$$

$$l_3 u\equiv \left.\frac{\partial^2 u(x)}{\partial x_1^2}\right|_{x_2=\gamma_2(x_1)}-\sum_{p=1}^{2}\sum_{j=1}^{2}\alpha_{3jp}(x_1)\left.\frac{\partial u(x)}{\partial x_j}\right|_{x_2=\gamma_p(x_1)}-$$

$$-\sum_{p=1}^{2}\alpha_{3p}(x_1)\,u\left(x_1,\gamma_p(x_1)\right)+\sum_{p=1}^{2}\sum_{j=1}^{2}\int_{a_1}^{b_1}\alpha_{3jp}(x_1,\eta_1)\left.\frac{\partial u(\eta)}{\partial\eta_j}\right|_{\eta_2=\gamma_p(\eta_1)} d\eta_1-$$

$$-\sum_{p=1}^{2}\int_{a_1}^{b_1}\alpha_{3p}(x_1,\eta_1)\;u\left(\eta_1,\gamma_p(\eta_1)\right)d\eta_1 = f_3(x_1),\quad x_1\in[a_1,b_1], \tag{3}$$

where all the data of equation (1) and boundary conditions (2), (3) are assumed to be continuous functions.

If we consider the data of boundary value problem (1)-(3) sufficiently smooth functions, then this problem is reduced to the second type Fredholm integral equation with respect to the function $u(x)$. Otherwise, we get the system of second type Fredholm integral equations with respect to the unknown function $u(x)$ and its derivatives. The kernel of these equations or of the obtained system doesn't contain singularities.

**Fundamental solutions and its basic properties.** Applying the Fourier transformations [6], [15], for the principal part of equation (1)(for the first two terms) we get the fundamental solution in the form

$$U(x-\xi)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{e^{i(\alpha,x-\xi)}}{\alpha_2\left(\alpha_1^2+\alpha_2^2\right)}d\alpha, \tag{4}$$

where $x-\xi=(x_1-\xi_1,x_2-\xi_2)$, and $(\alpha,x-\xi)=\alpha_1(x_1-\xi_1)+\alpha_2(x_2-\xi_2)$, $\mathbb{R}^2$ is a real plane.

Then, by means of Hormander's bilateral ladder method [14] for the integral (4) we obtain:

$$U(x-\xi)=\frac{x_2-\xi_2}{2\pi}\left[\ln\sqrt{|x_1-\xi_1|^2+(x_2-\xi_2)^2}-1\right]+$$

$$+\frac{|x_1-\xi_1|}{2\pi}\mathrm{arctg}\frac{x_2-\xi_2}{|x_1-\xi_1|}. \tag{5}$$

By means of differentiation of (4) or (), one can easily get

$$\frac{\partial^3 U(x-\xi)}{\partial x_2^3}+\frac{\partial^3 U(x-\xi)}{\partial x_1^2\partial x_2}=\delta(x-\xi)\ , \tag{6}$$

where

$$\frac{\partial U(x-\xi)}{\partial x_1}=\frac{e(x_1-\xi_1)}{\pi}\mathrm{arctg}\frac{x_2-\xi_2}{|x_1-\xi_1|} \tag{7}$$

$$\frac{\partial U(x-\xi)}{\partial x_2}=\frac{1}{2\pi}\ln\sqrt{|x_1-\xi_1|^2+(x_2-\xi_2)^2}\ , \tag{8}$$

$$\frac{\partial^2 U(x-\xi)}{\partial x_1^2}=e(x_2-\xi_2)\,\delta(x_1-\xi_1)-\frac{1}{2\pi}\frac{x_2-\xi_2}{|x_1-\xi_1|^2+(x_2-\xi_2)^2}\ , \tag{9}$$

$$\Delta_x U(x-\xi)=e(x_2-\xi_2)\,\delta(x_1-\xi_1). \tag{10}$$

$e(t)$ is a Heaviside's symmetric unique function, $\delta(x-\xi)=\delta(x_1-\xi_1)\,\delta(x_2-\xi_2)$ is Dirac's two-dimensional delta function [6], [15].

**Basic relations.** Using fundamental solution (), its property (6) and considering equations (1), we get Green's second formula [3], [6], [11], [13], [15]. From these formulas we get representations for any solution of equation (1) and expressions for the boundary values of this solution

$$\int_\Gamma \frac{\partial^2 u(x)}{\partial x_2^2}U(x-\xi)\cos(\nu,x_2)\,dx+\int_\Gamma \frac{\partial^2 u(x)}{\partial x_1^2}U(x-\xi)\cos(\nu,x_2)\,dx-$$

$$-\int_\Gamma \frac{\partial u(x)}{\partial x_2}\frac{\partial U(x-\xi)}{\partial x_2}\cos(\nu,x_2)\,dx-\int_\Gamma \frac{\partial u(x)}{\partial x_1}\frac{\partial U(x-\xi)}{\partial x_2}\cos(\nu,x_1)\,dx+$$

$$+\int_\Gamma u(x)\,\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_1)\,dx+\int_\Gamma u(x)\,\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_2)\,dx+$$

$$+\int_D l_0 u\cdot U(x-\xi)\,dx-\int_D f(x)\,U(x-\xi)\,dx=\begin{cases} u(\xi)\,, & \xi\in D,\\ \frac{1}{2}u(\xi)\,, & \xi\in\Gamma,\end{cases} \tag{11}$$

where

$$l_0 u\equiv\sum_{k=0}^{2}a_{2k}(x)\,\frac{\partial^2 u(x)}{\partial x_1^k\partial x_2^{2-k}}+\sum_{k=1}^{2}a_{1k}(x)\,\frac{\partial u(x)}{\partial x_k}+a_0(x)\,u(x)+$$

$$+\sum_{m=0}^{2}\sum_{n=1}^{2}\int_{a_1}^{b_1} K_{2mn}(x,\eta_1)\ \frac{\partial^2 u(\eta)}{\partial\eta_1^m\partial\eta_2^{2-m}}\bigg|_{\eta_2=\gamma_n(\eta_1)} d\eta_1+$$

$$+\sum_{m=1}^{2}\sum_{n=1}^{2}\int_{a_1}^{b_1} K_{1mn}(x,\eta_1)\ \frac{\partial u(\eta)}{\partial\eta_m}\bigg|_{\eta_2=\gamma_n(\eta_1)} d\eta_1+$$

$$+\sum_{n=1}^{2}\int_{a_1}^{b_1} K_{0n}(x,\eta_1)\ u(\eta_1,\gamma_n(\eta_1))\,d\eta_1\,. \tag{12}$$

Then applying the schemes of the papers [1], [2], [5], [10], we obtain the remaining basic relations that give representations both for the derivative of the unknown function and boundary values of these derivatives.

$$-\int_\Gamma \frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial U(x-\xi)}{\partial x_2}\cos(\nu,x_2)\,dx-\int_\Gamma \frac{\partial^2 u(x)}{\partial x_1\partial x_2}\frac{\partial U(x-\xi)}{\partial x_2}\cos(\nu,x_1)\,dx+$$

$$+\int_\Gamma \frac{\partial u(x)}{\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_2)\,dx+\int_\Gamma \frac{\partial u(x)}{\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_1)\,dx-$$

$$-\int_D l_0u\cdot\frac{\partial U(x-\xi)}{\partial x_2}dx+\int_D f(x)\,\frac{\partial U(x-\xi)}{\partial x_2}dx=\begin{cases}\dfrac{\partial u(\xi)}{\partial \xi_2}, & \xi\in D,\\[2ex] \dfrac{1}{2}\dfrac{\partial u(\xi)}{\partial \xi_2}, & \xi\in\Gamma,\end{cases} \tag{13}$$

$$-\int_\Gamma \frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial U(x-\xi)}{\partial x_1}\cos(\nu,x_2)\,dx-\int_\Gamma \frac{\partial^2 u(x)}{\partial x_1^2}\frac{\partial U(x-\xi)}{\partial x_1}\cos(\nu,x_2)\,dx+$$

$$+\int_\Gamma \frac{\partial u(x)}{\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_2)\,dx+\int_\Gamma \frac{\partial u(x)}{\partial x_1}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_1)\,dx-$$

$$-\int_\Gamma \frac{\partial u(x)}{\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_1)\,dx-\int_\Gamma \frac{\partial u(x)}{\partial x_1}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_1)\,dx-$$

$$-\int_D l_0u\cdot\frac{\partial U(x-\xi)}{\partial x_1}dx+\int_D f(x)\,\frac{\partial U(x-\xi)}{\partial x_1}dx=\begin{cases}\dfrac{\partial u(\xi)}{\partial \xi_1}, & \xi\in D,\\[2ex] \dfrac{1}{2}\dfrac{\partial u(\xi)}{\partial \xi_1}, & \xi\in\Gamma,\end{cases} \tag{14}$$

$$\int_\Gamma \frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\cos(\nu,x_2)\,dx+\int_\Gamma \frac{\partial^2 u(x)}{\partial x_1^2}\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\cos(\nu,x_2)\,dx-$$

$$-\int_\Gamma \frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_1)\,dx+\int_\Gamma \frac{\partial^2 u(x)}{\partial x_1\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_2)\,dx-$$

$$-\int_\Gamma \frac{\partial^2 u(x)}{\partial x_1\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_1)\,dx+\int_\Gamma \frac{\partial^2 u(x)}{\partial x_1^2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_2)\,dx+$$

$$+\int_D l_0u\cdot\frac{\partial^2 U(x-\xi)}{\partial x_1^2}dx-\int_D f(x)\,\frac{\partial^2 U(x-\xi)}{\partial x_1^2}dx=\begin{cases}\dfrac{\partial^2 u(\xi)}{\partial \xi_1^2}, & \xi\in D,\\[2ex] \dfrac{1}{2}\dfrac{\partial^2 u(\xi)}{\partial \xi_1^2}, & \xi\in\Gamma.\end{cases} \tag{15}$$

Notice that both in[1], [2], [14] and in the remaining two expressions, integration by parts is reduced so that the derivative higher than third order in domain $D$ (both for $u(x)$ and for $U(x-\xi)$) and a derivative higher than second order on boundary $\Gamma$ don't appear in the integrand.

$$\int_\Gamma \frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial U^2(x-\xi)}{\partial x_2^2}\cos(\nu,x_2)\,dx+\int_\Gamma \frac{\partial^2 u(x)}{\partial x_1\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_1)\,dx-$$

$$-\int_\Gamma \frac{\partial^2 u(x)}{\partial x_1\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_2)\,dx+\int_\Gamma \frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_1)\,dx+$$

$$+\int_D l_0 u\cdot\frac{\partial^2 U(x-\xi)}{\partial x_2^2}dx-\int_D f(x)\frac{\partial^2 U(x-\xi)}{\partial x_2^2}dx=\begin{cases}\dfrac{\partial^2 u(\xi)}{\partial \xi_2^2}, & \xi\in D,\\[2mm] \dfrac{1}{2}\dfrac{\partial^2 u(\xi)}{\partial \xi_2^2}, & \xi\in\Gamma,\end{cases} \tag{16}$$

$$\int_\Gamma \frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_2)\,dx+\int_\Gamma \frac{\partial^2 u(x)}{\partial x_1\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_1)\,dx-$$

$$-\int_\Gamma \frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_1)\,dx+\int_\Gamma \frac{\partial^2 u(x)}{\partial x_1\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_2)\,dx+$$

$$+\int_D l_0 u\cdot\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}dx-\int_D f(x)\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}dx=\begin{cases}\dfrac{\partial^2 u(\xi)}{\partial \xi_1\partial\xi_2}, & \xi\in D,\\[2mm] \dfrac{1}{2}\dfrac{\partial^2 u(\xi)}{\partial \xi_1\partial\xi_2}, & \xi\in\Gamma.\end{cases} \tag{17}$$

Thus we establish the following

**Theorem 1.** *If $D\subset\mathbb{R}^2$ is a bounded, convex domain with Lyapunov line $\Gamma$-boundary, all the data of equation* (1)*, $a_{2k}(x)$, $k=\overline{0,2}$, $x\in D$; $a_{1k}(x)$, $k=1,2$, $x\in D$; $a_0(x)$, $x\in D$; $K_{2mn}(x,\eta_1)$, $m=\overline{0,2}$, $n=\overline{1,2}$, $x\in D$, $\eta_1\in(a_1,b_1)$; $K_{1mn}(x,\eta_1)$, $m=\overline{1,2}$, $n=\overline{1,2}$, $x\in D$, $\eta_1\in(a_1,b_1)$; $K_{on}(x,\eta_1)$, $x\in D$, $\eta_1\in(a_1,b_1)$ and $f(x)$, $x\in D$ are continuous functions, $[a_1,b_1]=np_xD=np_{x_1}\Gamma_1=np_{x_1}\Gamma_2$ and $\Gamma_1$ and $\Gamma_2$ are the parts of the boundary $\Gamma$ of domain $D$ obtained under orthogonal projection of domain $D$ on $x$, then each solution of equation* (1) *determined in domain $D$ satisfies the basic relations* (11)*,* (13)*-*(17)*.*

**Necessary conditions.** Considering the second expressions of the basic relations (11), (13)-(17), passing from the integrals of the boundary $\Gamma$ on the parts of this boundary $\Gamma_k\,(k=1,2)$ obtained under orthogonal projection of $D$ on the axis $x_1$ parallel to $x_2$, we get

$$u(\xi_1,\gamma_k(\xi_1))=...,\ \ k=1,2;\ \ \xi_1\in[a_1,b_1], \tag{18}$$

$$\left.\frac{\partial u(\xi)}{\partial \xi_j}\right|_{\xi_2=\gamma_k(\xi_1)}=...,\ \ j,k=1,2;\ \ \xi_1\in[a_1,b_1], \tag{19}$$

where $\xi_2=\gamma_k(\xi_1)$, $k=1,2$ are the equations of the parts $\Gamma_k$ of boundary $\Gamma$ and the "dots" denote the sums of nonsingular terms.

As it is seen from the fundamental solution (), for the boundary values of the second derivative we have

$$\left.\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} = \frac{1}{2\pi}\frac{\gamma'_p(\sigma_p(x_1,\xi_1))}{(x_1-\xi_1)\left[1+\gamma'^2_p(\sigma_p)\right]}, \quad p=1,2; \tag{20}$$

$$\left.\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} = \frac{1}{2\pi}\frac{1}{(x_1-\xi_1)\left[1+\gamma'^2_p(\sigma_p)\right]}, \quad p=1,2; \tag{21}$$

$$\left.\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} = -\frac{1}{2\pi}\frac{\gamma'_p(\sigma_p)}{(x_1-\xi_1)\left[1+\gamma'^2_p(\sigma_p)\right]}, \quad p=1,2; \tag{22}$$

$$\left.\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_q(\xi_1)}} = \delta(x_1-\xi_1)\, e\left(\gamma_p(x_1)-\gamma_q(\xi_1)\right), \quad p,q=1,2; \quad p\neq q, \tag{23}$$

where $\sigma_p(x_1,\xi_1)$ is located between $x_1$ and $\xi_1$.

Then the remaining necessary conditions will take the form

$$\left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_1(\xi_1)} - \left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_2(\xi_1)} - \left.\frac{\partial^2 u(\xi)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_2(\xi_1)} =$$

$$= -\frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\right|_{x_2=\gamma_1(x_1)}\frac{dx_1}{x_1-\xi_1} + \ldots \tag{24}$$

$$\left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_2(\xi_1)} - \left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_1(\xi_1)} - \left.\frac{\partial^2 u(\xi)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_1(\xi_1)} =$$

$$= \frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\right|_{x_2=\gamma_2(x_1)}\frac{dx_1}{x_1-\xi_1} + \ldots \tag{25}$$

$$\left.\frac{\partial^2 u(\xi)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_k(\xi_1)} = \frac{(-1)^{k-1}}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\right|_{x_2=\gamma_k(x_1)}\frac{dx_1}{x_1-\xi_1} + \ldots, \quad k=1,2; \tag{26}$$

$$\left.\frac{\partial^2 u(\xi)}{\partial \xi_1\partial \xi_2}\right|_{\xi_2=\gamma_k(\xi_1)} = \frac{(-1)^{k}}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_k(x_1)}\frac{dx_1}{x_1-\xi_1} + \ldots, \quad k=1,2. \tag{27}$$

By that we obtained the following statement.

**Theorem 2.** *Under the conditions of the theorem 1 each solution of the equation* (1) *satisfies to the regular necessary conditions* (18),(19).

**Theorem 3.** *Under the conditions of the theorem 1 each solution of the equation* (1) *satisfies to the singular necessary conditions (24)-(27).*

**Fredholm property.** Considering necessary singular conditions (27) and taking into account boundary conditions (2), we get

$$\left.\frac{\partial^2 u(\xi)}{\partial \xi_1\partial \xi_2}\right|_{\xi_2=\gamma_k(\xi_1)} = \frac{(-1)^{k}}{\pi}\int_{a_1}^{b_1}\frac{dx_1}{x_1-\xi_1}\left\{f_k(x_1) + \sum_{p=1}^{2}\sum_{j=1}^{2}\alpha_{kjp}(x_1)\left.\frac{\partial u(x)}{\partial x_j}\right|_{x_2=\gamma_p(x_1)} + \right.$$

$$+\sum_{p=1}^{2}\alpha_{kp}(x_1)\,u\left(x_1,\gamma_p(x_1)\right)+\sum_{p=1}^{2}\sum_{j=1}^{2}\int_{a_1}^{b_1}\alpha_{kjp}(x_1,\eta_1)\left.\frac{\partial u(\eta)}{\partial\eta_j}\right|_{\eta_2=\gamma_p(\eta_1)}+$$

$$\left.+\sum_{p=1}^{2}\int_{a_1}^{b_1}\alpha_{kp}(x_1,\eta_1)\,u\left(\eta_1,\gamma_p(\eta_1)\right)d\eta_1\right\}+.... \tag{28}$$

The first term in the right hand side is easily regularized, if

$$f_k(x)\in C^{(1)}[a_1,b_1],\quad f_k(a_1)=f_k(b_1)=0,\quad k=1,2, \tag{29}$$

given in [9].

Concerning the second and third terms in the right hand side of (28), they are regularized using regular relations (18), (19).

After substitution of (18), (19), it suffices to replace regular integrals in (18) and (19) by singular integrals contained in (28). Finally, as for the last two terms in the right hand side of (28), it suffices to interchange the integrals contained in it.

Finally, passing to necessary condition (24), substitute in its right hand side (where singular integrals are contained) instead of $\left.\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\right|_{x_2=\gamma_1(x_1)}$ its regular expression obtained by means of (28), then, in the left hand side of expression (24), instead of $\left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_2(\xi_1)}$ and $\left.\frac{\partial^2 u(\xi)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_2(\xi_1)}$ substitute their expressions from boundary conditions (2) and (3). Then we get the regular relation for $\left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_1(\xi_1)}$ as well.

Thus we proved

**Theorem 4.** *Under the conditions of theorem 1, (29), if* $\alpha_{kjp}(x_1)$, $k=\overline{1,3}$, $j=1,2$, $p=1,2$; $\alpha_{kp}(x_1)$, $k=\overline{1,3}$, $p=1,2$, $x_1\in[a_1,b_1]$; $\alpha_{kjp}(x_1,\eta_1)$, $k=\overline{1,3}$, $j=1,2$, $p=1,2$; $x_1\in[a_1,b_1]$, $\eta_1\in[a_1,b_1]$; $\alpha_{kp}(x_1,\eta_1)$, $k=\overline{1,3}$, $p=1,2$; $x_1\in[a_1,b_1]$, $\eta_1\in[a_1,b_1]$ *and* $f_3(x_1)$, $x_1\in[a_1,b_1]$ *are continuous functions, then for boundary values* $u(x)$ *and its derivative up to second order inclusively, we get a normal system of the second order integral equations whose Fredholm kernel formulas don't contain singularities (i.e. singularity in the trace formula is weak).*

If all the boundary values up to second order inclusively are determined by means of the above mentioned system of integral equations, then after substitution of these boundary values to the left hand side of (11), (13)-(17), for the unknown function $u(x)$ and its derivatives up to the second order inclusively for $\xi\in D$, from the first terms of (11), (13)-(17) we get a system of Fredholm normal type integral equations of second order and with regular kernels.

So finally it is proved

**Theorem 5.** *Under the conditions of theorem 4, boundary value problem* (1)-(3) *is of Fredholm.*

A boundary value problem for the second order composite type equation was investigated in [7].

Various special cases of boundary value problems for composite type equations of third order have been considered in [8].

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