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ON PERTURBED BASES OF EXPONENTIAL FUNCTIONS WITH COMPLEX COEFFICIENTS

Abstract

A perturbed double system of exponents with piecewise continuous complex-valued coefficients is considered. Its basicity in Lebesque spaces of functions is established under definite conditions on the coefficients. Such systems arise while considering spectral problems for discontinuous differential operators.

Introduction

Consider the following system of exponents

$$\left\{e^{i\lambda_n(t)}\right\}_{n\in Z}, \tag{1}$$

($Z-$are integers), where $\lambda_n(t)$ has the representation

$$\lambda_n(t) = nt - \alpha(t)\ sign\,n + \beta_n(t)\,, as\ n \to \infty. \tag{2}$$

Let us assume that the following conditions are fulfilled.

a) $\alpha(t)$ is a piecewise- continuous function on $[-\pi,\pi]$, $\{t_k\}_1^r : -\pi = t_0 < t_1 < ... < t_r < t_{r+1} = \pi-$ are its discontinuity points of first kind, with respect to the function $\beta_n(t)$ we have

$$\beta_n(t) = O\left(\frac{1}{n^{\gamma_k}}\right), t \in (t_k, t_{k+1})\,, k = \overline{0,r}, \{\gamma_k\}_1^r \subset (0,+\infty)\,. \tag{3}$$

Systems in the form of (1) arise as the root functions of ordinary discontinuous differential operators. In order to bring clearness to this circumstance, consider the following system of exponents

$$\left\{e^{i\lambda_n t}\right\}_{n\in Z}, \tag{4}$$

where $\{\lambda_n\} \subset C-$ is a sequence of complex numbers. Systems (4) are model ones while studying spectral properties of differential operators. Under suitable choice of the bounded variation function $\sigma(t)$ on the segment $[-a,\,a]$, they are eigenfunctions of first order differential operator $Du = \frac{du}{dt}$ with an integral condition of the form

$$\int_{-a}^{a} u(t)\ d\sigma(t) = 0.$$

For this reason, many mathematicians appealed to study of basis properties of the systems form (4) in different spaces of functions. If the operator D is considered in the Lebesgue space $L_p(-a\,,a)$, $1 \le p < +\infty$, then its natural domain of definition is the Sobolev space $W_p^1(-a,\,a)$ i.e. the space consisting of absolutely continuous on $[-a,\,a]$ functions, whose derivatives belong to $L_p(-a\,,a)$ and the relation

$$Du = \frac{du}{dt} = \lambda\, u(t)\,, \tag{5}$$

holds a.e. on all the segment $[-a,\,a]$.

Apparently, the first results for basis properties of the systems of the form (4) in the spaces L_p, $1 \le p \le +\infty$, $(L_\infty \equiv C[-a,a])$ belong to the famous mathematicians Paley P.- N.Wiener [1] and N.Levinson [2]. Further, this direction was developed in the investigations of many mathematicians. For more detailed information see the monographs of R. Young [3] and A.M.Sedletskii [4] (and also the papers [5-8]) and their references. There is also the survey paper [9].

Many problems of mechanics and mathematical physics reduce to discontinuous differential operators, i.e. to the case when the domain of definition of a differential operator is not connected. As an example, consider the following Cartesian product $\tilde{W}_p^k = W_p^k(-1,0) \times W_p^k(0,1)$. Let us identify $L_p(-1,1)$ with the Cartesian product $\tilde{L}_p = L_p(-1,0) \times L_p(1,0)$. Consider the matrix differential operator

$$T = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

where D_{11}, D_{21} (D_{12}, D_{22}) are some differential operators with domains of definition $W_p^1(-1,0)$ $\left(W_p^1(0,1)\right)$. Let $\bar{u} = (u_1, u_2) \in \tilde{W}_p^k$ be some vector. By definition $T\bar{u} = \bar{W} = (W_1; W_2))$, where $W_i = D_{i1}u_1 + D_{i2}u_2$, $i = 1,2$. Such a consideration allows to extend the problem's statement by spectral parameter (5) with respect to the spectral parameter λ. Namely, instead of the scalar spectral parameter λ, we consider the spectral matrix $\Lambda(\lambda)$:

$$\Lambda(\lambda) = \begin{pmatrix} \lambda & \lambda_{12}(\lambda) \\ \lambda_{21}(\lambda) & \lambda_{22}(\lambda) \end{pmatrix},$$

where $\lambda_{ij}(\lambda)$ are some analytic functions on λ. Thus, $\Lambda(\lambda)$ is said to be an eigen matrix of the operator T, if there exists a nonzero $\bar{u} \ne 0 :\ T\bar{u} = \Lambda(\lambda)\,\bar{u}$. In this case $\bar{u}$ is called an eigenvector responding to the eigen matrix. Consider a simple case, when D_{12}, D_{21} are zero operators and $D_{11} = D\Big/_{W_p^1(-1,0)}$, $D_{22} = D\Big/_{W_p^1(0,1)}$ ($D/_M-$ is contraction of the operator D on M). Let $\lambda_{12}(\lambda) \equiv \lambda_{21}(\lambda) \equiv 0$, $\lambda_{22}(\lambda) = \mu(\lambda)$. Then, it is obvious that the function

$$u(t) = \begin{cases} e^{\lambda t}, & -1 \le t < 0, \\ e^{\mu(\lambda)\,t}, & 0 \le t \le 1, \end{cases}$$

is an eigen element of the discontinuous operator D, responding to the eigen matrix

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu(\lambda) \end{pmatrix}.$$

Naturally, for the operator T we can consider the different conditions (for example, boundary conditions) that determine the structure of the spectrum T. The following system is a trivial example of the case under consideration

$$s_n(t) = \begin{cases} \sin nt, & 0 < t < \frac{\pi}{2}, \\ \cos nt, & \frac{\pi}{2} < t < \pi. \end{cases}$$

Let $J_1 = \left(0, \frac{\pi}{2}\right)$,$J_2 = \left(\frac{\pi}{2}, \pi\right)$. It is obvious that $\{s_n\}$ are the eigen functions of the following spectral problem with a spectrum in boundary conditions

$$\begin{cases} u''(t) + \lambda^2 u(t) = 0\,, \ t \in J_1 \cup J_2\,, \\ u(0) = u'(\pi) = 0, \\ u'\left(\frac{\pi}{2} - 0\right) = \lambda u\left(\frac{\pi}{2} + 0\right)\,,\ -\lambda u\left(\frac{\pi}{2} - 0\right) = u'\left(\frac{\pi}{2} + 0\right). \end{cases}$$

It should be noted that the systems of the form (1) arise as eigen functions of appropriate differential operators while solving many problems of mechanics and mathematical physics by the method of separation of variables. Concerning these issues see the papers [1-3].

These examples very clearly demonstrate expediency of study of basis properties of the systems form (1). Present paper is devoted to investigation of basicity of system (1) in $L_p \equiv L_p(-\pi, \pi)$, $1 < p < +\infty$.

1. Necessary information and main assumptions

In further we'll need the following notion and facts from the theory of bases.

The Banach space will be called a B- space, the Hilbert case a H- space. X^* is a space conjugate to space X.

Definition 1. *The system $\{x_n\}_{n\in N} \subset X$ in B-space X is said to be ω- linearly independent, if from $\sum_{n=1}^{\infty} a_n x_n = 0 \Rightarrow a_n = 0$, $\forall n \in N$.*

It holds the following

Lemma 1. *Let X be B- space with the basis $\{x_n\}_{n\in N}$ and $F : X \to X$ be a Fredholm operator. Then the following properties of the system $\{y_n = Fx_n\}_{n\in N}$ in X are equivalent:*

1. *$\{y_n\}_{n\in N}$ is complete;*
2. *$\{y_n\}_{n\in N}$ is minimal;*
3. *$\{y_n\}_{n\in N}$ is ω- linearly independent;*
4. *$\{y_n\}_{n\in N}$ is a basis isomorphic to $\{x_n\}_{n\in N}$.*

We will need the following notion

Definition 2. *The systems $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ in a B- space X with the norm $\|\cdot\|$ are said to be p-close, if $\sum_n \|x_n - y_n\|^p < +\infty$.*

Definition 3. *The minimal system $\{x_n\}_{n\in N} \subset X$ in a B-space X with conjugated $\{x_n^*\}_{n\in N} \subset X^*$ is said to be a p- system if for $\forall x \in X$: $\{x_n^*(x)\}_{n\in N} \in l_p$, where l_p is an ordinary space of sequences $\{a_n\}_{n\in N}$ of scalars with the norm $\|\{a_n\}_{n\in N}\|_{l_p} = \left(\sum_n |a_n|^p\right)^{\frac{1}{p}}$.*

In the case of basicity, such a system will be called a p-basis.

The following lemma is also valid.

Lemma 2. *Let X be a B-space with q-basis $\{x_n\}_{n\in N}$ and the system $\{y_n\}_{n\in N} \subset X$ be p- close to it: $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < +\infty$. Then the expression*

$$Fx = \sum_n x_n^*(x)\, y_n,$$

generates a Fredholm operator in X, where $\{x_n^*\}_{n\in N} \subset X^*$ is a system conjugated to $\{x_n\}_{n\in N}$.

One can see these or other facts in the monographs [4-7], and also in the papers [8-12].

2. Basic results

At first we consider the system of exponents

$$\left\{e^{i\mu_n(t)}\right\}_{n\in Z}, \tag{6}$$

corresponding to the main part of (2), where $\mu_n(t) = nt - \alpha(t)\, sign\, n$, $n \in Z$. For the basicity of system (6) in L_p the results of the paper [13] will be used. Represent system (6) in the form

$$\left\{e^{-i\alpha(t)}e^{int};\, e^{i\alpha(t)}e^{-i(n+1)t}\right\}_{n\in Z_+}, \tag{7}$$

(Z_+- are non-negative integers). Let us denote by $\{\alpha_k\}_1^r : \alpha_k = \alpha(t_k + 0) - \alpha(t_k - 0)$ a jump of a function $\alpha(t)$ at the points $\{t_k\}_1^r$. Let the following condition be fulfilled:

) $\frac{\alpha_k}{\pi} - \frac{1}{p} \notin Z$, $\forall k = \overline{1,r}$.

Finding $\{n_i\}_1^r \subset Z$ from the following inequalities $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$:

$$-\frac{1}{q} < \frac{\alpha_i}{\pi} + n_{i-1} - n_i < \frac{1}{p}, i = \overline{1,r}\,,\ n_0 = 0. \tag{8}$$

assume

$$\omega = \frac{\alpha(-\pi) - \alpha(\pi)}{\pi} + n_r. \tag{9}$$

Based on Theorem 1 of the paper [13] we can directly conclude the following:

Statement 1. *Let the conditions),) be fulfilled for the function* $\alpha(t)$*. Suppose that* $\omega \neq \frac{1}{p}$*. The system* (7) *forms a basis in* L_p, $1 < p < +\infty$(*for* $p = 2$ *a Riesz basis) iff it holds the inequality* $-\frac{1}{q} < \omega < \frac{1}{p}$.

We'll use the following statement obtaining from the results of the paper [14].

Statement 2. *If system* (7) *forms a basis in* L_p, $1 < p < +\infty$, *then it is isomorphic to the classic system of exponents* $\left\{e^{int}\right\}_{n\in Z}$.

So, let system (6) form a basis in L_p, $1 < p < +\infty$. Denote by $\{\vartheta_n\}_{n\in Z} \subset L_q$ a system biorthogonal to it. Let $f \in L_p$ and $\{f_n\}_{n\in Z}$ be its biorthogonal coefficients by system (6), i.e.

$$f_n = \int_{-\pi}^{\pi} f(t)\,\overline{\vartheta_n(t)}dt, n \in Z,$$

where $(\bar{\cdot})$ is complex conjugation.

The following theorem can be directly concluded from Statement 2.

Theorem 1. *Let system* (6) *form a basis in* L_p, $1 < p < +\infty$. *Then there hold:*

1. Let $1 < p \le 2$ *and* $f \in L_p$. *Then* $\{f_n\}_{n\in Z} \in l_q$, *and*

$$\left\|\{f_n\}_{n\in Z}\right\|_{l_q} \le m_p \left\|f\right\|_p,$$

is fulfilled, where m_p- *is a constant independent of* f , $\|\cdot\|_p-$*is an ordinary norm in* L_p.

2. Let $p > 2$ *and the sequence of numbers* $\{a_n\}_{n\in Z}$ *belong to* l_q. *Then* $\exists f \in L_p$ *such that* $f_n = a_n$, $\forall n \in Z$, *moreover*

$$\left\|f\right\|_p \le M_p \left\|\{f_n\}_{n\in Z}\right\|_{l_q},$$

where M_p- is a constant independent of $\{f_n\}_{n\in Z}$.

Now, study the basicity of system (1) *in L_p. We have*

$$\left|e^{i\lambda_n(t)}-e^{i\mu_n(t)}\right|=\left|e^{i\beta_n(t)}-1\right|=\left|\sum_{k=1}^{\infty}\frac{\beta_n^k(t)}{k!}\right|\le\sum_{k=1}^{\infty}\frac{Mn^{-\gamma}}{k!}=cn^{-\gamma},$$

where $\gamma=\min_k \gamma_k$, $c-$is a constant independent of n. The last inequality follows from (3). Consider the different cases.

I. Let $1<p\le 2$, $\gamma>\frac{1}{p}$.We have

$$\sum_n\left\|e^{i\lambda(t)}-e^{i\mu_n(t)}\right\|_p^p\le c_p\sum_n\frac{1}{|n|^{\gamma p}}<+\infty.$$

Assume that all the conditions of Statement 1 are fulfilled. Then, system (6) forms a basis in L_p. Thus, from Theorem 1 it follows that it forms a q-basis in L_p in this case. Let $\{\vartheta_n\}_{n\in Z}$ be a system biorthogonal to it. Consider the operator $F:L_p\to L_p$:

$$Ff=\sum_n\vartheta_n(f)\,e^{i\lambda_n(t)},\, f\in L_p, \tag{10}$$

where $\vartheta_n(f)=\int_{-\pi}^{\pi}f(t)\,\overline{\vartheta_n(t)}dt$, $n\in Z$. By Lemma 2 operator (10) is Fredholm in L_p. It is easy to see that $F\left[e^{i\mu_n(t)}\right]=e^{i\lambda_n(t)}$, $\forall n\in Z$. Then, the statement of Lemma 1 is valid for system (6).

II. Let $2<p<+\infty$, $\gamma>\frac{1}{q}$. It is clear that $q<p$. Consequently, for $\forall f\in L_p$ it is valid $\|f\|_q\le c_p\|f\|_p$, where c_p depends only on p. Assume that all the conditions of Statement 1 are fulfilled. Consequently, system (6) forms a basis in L_p. It is clear that $f\in L_q$ and $1<q<2$. Then, from Theorem 1 we obtain that $\{f_n\}_{n\in Z}\in l_p$, where $\{f_n\}_{n\in Z}$ are the orthogonal coefficients of f by system (6). From the same theorem we obtain:

$$\left\|\{f_n\}_{n\in Z}\right\|_{l_p}\le m_q\|f\|_q\le M_p\|f\|_p,\forall f\in L_p,$$

where the constant M_p is independent of f. Thus, system (6) forms a p-basis in L_p. It is easy to see that systems (1) and (6) are q-close in L_p. Consider operator (10). Further, we behave similarly to case I. Hence the validity of the following theorem is proved.

Theorem 2. *Let asymptotic formula* (2), *hold, the function $\alpha(t)$ satisfy the conditions),) and for the function $\beta_n(t)$ the relations (3) be valid. Assume that it holds*

$$-\frac{1}{q}<\omega<\frac{1}{p},\gamma>\max\left\{\frac{1}{p};\frac{1}{q}\right\},$$

where $\gamma=\min_k\gamma_k$, ω is defined from expressions (8), (9). *Then, the following properties for system (1) in L_p are equivalent :*

1. *complete;*

2. *minimal;*

3. *ω-linearly independent;*

4. *forms a basis isomorphic to* $\left\{e^{int}\right\}_{n\in Z}$.

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