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INVESTIGATION OF GREEN FUNCTION OF HIGHER ORDER OPERATOR-DIFFERENTIAL EQUATION ON FINITE SEGMENT

Abstract

In the present paper the Green function of a second order operator-differential equation on a finite segment is studied.

Let H be a seperable Hilbert space. Denote by H_1 a Hilbert space of strongly measurable on the segment $[0,\pi]$ functions $f(x)$ with the values from H for which

$$\int_0^\pi \|f\|_H^2 \, dx < \infty$$

The scalar product of the elements $f(x),\ g(x) \in H_1$ is defined by the equality

$$[f,g]_{H_1} = \int_0^\pi (f(x), g(x))_H dx$$

In the space $H_1 = L_2 [H; 0 \le x \le \pi]$ consider the operator L generated by the differential expression

$$l(y) = (-1) y^{(2n)} + \sum_{j=2}^{2n} Q_j(x) y^{(2n-j)},\ 0 \le x \le \pi \tag{1}$$

and the boundary conditions of Sturm type

$$\begin{cases} y^{(l_1)}(0) = y^{(l_2)}(0) = ... = y^{(l_n)}(0) = 0 \\ y^{(\widetilde{l}_1)}(\pi) = y^{(\widetilde{l}_2)}(\pi) = ... = y^{(\widetilde{l}_n)}(\pi) = 0 \end{cases} \tag{2}$$

Here $0 \le l_1 < l_2 < ... < l_n \le 2n-1,\quad 0 \le \widetilde{l}_1 < \widetilde{l}_2 < ... < \widetilde{l}_n \le 2n-1,\quad y \in H_1$, and the derivatives are understood in the strong sense. Everywhere by $Q(x)$ we'll denote $Q_{2n}(x)$.

Let D' be an aggregate of all the functions of the form $\sum_{k=1}^{p} \varphi_k(x) f_k$, where $\varphi_k(x)$ are finite $2n$-times continuously differentiable scalar functions, and $f_k \in$ $\in D\{Q\}$.

Determine the operator L' generated by the expression (1) and boundary conditions (2) with domain of definition D'. When specific conditions are fulfilled, the operator L' is a positive symmetric operator in H_1. We'll assume that the closure of the operator L is a self-adjoint and lower semi-bounded operator in H_1.

In the paper we study the Green function of the operator L. Note that the Green function of the Sturm-Liouville equation with self-adjoint operator coefficients was first studied by B.M. Levitan [1]. The Green function and asymptotic behavior of the eigen values of the operator L generated by the expression

$$l(y) = -(P(x)y)' + Q(x)y$$

in the self-adjoint case was studied by E. Abdukadyrov [2], E.G. Kleiman [3], M.G. Dushdurov [4], G.I. Kasumova [5] investigated the Green function of the Sturm-Liouville operator in the case when $Q(x)$ for each x is a normal operator in H.

In [6] M.Bairamoglu studied the Green function and asymptotic behavior of the eigen values of a higher order operator equation given on the all axis. The case of a semi-axis was considered in the papers of G.I. Aslanov [5], A.A. Abudov, G.I. Aslanov (8), G.I. Kasumova [9].

For the operators $Q_j(x)$, $\quad j = \overline{2, 2n}$ we'll assume the followings.

1) The operators $Q(x)$ for almost all $x \in [0, \pi]$ are self-adjoint in H, there exists the set $D\{Q(x)\}$ common for all x, on which the operators $Q(x)$ are defined and symmetric (in such a way we'll prove that the operators $Q(x)$ may be unbounded in H).

2) The operators $Q(x)$ are uniformly lower bounded, i.e. for all $f \in D$ the inequality

$$(Q(x)f,\ f) > c(f, f), \quad c > 0$$

is fulfilled.

3) For $|x - \xi| \le 1$

$$\left\| [Q(\xi) - Q(x)]\, Q^{-a}(x) \right\| < A\,|x - \xi|\,, \text{ where } 0 < a < \frac{2n+1}{2n}, \quad A > 0$$

$$\left\| Q^{-\frac{1}{2n}}(x), Q^{\frac{1}{2n}}(\xi) \right\| < C_1, \qquad \left\| Q^{\frac{1}{2n}}(x), Q^{-\frac{1}{2n}}(\xi) \right\| < C_2,$$

C_1 and C_2 are positive constants.

4) For $|x - \xi| > 1$

$$\left\| Q(\xi) \exp\left[-\frac{\operatorname{Im}\omega_1}{2}\, |x - \xi|\, Q^{\frac{1}{2n}}(x) \right] \right\| < B,$$

where $\operatorname{Im}\omega_1 = \min\left\{ \operatorname{Im}\omega_i > 0,\ \operatorname{Im}\omega_i^{2n} = -1 \right\}, \quad B = const > 0$

5) $\left\| Q_j(x) Q^{\frac{1-j}{2n} + \varepsilon}(x) \right\| < C, \quad j = 1, 2, ..., 2n-1, \quad \varepsilon > 0$

Some other restrictions on $Q(x)$ we'll be shown later if it is necessary.

The following theorem is the main result of this paper.

Theorem. *If the conditions 1)-5) are fulfilled, then for sufficiently large $\mu > 0$ there exists an inverse operator $R_\mu = (L + \mu E)^{-1}$ being an integral operator with the operator kernel $G(x, \eta; \mu)$ that will be called the Green (operator) function of the operator L. $G(x, \eta; \mu)$ is an operator function in H that depends on two variables, x, η $(0 \le x,\ \eta \le \pi)$, the parameter μ, and satisfies the conditions:*

a) $\dfrac{\partial^k G(x, \eta; \mu)}{\partial \eta^k}$ $\quad k = \overline{0, 2n-2}$ *is strongly continuous in variables (x, η);*

b) There exists a strong derivative $\dfrac{\partial^{2n-1} G(x, \eta; \mu)}{\partial \eta^{2n-1}}$, *moreover*

$$\frac{\partial^{2n-1} G(x, x+0;\ \mu)}{\partial \eta^{2n-1}} - \frac{\partial^{2n-1} G(x, x-0;\ \mu)}{\partial \eta^{2n-1}} = (-1)^n E;$$

c) $(-1)^n \dfrac{\partial^{2n}}{\partial \eta^{2n}} + \sum\limits_{j=2}^{2n} G_\eta^{(2n-j)}(x,\eta;\mu)Q_j(\eta) + \mu G(x,\eta;\mu) = 0$

$$\frac{\partial^{l_1} G}{\partial \eta^{l_1}}\Big|_{x=0} = \frac{\partial^{l_2} G}{\partial \eta^{l_2}}\Big|_{x=0} = ... = \frac{\partial^{l_n} G}{\partial \eta^{l_n}}\Big|_{x=0} = 0$$

$$\frac{\partial^{\tilde{l}_1} G}{\partial \eta^{\tilde{l}_1}}\Big|_{x=\pi} = \frac{\partial^{\tilde{l}_2} G}{\partial \eta^{\tilde{l}_2}}\Big|_{x=\pi} = ... = \frac{\partial^{\tilde{l}_n} G}{\partial \eta^{\tilde{l}_n}}\Big|_{x=\pi} = 0$$

d) $G^*(x,\eta;\mu) = G(\eta;x;\mu)$;

e) $\int\limits_0^\pi \|G(\eta;x;\mu)\|_H^2 \, d\eta < \infty$.

At first construct the Green function of the operator L_0, generated by the expression

$$l_0(y) = (-1)^n y^{(2n)} + Q(x)y + \mu y \tag{3}$$

and boundary conditions (2).

As is known [1], the Green function $C_0(x,\eta;\mu)$ of the operator L_0 satisfies the following integral equation

$$G_0(x,\eta;\mu) = G_1(x,\eta;\mu) - \int\limits_0^\pi G_1(x,\xi;\mu)\times$$

$$\times \left[Q(\xi) - Q(x)\right] G_0(\xi,x;\mu)d\xi, \tag{4}$$

where $G_1(x,\eta;\mu)$ is the Green function of the following problem:

$$(-1)^n y^{(2n)} + Q(\xi)y + \mu y = \delta(x-\xi) \tag{5}$$

$$\begin{cases} y^{(l_1)}(0) = y^{(l_2)}(0) = ... = y^{(l_1)}(0) = 0 \\ y^{(\tilde{l}_1)}(\pi) = y^{(\tilde{l}_2)}(\pi) = ... = y^{(\tilde{l}_1)}(\pi) = 0 \end{cases} \tag{6}$$

Here "ξ" is a fixed point from the segment $[0,\pi]$. The Green function $G_1(x,\eta,\xi,\mu)$ of problem (5)-(6) is represented in the form:

$$G_1(x,\eta,\xi,\mu) = g(x,\eta,\xi,\mu) + V(x,\eta,\xi,\mu), \tag{7}$$

where $g(x,\eta,\xi,\mu)$ is the Green function of the equation (5) on all the axis. It has the form:

$$g(x,\eta,\xi,\mu) = \frac{K_\xi^{1-2m}}{2ni} \sum_{\alpha=1}^{n} \omega_\alpha e^{i\omega_\alpha K_\xi |x-\eta|}, \tag{8}$$

where $K_\xi = \left[Q(\xi) + \mu E\right]^{\frac{1}{2\pi}}$.

Here ω_α are the roots from the (-1) degree of $2n$ lying in the upper half-plane. The function $V(x,\eta,\xi,\mu)$ is a solution of the homogeneous equation

$$(-1)^n V^{(2n)} + Q(\xi)V + \mu V = 0 \tag{9}$$

satisfying the boundary conditions

$$\begin{cases} V^{(l_j)}(x,\eta,\xi,\mu)\,|_{x=0} = -g^{(l_j)}(x,\eta,\xi,\mu)\,|_{x=0} \\ V^{(\widetilde{l}_j)}(x,\eta,\xi,\mu)\,|_{x=0} = -g^{(\widetilde{l}_j)}(x,\eta,\xi,\mu)\,|_{x=\pi} \end{cases} \tag{10}$$

For the general solution of (9) we get

$$V(x,\eta,\xi,\mu) = \frac{K_\xi^{1-2n}}{2ni}\sum_{k=1}^{2n} A_k(\eta,\xi,\mu)e^{i\omega_k K_\xi x}, \tag{11}$$

The coefficients $A_k(\eta,\xi,\mu)$ are determined from boundary conditions (10). As a result, for $A_k(\eta,\xi,\mu)$ we get the following system of equations:

$$\begin{cases} \sum\limits_{k=1}^{2n} A_k\omega_k^{l_j} = -\sum\limits_{\alpha=1}^{n}\omega_\alpha^{l_j+1}e^{i\omega_\alpha K_\xi\eta}, \quad j=1,2,...,n \\ \sum\limits_{k=1}^{2n} A_k\omega_k^{\widetilde{l}_j}e^{i\omega_k K_\xi\pi} = -\sum\limits_{\alpha=1}^{n}\omega_\alpha^{\widetilde{l}_j+1}e^{i\omega_\alpha K_\xi(\pi-\eta)}, \qquad j=1,2,...,n \end{cases} \tag{12}$$

Denote by $\triangle_0$, $\triangle_k$ the determinants of this system:

$$\triangle_0 = \begin{vmatrix} \omega_1^{l_1} & \omega_2^{l_1} & ... & \omega_{2n}^{l_1} \\ \omega_1^{l_2} & \omega_2^{l_2} & ... & \omega_{2n}^{l_2} \\ ... & ... & ... & ... \\ \omega_1^{l_n} & \omega_2^{l_n} & ... & \omega_{2n}^{l_n} \\ \omega_1^{\widetilde{l}_1}e^{i\omega_1K_\xi\pi} & \omega_2^{\widetilde{l}_2}e^{i\omega_2K_\xi\pi} & ... & \omega_{2n}^{\widetilde{l}_n}e^{i\omega_{2n}K_\xi\pi} \\ ... & ... & ... & ... \\ \omega_1^{\widetilde{l}_1}e^{i\omega_1K_\xi\pi} & \omega_2^{\widetilde{l}_n}e^{i\omega_2K_\xi\pi} & ... & \omega_{2n}^{\widetilde{l}_n}e^{i\omega_2K_\xi\pi} \end{vmatrix}$$

$$\triangle_k = \begin{vmatrix} \omega_1^{l_1} & ... & \omega_{k-1}^{l_1} - \sum\limits_{\alpha=1}^{n}\omega_\alpha^{l_j+1}e^{i\omega_\alpha K_\xi\pi} & \omega_{k+1}^{l_1} & ... & \omega_{2n}^{l_1} \\ \omega_1^{l_2} & ... & \omega_{k-1}^{l_2} - \sum\limits_{\alpha=1}^{n}\omega_\alpha^{l_2+1}e^{i\omega_\alpha K_\xi\pi} & \omega_{k+1}^{l_2} & ... & \omega_{2n}^{l_2} \\ ... & ... & ... & ... & ... & ... \\ \omega_1^{l_n} & ... & \omega_{k-1}^{l_n} - \sum\limits_{\alpha=1}^{n}\omega_\alpha^{l_n+1}e^{i\omega_\alpha K_\xi\pi} & \omega_{k+1}^{l_1} & ... & \omega_{2n}^{l_2} \\ \omega_1^{\widetilde{l}_1}e^{i\omega_1K_\xi\pi} & ... & \omega_{k-1}^{\widetilde{l}_1}e^{i\omega_{k-1}K_\xi\pi} - \sum\limits_{\alpha=1}^{n}\omega_\alpha^{\widetilde{l}_j+1}e^{i\omega_\alpha K_\xi(\pi-\eta)} & \omega_{k+1}^{\widetilde{l}_j}e^{i\omega_{k+1}K_\xi\pi} & ... & \omega_{2n}^{\widetilde{l}_1}e^{i\omega_{2n}K_\xi\pi} \\ ... & ... & ... & ... & ... & ... \\ \omega_1^{\widetilde{l}_n}e^{i\omega_1K_\xi\pi} & ... & \omega_{k-1}^{\widetilde{l}_n}e^{i\omega_{k-1}K_\xi\pi} - \sum\limits_{\alpha=1}^{n}\omega_\alpha^{\widetilde{l}_n+1}e^{i\omega_\alpha K_\xi(\pi-\eta)} & \omega_{k+1}^{\widetilde{l}_n}e^{i\omega_{k+1}K_\xi\pi} & ... & \omega_{2n}^{\widetilde{l}_n}e^{i\omega_{2n}K_\xi\pi} \end{vmatrix}$$

The solution of the system (12) is written in the form $A_k = \dfrac{\triangle_k}{\triangle_0}$. Denote by M_r some minor of the determinant $\triangle_0$ containing the first n-tuples and any n columns. Its cofactor $W_r = \widetilde{W}_r e^{iK_\xi\pi(\omega_{r_1}+...+\omega_{r_n})}$ contains the last $n-$tuples n columns with the remaining numbers $r_1, r_2, ..., r_n$. Write the expressions for M_r and $\widetilde{M}_r$:

$$M_r = \begin{vmatrix} \omega_{r_1}^{l_1} & \omega_{r_2}^{l_1} & ... & \omega_{r_n}^{l_1} \\ \omega_{r_1}^{l_2} & \omega_{r_2}^{l_2} & ... & \omega_{r_2}^{l_n} \\ ... & ... & ... & ... \\ \omega_{r_1}^{l_n} & \omega_{r_2}^{l_n} & ... & \omega_{r_n}^{l_n} \end{vmatrix},$$

$$\widetilde{M}_r = \begin{vmatrix} \omega_{r_{1+n}}^{\widetilde{l}_1} & \omega_{r_2+n}^{\widetilde{l}_1} & ... & \omega_{r_n+n}^{\widetilde{l}_1} \\ \omega_{r_1+n}^{\widetilde{l}_2} & \omega_{r_2+n}^{\widetilde{l}_2} & ... & \omega_{r_n+n}^{\widetilde{l}_2} \\ ... & ... & ... & ... \\ \omega_{r_n+n}^{\widetilde{l}_n} & \omega_{r_n+n}^{\widetilde{l}_n} & ... & \omega_{r_n+n}^{\widetilde{l}_n} \end{vmatrix}$$

Using the Laplace theorem, we can expand the determinant $\triangle_0$ in the following way:

$$\triangle_0 = \sum_r M_r \widetilde{M}_r e^{iK_\xi \pi (\omega r_1 + \omega r_2 + ... + \omega_{r_n})} \tag{13}$$

Introduce denotations for the right sides of the system (12):

$$b_j(\eta) = -\sum_{\alpha=1}^{n} \omega_\alpha^{l_j+1} e^{i\omega_\alpha K_\xi \eta},$$
$$b_{j+n}(\eta) = -\sum_{\alpha=1}^{n} \omega_\alpha^{\widetilde{l}_j+1} e^{i\omega_\alpha K_\xi (\pi-\eta)} \qquad j = 1, 2, ..., n \tag{14}$$

If we at first expand the determinant $\triangle_k$ in the elements of the $k-$th column and expand the obtained minors by the Laplace theorem, we get:

$$\triangle_k = \sum_{j=1}^{n} \left[b_j(\eta) \sum_r M_{jr} \widetilde{M}_r e^{iK_\xi \pi \sum' \omega' r_s} + \right.$$
$$\left. + b_{j+n}(\eta) \sum_r M_{jr} \widetilde{M}_r e^{iK_\xi \pi \sum'' \omega' r_s} \right]$$

The sum $\sum' \omega_{r_s}$ contains n addends except ω_k, the sum $\sum'' \omega_{r_s}$ contains $n-1$ addends and ω_k also is not contained in this sum.

So, the Green function of the equation with boundary conditions (6) is of the form:

$$G_1(x, \eta, \xi, \mu) = \frac{K_\xi^{1-2n}}{2ni} \left[\sum_{\alpha=1}^{n} \omega_\alpha^{i\omega_\alpha K_\xi |x-\eta|} + \sum_{k=1}^{2n} A_k e^{i\omega_k K_\xi x} \right] \tag{15}$$

Since for $k = \overline{1, 2n}$ and any $x \in [0, \pi]$

$$\operatorname{Re}\left[iK_\xi \pi \ \left(\sum' \omega_{r_s} + \sum \omega_s \right) + iK_\xi \omega_k x \right] \le 0,$$
$$\operatorname{Re}\left[iK_\xi \pi \ \left(\sum'' \omega_{r_s} + \sum \omega_s \right) + iK_\xi \omega_k x \right] \le 0,$$

and as $\mu \to \infty$ the following estimates hold

$$\|b_j(\eta)\|_H = \left\| \sum_{\alpha=1}^{n} \omega_\alpha^{l_j+1} e^{i\omega_\alpha K_\xi \eta} \right\|_H \le C_1$$

$$\|b_{j+n}(\eta)\|_H = \left\| \sum_\alpha \omega_\alpha^{l_j+1} e^{i\omega_\alpha K_\xi(\pi-\eta)} \right\|_H \leq C_2,$$

we get that as $\mu \to \infty$ for the Green function of the problem (5)-(6) it holds the asymptotic equality

$$G_1(x,\eta,\xi,\mu) = \frac{K_\xi^{1-2n}}{2ni} \sum_{\alpha=1}^{n} \omega_\alpha e^{i\omega_\alpha K_\xi(x-\eta)}(E + r(x,\eta,\xi,\mu)), \tag{16}$$

moreover, for $\mu \to \infty$ there is $\|r(x,\eta,\xi,\mu)\| = o(1)$ uniformly with respect to (x,η).

As it was noted above, the Green function $G_0(x,\eta,\mu)$ of the operator L_0 satisfies the integral equation (4). For investigating the solution of the integral equation (4), following the paper [1] we introduce the Banach spaces $X_1, X_2, X_3^{(p)}, X_2^{(s)}, X_4^{(s)}$ and X_5 $(p \geq 1,\ s \geq 0)$ whose elements are the operator functions $A(x,\eta)$ in the space H, and the norms are determined in the following way:

$$\|A(x,\eta)\|_{X_1}^2 = \int_0^\pi \left\{ \int_0^\pi \|A(x,\eta)\|_H^2 \, d\eta \right\} dx,$$

$$\|A(x,\eta)\|_{X_2}^2 = \int_0^\pi \left\{ \int_0^\pi \|A(x,\eta)\|_2^2 \, d\eta \right\} dx.$$

(Here $\|A(x,\eta)\|_2$ denotes the Hilbert-Schmilt norm (absolute norm) of the operator function $A(x,\eta)$ in H).

$$\|A(x,\eta)\|_{X_3^{(p)}} = \left[\sup_{0\leq x\leq\pi} \int_0^\pi \|A(x,\eta)\|_H^p \, d\eta \right]^{\frac{1}{p}},$$

$$\|A(x,\eta)\|_{X_2^{(s)}} = \int_0^\pi dx \left\{ \int_0^\pi \|A(x,\eta)Q^s(\eta)\|_2^2 \, d\eta \right\},$$

$$\|A(x,\eta)\|_{X_4^{(s)}} = \sup_{0\leq x\leq\pi} \int_0^\pi \|A(x,\eta)Q^s(\eta)\|_H \, d\eta,$$

$$\|A(x,\eta)\|_{X_5} = \sup_{0\leq x\leq\pi} \sup_{0\leq\eta\leq\pi} \|A(x,\eta)\|_H .$$

Determine the following integral operator:

$$NA(x,\eta) = \int_0^\pi G_1(x,\xi,\mu) \left[Q(\xi) - Q(x)\right] A(\xi,\eta) d\xi \tag{17}$$

The kernel $G_1(x,\xi,\mu)\{Q(\xi) - Q(x)\}$ is a bounded operator in H with respect to (x,ξ), $0 < x,\ \xi < \pi$ for $\mu > 0$. Indeed,

$$\|G(x,\xi,\mu)\| \, \|\{Q(\xi) - Q(x)\}\|_H =$$

$$=\frac{1}{2n}\left\|[Q(x)+\mu E]^{\frac{1-2n}{2n}}\sum_{\alpha=1}^{n}\omega_\alpha e^{i\omega_\alpha[Q(\xi)+\mu E]^{\frac{1}{2n}}|x-\eta|}\times\right.$$

$$\left.\times(E+r(x,\xi;\mu))+[Q(\xi)-Q(x)]\right\|_H\leq$$

$$\leq\frac{1+0(1)}{2n}\left\|[Q(\xi)+\mu E]^{\frac{1-2n}{2n}}e^{i\omega_\alpha[Q(\xi)+\mu E]^{\frac{1}{2n}}|x-\eta|}\left|Q(\xi)-Q(x)\right|\right\|\leq$$

$$\leq\frac{1+o(1)}{2n}\left\|[Q(\xi)+\mu E]^{\frac{1-2n}{2n}}[Q(\xi)-Q(x)]\right\|_H\leq C.$$

For $|x-\xi|>1$:

$$\|G(x,\xi,\mu)\{Q(\xi)-Q(x)\}\|_H\leq$$

$$\leq\frac{1+o(1)}{2}\left\|e^{-\operatorname{Im}\omega_1[Q(x)+\mu E]^{\frac{1}{2n}}|x-\xi|}[Q(\xi)-Q(x)]\right\|\leq$$

$$\leq C\left\|\exp\left\{-\frac{\operatorname{Im}\omega_1}{2}(Q(x)+\mu E)^{\frac{1}{2n}}|x-\xi|\right\}Q(\xi)\right\|+$$

$$+C\left\|Q(x)\exp\left\{-\frac{\operatorname{Im}\omega_1}{2}(Q(x)+\mu E)^{\frac{1}{2n}}|x-\xi|\right\}\right\|\leq C.$$

Therefore it makes sense to consider the operator N generated by the kernel $G_1(x,\xi,\mu)Q(\xi)-Q(x)$

It holds the following important lemma.

Lemma 1. *If the operator-valued function $Q(x)$ satisfies the conditions 1)-5), then for sufficiently large $\mu>0$ the operator N is contractive in the spaces $X_1,X_2,X_3^{(p)},X_2^{(s)},X_4^{(s)},X_5$.*

In all the considered Banach spaces, the equation (4) has a unique solution that may be obtained by means of the iterative method if the operator function $G_1(x,\eta,\mu)$ belongs to the appropriate space.

In the space H estimate the norm $G_1(x,\eta,\mu)$:

$$\|G_1(x,\eta,\mu)\|_H\leq\frac{1+o(1)}{2}\left\|[Q(x)+\mu E]^{\frac{1-2n}{2n}}\right\|_H\max_\alpha\left\|e^{i\omega_\alpha K_\xi|x-\eta|}\right\|=$$

$$=\frac{1+o(1)}{2}\left\|\int_1^\infty(\lambda+\mu)^{\frac{1-2n}{2n}}dE_\lambda(x)\right\|_H\max\left\|\int_1^\infty e^{i\omega_\alpha}(\lambda+\mu)^{\frac{1}{2n}|x-\eta|}dE_\lambda(x)\right\|_H\leq$$

$$\leq\frac{1+o(1)}{2}(1+\mu)^{\frac{1-2n}{2n}}\exp\left[-\operatorname{Im}\omega_1(1+\mu)^{\frac{1}{2n}}|x-\eta|\right]$$

ω_1 is the nearest point among $\omega_1,\omega_1,...,\omega_n$ to the real axis. Hence we have:

$$\int_1^\pi\|G_1(x,\eta,\mu)\|_H^2\,d\eta\leq\frac{(1+o(1))^2}{4}(1+\mu)^{\frac{1-2n}{n}}\int_1^\pi e^{-2\operatorname{Im}\omega_1(1+\mu)^{\frac{1}{2n}}|x-\eta|}d\eta\leq$$

$$\leq\frac{(1o(1))^2\left(e^{-2\operatorname{Im}\omega_1(1+\mu)^{\frac{1}{2n}}\pi}-1\right)}{8\operatorname{Im}\omega_1}(1+\mu)^{\frac{1-4n}{2n}}\qquad(18)$$

Additionally assume that the following condition (6) is also fulfilled along with conditions 1)-5).

6) Almost for all $x\in[0,\pi]$, $Q(x)$ is inverse to the completely continuous operator. Denote by $\beta_1(x),\beta_2(x),...,\beta_n(x)...$ its eigen values in the increasing order, i.e. $\beta_1(x)\le\beta_2(x)\le...\le\beta_n(x)\le...$ and assume that the series $\sum_{k=1}^{\infty}\beta_k^{\frac{1-4n}{2n}}(x)$ converges almost everywhere and its sum $F(x)\in L_1[0,\pi]$.

Using this condition, estimate the absolute norm $\|G_1(x,\eta,\mu)\|_2^2$ (Hilbert-Schmidt norm)

$$\|G_1(x,\eta,\mu)\|_2^2=\frac{(1+o(1))^2}{4n^2}\sum_{j=1}^{\infty}\left|\sum_{\alpha=1}^{n}\left(\beta_j(x)+\mu\right)^{\frac{1-2n}{2n}}\omega_\alpha e^{i\omega_\alpha(\beta_j(x)+\mu)^{\frac{1}{2n}}|x-\eta|^2}\right|\le$$

$$\le\frac{(1+o(1))^2}{4n^2}\sum_{j=1}^{\infty}\left\{(\beta_j(x)+\mu)^{\frac{2-4n}{2n}}\left|\sum_{\alpha=1}^{n}\omega_\alpha e^{i\omega_\alpha(\beta_j(x)+\mu)^{\frac{1}{2n}}|x-\eta|^2}\right|\right\}\le$$

$$\le\frac{(1+o(1))^2}{4n}\sum_{j=1}^{n}(\beta_j(x)+\mu)^{\frac{2-4n}{2n}}e^{-2\operatorname{Im}\omega_1(\beta_j(x)+\mu)^{\frac{1}{2n}}|x-\eta|}$$

Hence

$$\int_1^{\pi}\|G_1(x,\eta,\mu)\|_2^2\,d\eta\le\frac{(1+o(1))^2}{4n}\sum_{j=1}^{n}(\beta_j(x)+\mu)^{\frac{1-2n}{n}}\int_1^{\pi}e^{-2\omega_1(\beta_j(x)+\mu)^{\frac{1}{2n}}|x-\eta|}d\eta\le$$

$$\le\frac{(1+o(1))^2}{8n\operatorname{Im}\omega_1}\sum_{j=1}^{n}(\beta_j(x)+\mu)^{\frac{1-4n}{2n}}=\frac{(1+o(1))^2}{8n\operatorname{Im}\omega_1}F(x)$$

Integrating in the segment $[0,\pi]$ with respect to x, we get:

$$\int_0^{\pi}\left\{\int_0^{\pi}\|G_1(x,\eta,\mu)\|_2^2\,d\eta\right\}dx\le\frac{(1+o(1))^2}{8n\operatorname{Im}\omega_1}\int_0^{\pi}F(x)dx<\infty \qquad (19)$$

From estimates (18) and (19) we get that the function $G_1(x,\eta,\mu)$ belongs to the spaces $X_3^{(2)}$ and X_2 only if the operator function $Q(x)$ satisfies conditions 1)-6).

Therefore for sufficiently large $\mu>0$, the function $G_0(x,\eta,\mu)$ is also an element of the spaces $X_3^{(2)}$ and X_2.

Using the obvious form of the function $G_1(x,\eta,\mu)$, we easily prove the following lemma.

Lemma 2. *Let the operator function $Q(x)$ satisfy conditions 1), 2) and 6), and for $|x-\eta|\le1$ it hold the following estimation*

$$\left\|Q^{-\frac{1}{2n}}(x)Q^{\frac{1}{2n}}(\eta)\right\|<C,\qquad C=const$$

Then the function $G_1(x,\eta,\mu)$ belongs to the space $X_4^{\left(\frac{1}{2n}\right)}$.

Lemma 3. *Let $Q(x)$ satisfy conditions 1), 2) and 6). Besides, let $|x-\eta| \leq 1$*

$$\left\| Q^{\frac{1}{2n}}(x) Q^{-\frac{1}{2n}}(\eta) \right\| < C, \qquad C = const$$

Then the operator-valued function $\dfrac{\partial^{2n} G_1}{\partial \eta^{2n}}$ $(x \neq \eta)$ belongs to the space $X_4^{\left(-\frac{1}{2n}\right)}$.

It is proved that the solution of the integral equation (9) is the Green function of the operator L_0, i.e. satisfies all the main properties of the Green function.

The Green function $G_1(x,\eta,\mu)$ of the operator L generated by the expression (1) and boundary conditions (2) is sought in the form

$$G(x,\eta,\mu) = G_1(x,\eta,\mu) - \int_0^\pi G_1(x,\xi,\mu)\rho(\xi,\eta)\,d\xi \tag{20}$$

Using the main properties of the Green function $G_1(x,\eta,\mu)$, for determining $\rho(x,\eta)$ we get the following integral equation

$$\rho(x,\eta) + \sum_{j=1}^{2n} Q_j(x) \frac{\partial^{2n-j} G_1(x,\eta,\mu)}{\partial x^{2n-j}} - \sum_{j=2}^{2n} Q_j(x) \int_0^\pi \frac{\partial^{2n-j} G_1}{\partial x^{2n-j}} \rho(\xi,\eta) d\eta = 0 \tag{21}$$

If we denote

$$F(x,\eta,\mu) = -\sum_{j=1}^{2n} Q_j(x) \frac{\partial^{2n-j} G_1}{\partial x^{2n-j}},$$

equation (21) is written in the form:

$$\rho(x,\eta) = F(x,\eta,\mu) - \int_0^\pi F(x,\xi,\mu)\rho(\xi,\eta) d\xi \tag{22}$$

Using the asymptotic estimations (16), for the function $G_1(x,\eta,\xi,\mu)$ we can get the following estimation for the norm of the operator function $F(x,\eta,\xi)$

$$\|F(x,\eta,\xi)\|_H \leq C\mu^{-\varepsilon} e^{-\operatorname{Im}\omega_1 \sqrt[2n]{\mu}|x-\eta|}$$

Hence

$$\sup_{0\leq x\leq \pi} \int_0^\pi \|F(x,\eta,\mu)\|_H^2 \leq C\mu^{-2\varepsilon}$$

From this estimation it follows that the function $F(x,\eta,\mu)$ is an element of the space $X_3^{(2)}$ and as $\mu \to \infty$ it converges to zero with respect to the norm of the space $X_3^{(2)}$. Hence it follows that the solution of equation (22) as $\mu \to \infty$ asymptotically behaves as the function $F(x,\eta,\mu)$. As a result, from the integral relation (20) we get the following asymptotic equality

$$G(x,\eta,\mu) = G_1(x,\eta,\mu)\left[E + \alpha(x,\eta,\mu)\right] \tag{23}$$

where $\|\alpha(x,\eta,\mu)\|_H$ as $\mu \to \infty$.

From estimations (8), (16) and (23) we finally get:

$$G(x,\eta,\mu) = g(x,\eta,\mu)\left[E+\beta(x,\eta,\mu)\right],$$

where $\|\beta(x,\eta,\mu)\|_H = o(1)$ as $\mu \to \infty$.

Above we showed that for the function $g(x,\eta,\mu)$ the following estimation is fulfilled

$$\int_0^\pi \left\{ \int_0^\pi \|G(x,\eta,\mu)\|_2^2\, d\eta \right\} dx < \infty$$

Hence it follows that the integral operator with the kernel $G(x,\eta,\mu)$ is an operator of Hilbert-Schmidt type. Since the function $G(x,\eta,\mu)$ is a kernel of the operator $R_\lambda = (L+\mu E)^{-1}$, we get that the operator L has a discrete spectrum $\lambda_1, \lambda_2, ..., \lambda_n...$ with a unique limit point at the infinity.

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