Lala R. ALIYEVA

EQUIVALENT NORMS IN MEAN OSCILLATION SPACES

Abstract

In the paper, some properties of the functions from the space $BMO^k_{\varphi,\theta}$ are investigated in terms of Φ-oscillation. Equivalent conditions for belonging of the function to the space $BMO^k_{\varphi,\theta}$ in terms of Φ-oscillation and harmonic oscillation are found.

1. Let R^n be an n-dimensional Euclidean space of the points $x = (x_1, x_2, ..., x_n)$, $B(a,r) := \{x \in R^n : |x-a| \le r\}$ be a closed bar in R^n of radius $r > 0$ with the center at the point $a \in R^n$, N a set all natural numbers; $v = (v_1, v_2, ..., v_n)$, $x^v = x_1^{v_1} \cdot x_2^{v_2} \cdots x_n^{v_n}$, $|v| = v_1 + v_2 + ... + v_n$ where $v_1, v_2, ..., v_n$ are non-negative integrals. Denote by $L_{loc}(R^n)$ an aggregate of all locally summable in R^n functions.

Let $f \in L_{loc}(R^n)$, $k \in N \cup \{0\}$. Consider the polynomial (see [2], [4])

$$P_{k,B(a,r)} f(x) := \sum_{|v|\le k} \Big(\frac{1}{|B(a,r)|} \int\limits_{B(a,r)} f(t) \varphi_v \Big(\frac{t-a}{r}\Big) dt \Big) \varphi_v \Big(\frac{x-a}{r}\Big),$$

where $|B(a,r)|$ denotes the volume of the ball $B(a,r)$, and $\{\varphi_v\}$, $|v| \le k$ is an orthonormed system obtained from application of the orthogonalization process with respect to the scalar product

$$(f,g) := \frac{1}{|B(0,1)|} \int\limits_{B(0,1)} f(t)\, g(t)\, dt$$

to the system of power functions $\{x^v\}$, $|v| \le k$ arranged in partially lexicographic order (see [6]).

The modulus of the k -th order ($k \in N$) mean oscillation of the locally summable function f is defined by the equality

$$M_f^k(\delta) := \sup \{\Omega_k(f, B(x,r)) : 0 < r < \delta, \quad x \in R^n\} \quad (\delta > 0),$$

where $\Omega_k(f, B(x,r)) := \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \left| f(t) - P_{k-1,B(x,r)} f(t) \right| dt \quad (x \in R^n, r > 0)$.

By Φ denote the class of all positive monotonically increasing on $(0, +\infty)$ functions $\varphi(t)$ such that $\varphi(+0) = 0$. By definition, we'll consider the function $\varphi(t) \equiv 1$ an element of the class Φ. Denote by Φ_k the aggregate of all the functions $\varphi \in \Phi$ such that $\frac{\varphi(t)}{t^k}$ almost decreases.

Let $\varphi \in \Phi_k$, $k \in N$, $1 \le \theta \le \infty$. Denote by $BMO^k_{\varphi,\theta}$ an aggregate of all the functions $f \in L_{loc}(R^n)$ for which $\|f\|_{BMO^k_{\varphi,\theta}} + < \infty$, where

$$\|f\|_{BMO^k_{\varphi,\theta}} := \left(\int\limits_0^\infty \left(\frac{M_f^k(t)}{\varphi(t)} \right)^\theta \frac{dt}{t} \right)^{\frac{1}{\theta}}, \quad for \quad 1 \le \theta < \infty,$$

$$\|f\|_{BMO^k_{\varphi,\infty}} := \sup\left\{\frac{M^k_f(t)}{\varphi(t)} : t > 0\right\}.$$

Note that the spaces $BMO^k_{\varphi,\theta}$ were first introduced in [3]. These spaces are Banach for the norm indicated above.

2. Let $\alpha > 0,\ r > 0$, and

$$\Psi^{(\alpha)}(x) = c_n^{(\alpha)} \cdot \frac{1}{1+|x|^{n+\alpha}}, \quad \Psi_r^{(\alpha)}(x) = r^{-n}\Psi^{(\alpha)}\left(\frac{x}{r}\right),$$

where $c_n^{(\alpha)}$ is chosen so that the condition

$$\int_{R^n} \Psi^{(\alpha)}(x)\,dx = 1.$$

is fulfilled.

It is easy to see that for any $r > 0$ it holds the equality

$$\int_{R^n} \Psi_r^{(\alpha)}(x)\,dx = 1.$$

For the function $f \in L_{loc}(R^n)$ we introduce the following denotation

$$\Omega_{k,\alpha}(f, B(x;r)) := \int_{R^n} \Psi_r^{(\alpha)}(x-t)\left|f(t) - P_{k-1,B(x,r)}f(t)\right| dt \quad (x \in R^n, r > 0),$$

$$H_f^{k,\alpha}(\delta) := \sup\left\{\Omega_{k,\alpha}(f, B(x,r)) : 0 < r \le \delta, \quad x \in R^n\right\} \quad (\delta > 0).$$

Obviously, the function $H_f^{k,\alpha}(\delta)$ monotonically increases with respect to the argument $\delta \in (0, +\infty)$.

The following statements are proved in [7].

Proposition A. Let $f \in L_{loc}(R^n)$, $\alpha > 0$, $k \in N$, $k < \alpha+1$. Then the following inequality is true:

$$H_f^{k,\alpha}(\delta) \le c \cdot \delta^\alpha \int_\delta^\infty \frac{M_f^k(t)}{t^{\alpha+1}} dt, \quad \delta > 0, \tag{1}$$

where $c > 0$ is independent of f and δ.

Proposition B. Let $f \in L_{loc}(R^n)$, $\alpha > 0$, $k \in N$. Then the following inequality is true

$$M_f^k(\delta) \le c \cdot H_f^{k,\alpha}(\delta), \ (\delta > 0), \tag{2}$$

where $c > 0$ is independent of f and δ.

Let $P(x)$ be a Poisson kernel for R^n, i.e. $P(x) = c_n \cdot \frac{1}{\left(1+|x|^2\right)^{\frac{n+1}{2}}}$, where $c_n = \left(\frac{n+1}{2}\right) \cdot \pi^{-\frac{n+1}{2}}$. It is easy to verify that $P(x) \approx \Psi^{(1)}(x)$, $x \in R^n$. Note that for the non-negative functions $F(x)$ and $G(x)$ $(x \in X)$ the notation $F(x) \approx G(x)$ $(x \in X)$ means the following: there exist positive constants c_1 and c_2 such that for all $x \in X$ it holds the inequality

$$c_1 \cdot F(x) \le G(x) \le c_2 \cdot F(x).$$

For $f \in L_{loc}(R^n)$ we assume

$$H_f(\delta) := \sup_{\substack{0<r\leq\delta \\ x\in R^n}} \int_{R^n} P_r(x-t)\,|f(t) - P_r f(x)|\,dt, \quad \delta > 0,$$

where $P_r(x) := r^{-n} P\left(\frac{x}{r}\right) \quad (r>0)$, $P_r f(x) := (P_r * f)(x) = \int_{R^n} P_r(x-t) f(t)\,dt$.

$H_f(\delta)$ is called a harmonic oscillation modulus (see. [1]). In [7] it is proved that $H_f(\delta) \approx H_f^{1,1}(\delta)$ $(\delta > 0)$, where the constants with respect to "$\approx$" are independent of f and δ.

In the sequel, by (α) we'll denote the greatest integer that is less than the number α.

Let $f \in L_{loc}(R^n)$, $\alpha > 0$, $k \in N$, $\varphi \in \Phi_k$, and the following integral converge

$$\int_1^\infty \frac{\varphi(t)}{t^{\alpha+1}}\,dt.$$

We'll use the following denotation

$$A_{\varphi,\theta}^{k,\alpha}(f) := \left(\int_0^\infty \left(\frac{H_f^{k,\alpha}(t)}{\varphi(t)} \right)^\theta \frac{dt}{t} \right)^{\frac{1}{\theta}} \qquad for \quad 1 \leq \theta < \infty,$$

$$A_{\varphi,\infty}^{k,\alpha}(f) := \sup\left\{ \frac{H_f^{k,\alpha}(t)}{\varphi(t)} : t > 0 \right\}.$$

3. Theorem 1. *Let $f \in L_{loc}(R^n)$, $\alpha > 0$, $k \in N$, $\varphi \in \Phi_k$. Then, if $A_{\varphi,\infty}^{k,\alpha}(f) < +\infty$ then $f \in BMO_{\varphi,\theta}^k$, and the following inequality is true*

$$\|f\|_{BMO_{\varphi,\theta}^k} \leq c \cdot A_{\varphi,\theta}^{k,\alpha}(f),$$

where the constant $c > 0$ is independent of f.

Proof. Let at first $\theta = \infty$. If $A_{\varphi,\theta}^{k,\alpha}(f) < +\infty$, this means that

$$H_f^{k,\alpha}(\delta) \leq A_{\varphi,\infty}^{k,\alpha}(f) \cdot \varphi(\delta), \quad \delta > 0.$$

Hence, by inequality (2) we get

$$M_f^k(\delta) \leq c \cdot H_f^{k,\alpha}(\delta) \leq c \cdot A_{\varphi,\infty}^{k,\alpha}(f) \cdot \varphi(\delta), \quad \delta > 0.$$

The latter means that $f \in BMO_{\varphi,\infty}^k$, and furthermore

$$\|f\|_{BMO_{\varphi,\infty}^k} \leq c \cdot A_{\varphi,\infty}^{k,\alpha}(f), \quad \delta > 0, \tag{3}$$

where c is a constant from inequality (2).

If $1 \leq \theta < \infty$, again in the case $A_{\varphi,\theta}^{k,\alpha}(f) < +\infty$ we apply inequality (2) and get

$$\|f\|_{BMO_{\varphi,\theta}^k} = \left(\int_0^\infty \left(\frac{M_f^k(t)}{\varphi(t)} \right)^\theta \frac{dt}{t} \right)^{\frac{1}{\theta}} \leq c \left(\int_0^\infty \left(\frac{H_f^{k,\alpha}(t)}{\varphi(t)} \right)^\theta \frac{dt}{t} \right)^{\frac{1}{\theta}} = c A_{\varphi,\theta}^{k,\alpha}(f), \tag{4}$$

i.e. in this case we get the required statement. The theorem is proved.

Theorem 2. *Let $f \in L_{loc}(R^n)$, $\alpha > 0$, $k = (\alpha) + 1$, $\varphi \in \Phi_k$, and the following condition be fulfilled*

$$\delta^{\alpha} \int_{\delta}^{\infty} \frac{\varphi(t)}{t^{\alpha+1}} dt = O(\varphi(\delta)), \quad \delta > 0. \tag{5}$$

Then if $f \in BMO^k_{\varphi,\theta}$, the following relations are true:

a) $\int\limits_{R^n} \frac{|f(x)|}{1+|x|^{n+\alpha}} dx < +\infty$,

b) $A^{k,\alpha}_{\varphi,\theta}(f) < +\infty$.

The following inequality is true:

$$A^{k,\alpha}_{\varphi,\theta}(f) \le c \cdot \|f\|_{BMO^k_{\varphi,\theta}}, \tag{6}$$

where the constant $c > 0$ is independent of f.

Proof. Let $f \in BMO^k_{\varphi,\theta}$. At first consider the case $\theta = \infty$. Then we have

$$M^k_f(r) \le c \cdot \|f\|_{BMO^k_{\varphi,\infty}} \cdot \varphi(r), \quad r > 0. \tag{7}$$

In this case, the validity of the statement a) follows from theorem 1 of [5]. Further, from inequalities (1), (5) and (7) we have

$$H^{k,\alpha}_f(\delta) \le c \cdot \delta^{\alpha} \int_{\delta}^{\infty} \frac{M^k_f(t)}{t^{\alpha+1}} dt \le c \cdot \|f\|_{BMO^k_{\varphi,\infty}} \cdot \delta^{\alpha} \int_{\delta}^{\infty} \frac{\varphi(t)}{t^{\alpha+1}} dt \le$$

$$\le c_1 \cdot \|f\|_{BMO^k_{\varphi,\infty}} \cdot \varphi(\delta), \quad \delta > 0,$$

where $c_1 > 0$ is independent of f and δ. Hence we get

$$A^{k,\alpha}_{\varphi,\infty}(f) = \sup \left\{ \frac{H^{k,\alpha}_f(\delta)}{\varphi(\delta)} : \delta > 0 \right\} \le c_1 \cdot \|f\|_{BMO^k_{\varphi,\infty}},$$

i.e. the statement b) of the theorem and inequality (6) hold in the case $\theta = \infty$.

Let now $1 \le \theta < \infty$ and $f \in BMO^k_{\varphi,\theta}$. Then for any $r \in (0, +\infty)$ we have

$$\left(\int_{r}^{\infty} \left(\frac{M^k_f(t)}{\varphi(t)} \right)^{\theta} \frac{dt}{t} \right)^{\frac{1}{\theta}} \ge \left(\int_{r}^{2r} \left(\frac{M^k_f(t)}{\varphi(t)} \right)^{\theta} \frac{dt}{t} \right)^{\frac{1}{\theta}} \ge \frac{M^k_f(r)}{\varphi(2r)} \cdot (\ln 2)^{1/\theta}. \tag{8}$$

Show that if condition (5) is fulfilled, the relation $\varphi(2r) \approx \varphi(r)$, $r > 0$ is true. Really, by the monotone increase of φ we have $\varphi(r) \le \varphi(2r)$, $r > 0$. On the other hand, by means of inequality (5) we get

$$c \cdot \varphi(r) \ge \delta^{\alpha} \int_{\delta}^{\infty} \frac{\varphi(t)}{t^{\alpha+1}} dt \ge \delta^{\alpha} \int_{2\delta}^{\infty} \frac{\varphi(t)}{t^{\alpha+1}} dt \ge$$

$$\geq \varphi(2r)\cdot\delta^{\alpha}\int\limits_{2\delta}^{\infty}t^{-1-\alpha}dt=\varphi(2r)\cdot\frac{1}{\alpha 2^{\alpha}},$$

i.e. $\varphi(2r)\leq c\cdot\alpha\cdot 2^{\alpha}\cdot\varphi(r)$, $r>0$. Thus, $\varphi(2r)\approx\varphi(r)$, $r>0$.

Further, from relation (8) we get

$$M_f^k(r)\leq(\ln 2)^{-\frac{1}{\theta}}\cdot c\cdot\varphi(r)\cdot\left(\int\limits_r^{\infty}\left(\frac{M_f^k(t)}{\varphi(t)}\right)^{\theta}\frac{dt}{t}\right)^{\frac{1}{\theta}}\leq$$

$$\leq c_1\cdot\varphi(r)\cdot\|f\|_{BMO^k_{\varphi,\theta}},r\in(0,+\infty).$$

Hence, in particular we have

$$\int\limits_1^{\infty}\frac{M_f^k(t)}{\varphi(t)}dt\leq c_1\cdot\|f\|_{BMO^k_{\varphi,\theta}}\int\limits_r^{\infty}\frac{\varphi(t)}{t^{\alpha+1}}dt<+\infty. \tag{9}$$

Therefore in this case also, by applying theorem 1 from [5], we get the validity of the statement a).

Introduce the denotation

$$G_f^{k,\alpha}(r):=r^{\alpha}\int\limits_r^{\infty}\frac{M_f^k(t)}{t^{\alpha+1}}dt$$

and prove that

$$\left(\int\limits_0^{\infty}\left(\frac{C_f^{k,\alpha}(t)}{\varphi(t)}\right)^{\theta}\frac{dt}{t}\right)^{\frac{1}{\theta}}\leq c\cdot\|f\|_{BMO^k_{\varphi,\theta}}, \tag{10}$$

where the constant $c>0$ is independent of f.

Let $g\in L^{\theta_1}(0,+\infty)$, $g(r)\geq 0$ $(r>0)$, $\frac{1}{\theta_1}+\frac{1}{\theta}=1$. Then changing the integration order, we get

$$\int\limits_0^{\infty}\frac{C_f^{k,\alpha}(t)}{t^{1/\theta}\varphi(t)}\cdot g(t)\,dt=\int\limits_0^{\infty}\left(\frac{1}{t^{1/\theta}\varphi(t)}\cdot t^{\alpha}\int\limits_t^{\infty}\frac{M_f^k(y)}{y^{\alpha+1}}dy\right)g(t)\,dt=$$

$$=\int\limits_0^{\infty}\frac{M_f^k(y)}{y^{\alpha+1}}\left(\int\limits_0^{y}\frac{t^{\alpha}g(t)}{t^{1/\theta}\cdot\varphi(t)}\right)dy. \tag{11}$$

It is known that if condition (5) is fulfilled, there exists a number $v\in(0,\alpha)$ such that $\frac{\varphi(t)}{t^v}$-almost decreases. Let $\beta=v-\alpha+\frac{1}{\theta}$. Then by means of (11) we get

$$\int\limits_0^{\infty}\frac{C_f^{k,\alpha}(t)}{t^{1/\theta}\varphi(t)}\cdot g(t)\,dt=\int\limits_0^{\infty}\frac{M_f^k(y)}{y^{\alpha+1}}\left(\int\limits_0^{y}\frac{g(t)}{\left(\frac{\varphi(t)}{t^v}\right)\cdot t^{\beta}}dt\right)dy\leq$$

$$\leq c\cdot\int\limits_0^\infty \frac{M_f^k(y)}{y^{\alpha+1}}\left(\frac{y^v}{\varphi(y)}\int\limits_0^y g(t)\,t^{-\beta}dt\right)dy=$$

$$=c\cdot\int\limits_0^\infty \frac{M_f^k(y)}{\varphi(y)}\left(y^{v-\alpha-1}\int\limits_0^y g(t)\cdot t^{-\beta}dt\right)dy=$$

$$=c\int\limits_0^\infty \frac{M_f^k(y)}{y^{1/\theta}\varphi(y)}\left(y^{\beta-1}\int\limits_0^y g(t)\cdot t^{-\beta}dt\right)dy, \tag{12}$$

where $c>0$ is a constant depending on φ and v only.

Further, we get $\beta=v+\frac{1}{\theta}-\alpha<\alpha+\frac{1}{\theta}-\alpha=\frac{1}{\theta}\leq 1$ i.e. $\beta<1$.

Considering this, for $\theta_1=\infty$ (i.e. $\theta=1$) from inequality (12) we get

$$\int\limits_0^\infty \frac{G_f^k(t)}{t^{1/\theta}\varphi(t)}\cdot g(t)\,dt\leq c\cdot\|g\|_{L^{\theta_1}(0,+\infty)}\cdot\int\limits_0^\infty \frac{M_f^k(y)}{y\cdot\varphi(y)}\left(y^{\beta-1}\int\limits_0^y t^{-\beta}dt\right)dy=$$

$$=c\cdot\frac{1}{1-\beta}\cdot\|g\|_{L^{\theta_1}(0,+\infty)}\cdot\|f\|_{BMO^k_{\varphi,\theta}}. \tag{13}$$

Consider now the case $1<\theta_1<\infty$. Then applying the Holder inequality, from (12) we get

$$\int\limits_0^\infty \frac{G_f^k(t)}{t^{1/\theta}\varphi(t)}\cdot g(t)\,dt\leq c\cdot\left(\int\limits_0^\infty\left(\frac{M_f^k(y)}{\varphi(y)}\right)^\theta\frac{dy}{y}\right)^{1/\theta}\times$$

$$\times\left(\int\limits_0^\infty\left(y^{\beta-1}\int\limits_0^y g(t)\cdot t^{-\beta}dt\right)^{\theta_1}dy\right)^{1/\theta_1}. \tag{14}$$

Introduce the denotation $r=(1-\beta)\,\theta_1-1$. Then we have $r=\left(1-v-\frac{1}{\theta}+\alpha\right)\theta_1-1>\left(1-v-\frac{1}{\theta}+v\right)\theta_1-1=\left(1-\frac{1}{\theta}\right)\theta_1-1=\frac{1}{\theta_1}\cdot\theta_1-1=1-1=0$, i.e. $r>0$. Applying the Hardy inequality (see [8])

$$\left(\int\limits_0^\infty\left(\int\limits_0^x |h(y)|\,dy\right)^{\theta_1}x^{-r-1}dx\right)^{1/\theta_1}\leq\frac{\theta_1}{r}\cdot\left(\int\limits_0^\infty (y\,|h(y)|)^{\theta_1}\,y^{-r-1}dy\right)^{1/\theta_1},$$

having taken $h(y)=g(y)\cdot y^{-\beta}$, from (14) we get

$$\int\limits_0^\infty \frac{G_f^{k,\alpha}(t)}{t^{1/\theta}\varphi(t)}\cdot g(t)\,dt\leq c\cdot\|f\|_{BMO^k_{\varphi,\theta}}\cdot\frac{\theta_1}{r}\times$$

$$\cdot\left(\int\limits_0^\infty\left(y\cdot g(y)\cdot y^{-\beta}\right)^{\theta_1}y^{(\beta-1)\theta_1}dy\right)^{1/\theta_1}=$$

$$= c \cdot \frac{\theta_1}{(1-\beta)\,\theta_1 - 1} \|f\|_{BMO^k_{\varphi,\theta}} \left(\int_0^\infty (g(y))^{\theta_1} dy \right)^{1/\theta_1} =$$

$$= c \cdot \frac{\theta_1}{(1-\beta)\,\theta_1 - 1} \|f\|_{BMO^k_{\varphi,\theta}} \|g\|_{L^{\theta_1}(0,+\infty)} . \qquad (15)$$

Inequalities (13) and (15) show that for $1 \le \theta < \infty$ the following inequality is true

$$\left(\int_0^\infty \left(\frac{G^{k,\alpha}_f(t)}{\varphi(t)} \right)^\theta \frac{dt}{t} \right)^{1/\theta} \le c \cdot \|f\|_{BMO^k_{\varphi,\theta}} ,$$

where $c > 0$ is independent of f.

Hence, by means of inequality (1) we get

$$A^{k,\alpha}_{\varphi,\theta}(f) = \left(\int_0^\infty \left(\frac{H^{k,\alpha}_f(t)}{\varphi(t)} \right)^\theta \frac{dt}{t} \right)^{1/\theta} \le c \cdot \left(\int_0^\infty \left(\frac{G^{k,\alpha}_f(t)}{\varphi(t)} \right)^\theta \frac{dt}{t} \right)^{1/\theta} \le c_1 \cdot \|f\|_{BMO^k_{\varphi,\theta}} ,$$

i.e. the statement b) of the theorem and inequality (6)are valid in the case $1 \le \theta < \infty$.

The theorem is proved.

Theorems 1 and 2 yield

Theorem 3. *Let $f \in L_{loc}(R^n)$, $\alpha > 0$, $k = (\alpha) + 1$, $\varphi \in \Phi_k$ and condition (5) be fulfilled.*

Then the following conditions 1) and 2) on f are equivalent:

1) $f \in BMO^k_{\varphi,\theta}$;

2) a) $\int_{R^n} \frac{|f(x)|}{1+|x|^{n+\alpha}} dx < +\infty$;

b) $A^{k,\alpha}_{\varphi,\theta}(f) < +\infty$.

Moreover, $\|f\|_{BMO^k_{\varphi,\theta}} \approx A^{k,\alpha}_{\varphi,\theta}(f)$, where the constants in the relation "$\approx$" are independent of f.

Let $BMO_{\varphi,\theta} := BMO^1_{\varphi,\theta}$, $BMO_\varphi := BMO_{\varphi,\infty}$. From the previous theorem we get the following statement in terms of modulus of harmonic oscillation $H_f(\delta)$.

Corollary 1. *Let $f \in L_{loc}(R^n)$, $\varphi \in \Phi_1$, and the following condition be fulfilled:*

$$\delta \cdot \int_\delta^\infty \frac{\varphi(t)}{t^2} dt = O(\varphi(\delta)), \quad \delta > 0. \qquad (16)$$

Then the following conditions 1) and 2) are equivalent:

1) $f \in BMO_{\varphi,\theta}$;

2) a) $\int_{R^n} \frac{|f(x)|}{1+|x|^{n+1}} dx < +\infty$;

b) $A_{\varphi,\theta}(f) < +\infty$, where

$$A_{\varphi,\theta}(f) := \left(\int_0^\infty \left(\frac{H_f(t)}{\varphi(t)} \right)^\theta \frac{dt}{t} \right)^{1/\theta} \qquad for \quad 1 \le \theta < \infty,$$

$$A_{\varphi,\infty}(f) := \sup\left\{\frac{H_f(t)}{\varphi(t)} : t > 0\right\}.$$

Moreover $\|f\|_{BMO_{\varphi,\theta}} \approx A_{\varphi,\theta}(f)$, *where the constants in the relation "≈" don't depend on* f.

Corollary 2. *Let* $f \in L_{loc}(R^n)$, $\varphi \in \Phi_1$, *and condition (16) be fulfilled. Then the following conditions 1) and 2) are equivalent:*

1) $f \in BMO_\varphi$;

2) a) $\int\limits_{R^n} \frac{|f(x)|}{1+|x|^{n+1}} dx < +\infty$,

b) $A := \sup\limits_{\substack{r>0 \\ x\in R^n}} \frac{1}{\varphi(r)} \int\limits_{R^n} P_r(x-t)\,|f(t) - P_r f(x)|\,dt < +\infty$.

Moreover,

$$c_1 \cdot \|f\|_{BMO_\varphi} \le A \le c_2 \cdot \|f\|_{BMO_\varphi},$$

where c_1 *and* c_2 *are some positive constants independent of* f.

References

[1]. Blasco O., Perez M.A. *On functions of integrable mean oscillation.* Rev. Mat. Complut., 2005, v. 18, No 2, pp. 465-477.

[2]. DeVore R., Sharpley R. *Maximal functions measuring smoothness. Memoir.* Amer. Math. Soc., 1984, v. 47, No 293, pp. 1-115.

[3]. Rzaev R.M. *On boundedness of multidimensional singular integral operator in spaces* $BMO^k_{\varphi,\theta}$ *and* $H^k_{\varphi,\theta}$. Proc. Azerb., Math. Soc., 1996, v. 2, pp. 164-175 (Russian).

[4]. Rzaev R.M. *A multidimensional singular integral operator in spaces defined by conditions on the k-th order mean oscillation.* Dokl. Akad. Nauk (Russia), 1997, v. 356, No 5, pp. 602-605 (Russian).

[5]. Rzaev R.M. *Some growth conditions for locally summable functions. Abstracts of International conference on mathematics and mechanics.* Baku, 2006, 147p.

[6]. Rzaev R.M., Aliyeva L.R. *On local properties of functions and singular integrals in terms of the mean oscillation.* Cent. Eur. J. Math., 2008, v. 6, No 4, pp. 595-609.

[7]. Rzaev R.M., Aliyeva L.R. *Mean oscillation, Φ- oscillation and harmonic oscillation.* Trans. NAS Azerb., 2010, v. 30, No 1, pp. 167-176.

[8]. Stein E.M. *Singular integrals and differentiability properties of functions.* Princeton, New Jersey, 1970.

Lala R. Aliyeva
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F. Agayev str., AZ 1141, Baku, Azerbaijan.
Tel.: (99412) 539 47 20 (off.).

Received January 14, 2011; Revised March 18, 2011