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SEMIGROUPS OF LOCAL HOMEOMORPHIC MAPPINGS

Abstract

In this paper semigroups of local homeomorphic mappings of topological spaces are studied.

1.1. Let x be a local compact Hausdorf space under the condition that for every two points $\xi, \eta \in X$ and for every neighbourhood V_ξ of ξ there exists an open homeomorphic mapping $a : X \overset{into}{\to} X$ under the condition that $aX \subseteq V_\xi$, $a\eta = \xi$. We'll denote the class of all such spaces $\widetilde{L}$. In particular, the finite dimensional Euclidean spaces and the cube $D^\tau, \tau \geq \aleph_0$ [2] belong to the class $\widetilde{L}$. It is clear that if Ω is an open subset of the space $X \in \widetilde{L}$, then $\Omega \in \widetilde{L}$. We'll denote $OH(X), X \in \widetilde{L}$ the semigroup of all open homeomorphic mappings $X \overset{into}{\to} X$ and $LH(X), X \in \widetilde{L}$ the semigroup of all local homeomorphic mappings $X \overset{into}{\to} X$. Let $X \in \widetilde{L}$ and $\{K_i\}_{i \in I}$ be a system of compacts of X under the condition that $\bigcup\limits_{i \in I} IntK_i = X$. We'll denote $OH_{\{K_i\}}(X)$ the set of all such open homeomorphic mappings $a : X \overset{into}{\to} X$ that $aX \subseteq K_{i_a}$, $i_a \in I$. Let D_X be an arbitrary subsemigroup of the semigroup $LH(X)$ such that $OH_{\{K_i\}}(X) \subseteq D_X \subseteq LH(X)$ and D_X^0 be the set of all such elements a of D_X that $\overline{aX}$ is a compact. Obviously, D_X^0 is an ideal [1] of D_X.

Theorem 1.2. *Let $X, Y \in \widetilde{L}$. If the semigroups D_X and D_Y are isomorphic, then X and Y are homeomorphic.*

Lemma 1.3. *Let x_0 be a solution of the equation $ax = b, a, b \in D_X$. If $x_0 \in D_X^0$ then $b \in D_X^0$. Besides $\overline{bX} \subseteq aX$.*

Lemma 1.4. *Let $\xi \in X$ and V_ξ be an arbitrary neighbourhood of the point ξ. There exists such an element $h \in D_X^0$ that $\xi \in hX, hX \subseteq V_\xi$ and $\varphi h \in D_Y^0$.*

Lemma 1.5. *Let be an arbitrary point of X and a be an element D_X^0 of under the condition $\xi \in aX, \varphi a \in D_Y^0$. We'll denote $\{a_\gamma\}_{\gamma \in \Gamma}$ the system of all elements of D_X^0 under the conditions:*

1) $\xi \in a_\gamma X, \gamma \in \Gamma$

2) for each $\gamma \in \Gamma$ there exists an element $c_\gamma \in D_X^0$ such that $ac_\gamma = a_\gamma$ and $\varphi(c_\gamma) \in D_Y^0$.

Lemma 1.6. *Let $\xi \in \Omega_X$ and V_ξ be an arbitrary neighbourhood of ξ. There exists $\gamma_0 \in \Gamma$ such that $a_{\gamma_0} X \subseteq V_\xi$.*

Let b an element of D_X^0 such that $\xi \in bX$, $\varphi b \in D_Y^0$ and $\{b_s\}_{s \in S}$ be a system of all elements of D_X^0 under the conditions 1), 2).

Lemma 1.7. *Let $\left\{a_{\gamma_m}\right\}_{m=1}^n$ be a finite subsystem of the system $\{a_\gamma\}_{\gamma \in \Gamma}$ and $\{b_{s_k}\}_{k=1}^l$ be a finite subsystem of the system arbitrary neighbourhood of ξ. There exists an element $c \in \{b_s\}_{s \in S}$, V_ξ be an D_X^0 under the conditions:*

a)$\xi \in cX \subseteq V_\xi$,

b) for each γ_m and s_k there exist $d_{\gamma_m}, l_{s_k} \in D^0_X$ such that

$$a_{\gamma_m} d_{\gamma_m} = c, b_{s_k} l_{s_k} = c, \varphi\left(d_{\gamma_m}\right), \varphi\left(l_{s_k}\right) \in D^0_\gamma,\ m = 1, 2, ..., n,\ k = 1, 2, ..., l.$$

It is clear that $c \in \{a_\gamma\}_{\gamma\in\Gamma}$ and $c \in \{b_s\}_{s\in S}$.

Lemma 1.8. *The set $\bigcap\limits_{\gamma\in\Gamma} \overline{a_\gamma X} = \bigcap\limits_{\gamma\in\Gamma} a_\gamma X$ consists of a unique point ξ.*

Lemma 1.9. *The system of sets $\left\{\overline{(\varphi a_\gamma) Y} : \gamma \in \Gamma\right\}$ is a centered system of closed sets of the compact $\overline{(\varphi a) Y}$ and $\bigcap\limits_{\gamma\in\Gamma} \overline{(\varphi a_\gamma) Y} = \bigcap\limits_{\gamma\in\Gamma} (\varphi a_\gamma) Y$.*

Lemma 1.10. *The set $\bigcap\limits_{\gamma\in\Gamma} \overline{(\varphi a_\gamma) Y} = \bigcap\limits_{\gamma\in\Gamma} (\varphi a_\gamma) Y$ consists of a unique point $\xi' \in Y$. The point ξ' doesn't depend on an element $b \in D^0_X$ such that $\xi \in bX$, $\varphi b \in D^0_Y$.*

We'll denote the mapping $f\xi = \xi'$. The analogous mapping for the isomorphism φ^{-1} we'll denote g.

Lemma 1.11. *The mappings f and g are bijective and $g = f^{-1}$.*

Proof. Let ξ' be an arbitrary point of Y and $a' \in D^0_Y$ such that $\xi' \in a'Y, \varphi^{-1}a' \in D^0_X$, $\{a'_s\}_{s\in S}$ be a system of all elements of D^0_Y under the conditions 1), 2) for the element a' and the point ξ'. By virtue of lemma 1.10 the set $\bigcap\limits_{s\in S} \left(\varphi^{-1}a'_\gamma\right) X$ consists of a unique point ξ and $g\xi' = \xi = X$. It follows from $\varphi^1 a'$. $\varphi^{-1}c'_s = \varphi^{-1}a'_s$, the condition 2, lemma 1.3 that $\overline{(\varphi^{-1}a'_s) X} \subseteq \left(\varphi^{-1}a'\right) X$ and $\xi \in \left(\varphi^{-1}a'\right) X$. Let $\{a_\gamma\}_{\gamma\in\Gamma}$ be the system of all elements of D^0_X under the conditions 1), 2) for the element $\varphi^{-1}a'$ and the point ξ. For each $\gamma \in \Gamma$ the equations $\left(\varphi^{-1}a'\right) x_\gamma = a_\gamma$ are solvable. It's clear that $\left\{\varphi^{-1}a'_s\right\}_{s\in S} \subseteq \{a_\gamma\}_{\gamma\in\Gamma}$ and $\bigcap\limits_{\gamma\in\Gamma} (\varphi a_\gamma) Y \subseteq \bigcap\limits_{s\in S} a'_\gamma Y = \xi'$.

Since $\bigcap\limits_{\gamma\in\Gamma} (\varphi a_\gamma) Y$ consists of a unique point, then $\bigcap\limits_{\gamma\in\Gamma} (\varphi a_\gamma) Y = \xi'$. Hence $f\xi = \xi'$. As $g\xi' = \xi$, then $gf\xi = \xi, \xi \in X$ and $fg\xi' = \xi', \xi \in Y$.

Lemma 1.12. *If $a \in D^0_X$, $\varphi a \in D^0_Y$, then*

$$faX = (\varphi aY).$$

Proof. Let $\{a_\gamma\}_{\gamma\in\Gamma}$ be a system consisting of all elements of D^0_X under the conditions 1), 2) for the element $a \in D^0_X$ and the point $\xi \in aX$, then $\bigcap\limits_{\gamma\in\Gamma} (\varphi a_\gamma) Y = f\xi$. From the condition 2 and lemma 1.3 it follows that $\varphi a \cdot \varphi c_\gamma = \varphi a_\gamma$, $f\xi \in \overline{(\varphi a_\gamma) Y} \subseteq (\varphi a) Y$. Thus $faX \subseteq \varphi aY$. Analogously if $\xi' \in (\varphi a) Y$, then $g\xi' = f^{-1}\xi' \in aX$. It follows $f^{-1} (\varphi a) Y \subseteq aX$ and so $(\varphi a) Y \subseteq faX$.

Lemma 1.13. *The mappings f and f^{-1} are continuous.*

Proof. Let V' be an arbitrary neighbourhood of the point $f\xi$. By virtue of lemma 1.4 there exists an element $a' \in D^0_Y$ such that $f\xi \in a'Y \subseteq V'$ and $\varphi^{-1}a' \in D^0_X$. From lemma 1.12 it follows that $f^{-1}(a'Y) = \left(\varphi^{-1}a'\right) X$ and $a'Y = f\left(\varphi^{-1}a'\right) X$. The set $\left(\varphi^{-1}a'\right) X$ is a neighbourhood of the point ξ and $f\left(\varphi^{-1}a'\right) X = a'Y \subseteq V'$. One can prove that the mapping f^{-1} is continuous in the same way.

Theorem 1.14. *Let $X, Y \in \widetilde{L}$. If the semigroups D_X and D_Y are isomorphic, then it holds:*

$$\varphi c = fcf^{-1}, c \in D_X,$$

where f is a homeomorphism X onto Y induced by the isomorphism φ of the semigroups D_X and D_Y.

Corollary 1.15. *Let $X \in \widetilde{L}$. For every automorphism ψ of the semigroup D_X it holds:*

$$\psi c = gcg^{-1}, c \in D_X,$$

where g is a homeomorphism X onto itself induced by the automorphism of the semigroup D_X.

Proof. Let ξ be an arbitrary point of X, a be an element of D_X^0 such that $\xi \in aX$, $\varphi a \in D_Y^0$ and $\{a_\gamma\}_{\gamma\in\Gamma}$ be a system of all elements of D_X^0 under the conditions 1), 2).

The following equalities take place:

$$\bigcap_{\gamma\in\Gamma} a_\gamma X = \xi, \quad \bigcap_{\gamma\in\Gamma} (\varphi a_\gamma) Y = f\xi,$$

$$\bigcap_{\gamma\in\Gamma} ca_\gamma X = c\xi, \quad \bigcap_{\gamma\in\Gamma} \varphi (ca_\gamma) Y = (\varphi c) f\xi,$$

$$ac_\gamma = a_\gamma, \gamma \in \Gamma, \varphi a \cdot \varphi c_\gamma = \varphi a_\gamma, \gamma \in \Gamma,$$

$$cac_\gamma = ca_\gamma, \gamma \in \Gamma, \varphi (ca) \cdot \varphi c_\gamma = \varphi (ca_\gamma), \gamma \in \Gamma,$$

$$c_\gamma \in D_X^0, \quad \varphi (c_\gamma) \in D_Y^0.$$

Since D_X^0 is an ideal of D_X and D_Y^0 is an ideal of D_Y, then $ca \in D_X^0$, $\varphi (ca) = \varphi c \cdot \varphi a \in D_Y^0$.

Let us denote $\{b_s\}_{s\in S}$ the system of all elements of D_X^0 under the conditions 1), 2) for the point $c\xi$ and the element ca. The system $\{ca_\gamma\}_{\gamma\in\Gamma}$ is a subsystem of the system $\{b_s\}_{s\in S}$. Since $\bigcap\limits_{\gamma\in\Gamma} \varphi (ca_\gamma) Y = (\varphi c) f\xi$, so $fc\xi = \bigcap\limits_{s\in S} (\varphi b_s) Y = \bigcap\limits_{\gamma\in\Gamma} \varphi (ca_\gamma) Y = (\varphi c) f\xi$, $\xi \in X$. By virtue $f^{-1}Y = X$ for every point $\xi \in X$ there exists a unique point $\xi' \in Y$, such that $f^{-1}\xi' = \xi$. We obtain the equality

$$(\varphi c) \xi' = fcf^{-1}\xi', \quad \xi' \in Y.$$

Corollary 1.16. *Let $X, Y \in \widetilde{L}$. If semigroups $LH(X)$ and $LH(Y)$ are isomorphic, then every isomorphism of the semigroups $LH(X)$ and $LH(Y)$ maps $OH(X)$ onto $OH(Y)$.*

Corollary 1.17. *Let $X \in \widetilde{L}$. The semigroups $LH(X)$ and $OH(X)$ are not isomorphic.*

References

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