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ASYMPTOTICS OF SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A SINGULARLY PERTURBED QUASILINEAR ONE-CHARACTERISTIC EQUATION

Abstract

On a rectangular domain, a boundary value problem is considered for a non-classic type third order quasilinear equation containing a small parameter at higher derivatives. The complete asymptotic expansion in small parameter of the solution of the problem under consideration with boundary layer functions near two sides of the rectangle is constructed, and the residual term is estimated.

By studying some real phenomena, where there are nonuniform transitions from one physical characteristics to another ones, it is necessary to investigate singularly perturbed boundary value problems. Such problems have drawn attention of many mathematicians. Nonclassical singularly perturbed equations have not been studied enough. In the papers [1], [2], the asymptotics of the solution of boundary values problems for linear equations of nonclassic type was constructed in a restangular domain with four viscous boundarued.

In the present paper, in the rectangle $D = \{(x, y)\,|0 \le x \le 1, \;\; 0 \le y \le 1\}$ we consider the following boundary value problem

$$L_\varepsilon U \equiv \varepsilon^2 \frac{\partial}{\partial x}(\triangle U) - \varepsilon^p \left(\frac{\partial U}{\partial y}\right)^p - \varepsilon \triangle U + \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + F(x, y, U) = 0, \qquad (1)$$

$$U\,|_{x=0} = 0, \quad U\,|_{x=1} = 0, \quad \frac{\partial U}{\partial x}\,|_{x=1} = 0, \quad (0 \le y \le 1), \qquad (2)$$

$$U\,|_{y=0} = 0, \quad U\,|_{y=1} = 0, \quad (0 \le x \le 1), \qquad (3)$$

where ε is a small parameter, $\triangle \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $p = 2k+1$, k is an arbitrary natural number, $F(t, x, U)$ is a given smooth function satisfying the condition

$$\frac{\partial F(x, yU)}{\partial U} \ge \gamma^2 > 0, \quad (x, y, U) \in (D \backslash \{(x, y) \in D\,|x = y\}) \times (-\infty, +\infty). \qquad (4)$$

Therewith, $F(t, x, U)$ may depend on U both linearly, i.e.

$$F(t, x, U) = a(x, y)U - f(x, y), \;\; a(x, y) \ge \gamma^2 > 0, \qquad (5)$$

and nonlinearly.

Obviously, if $F(x, y, 0) \equiv 0$, problem (1)-(3) has only a trivial solution. Therefore we suppose

$$F(x, y, 0) \ne 0 \quad \text{as} \quad (x, y) \in D. \qquad (6)$$

Our aim is to construct the asymptotic expansion of the solution of boundary value problem (1)-(2) in small parameter $\varepsilon > 0$.

Before to construct the asymptotics of the solution of problem (1)-(3), formulate some statements that we'll need later on

Lemma 1. *Let $F(t,x,U)\in C^m\left(D\times(-\infty,\infty)\right)$satisfy conditions $(4),(6)$ the condition*

$$\frac{\partial^i f(x,y)}{\partial x^{i_1}\partial y^{i_2}}\left|_{x=y}\right.=0;\quad i=i_1+i_2;\quad i=0,1,...,m, \tag{7}$$

in the case of linear dependence of F on U, and the condition

$$F(t,x,U)\left|_{x=y}\right.=0,\ \frac{\partial^i F(x,y,0)}{\partial x^{i_1}\partial y^{i_2}\partial U^{i_3}}\left|_{x=y}\right.=0;\quad i=i_1+i_2+i_3;\quad i=0,1,...,m, \tag{8}$$

in the case of nonlinear dependence of F on U, where m is a natural number. Then the problem

$$\frac{\partial W}{\partial x}+\frac{\partial W}{\partial y}+F(x,y,W)=0,\quad W\left|_{x=0}\right.=0,\quad W\left|_{y=0}\right.=0 \tag{9}$$

has a unique solution, moreover $W(x,y)\in C^m(D)$ and

$$\frac{\partial^i W(x,y)}{\partial x^{i_1}\partial y^{i_2}}\left|_{x=y}\right.=0;\quad i=i_1+i_2;\quad i=0,1,...,m. \tag{10}$$

Lemma 2. *Let $\varphi(x)\in C^k[0,1]$. Then for each fixed value of $x\in[0,1]$, the problem*

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\eta}{\partial\xi}\right)^p+\frac{\partial^2\eta}{\partial\xi^2}+\frac{\partial\eta}{\partial\xi}=0, \tag{11}$$

$$\eta\left|_{\xi=0}\right.=\varphi(x),\qquad \lim_{\xi\to+\infty}\eta=0 \tag{12}$$

has a unique solution, moreover with respect to $\eta(x,\xi)$ is infinitely ξ differentiable, and with respect to x has continuous derivatives to the k-th order, inclusively. The following estimations of the form

$$\left|\frac{\partial^i\eta(x,\xi)}{\partial x^{i_1}\partial\xi^{i_2}}\right|\le Ce^{-\xi};\ i=i_1+i_2;\ i_1=0,1,...,k \tag{13}$$

are valid uniformly with respect to $x\in[0,1]$.

Lemma 3. *Let $\psi(x)\in C^k[0,1]$, $h(x,\xi)$ be a known function having continuous derivatives with respect to x up to the k-th order inclusively, be infinitely differentiable with respect to the variable $\xi\in[0,+\infty)$, and for each fixed value of $x\in[0,1]$ satisfy the estimation of the form*

$$|h(x,\xi)|\le C_1(a_0+a_1\xi+a_2\xi^2+...+a_{i-1}\xi^{i-1})e^{-\xi} \tag{14}$$

where $C_1>0$, $a_0>0$, $a_1\ge 0,...,a_{i-1}\ge 0$ are constants, i is any fixed natural number. If the function $\eta(x,\xi)$ is the solution of problem (11), (12), then for each fixed value of $x\in[0,1]$ problem

$$p\frac{\partial}{\partial\xi}\left[\left(\frac{\partial\eta}{\partial\xi}\right)^{p-1}\frac{\partial\mu}{\partial\xi}\right]+\frac{\partial^2\mu}{\partial\xi^2}+\frac{\partial\mu}{\partial\xi}=h(x,\xi), \tag{15}$$

$$\mu_{|_{\xi=0}}=\psi(x),\quad \lim_{\xi\to+\infty}\mu=0 \tag{16}$$

has a unique solution, moreover $\mu(x,\xi)$ with respect to x has continuous derivatives up to the k-th order inclusively, and be infinitely differentiable with respect to ξ. Then, the following estimates are true

$$\left|\frac{\partial^s \mu(x,\xi)}{\partial x^{s_1}\partial \xi^{s_2}}\right| \le C_2\left(b_0+b_1\xi+b_2\xi+b_2\xi^2+...+b_i\xi^i\right)e^{-\xi}, \tag{17}$$

where $C_2>0$, $b_0,b_1,b_2,...,b_i$ *are non-negative constants,* $s=s_1+s_2$; $s_1=0,1,...,k$; $s_2=0,1,...$.

The proof of lemma 1-3 are given in the paper [7] (see theorems 1-3).

Now, construct an asymptotes in small parameter of the solution of problem (1)-(3).

At first find the approximate solution of equation (1) in the form

$$W=W_0+\varepsilon W_1+...+\varepsilon'' W_n, \tag{18}$$

moreover the functions $W_i(t,x)$; $i=0,1,...,n$ will be chosen so that

$$L_\varepsilon W=0(\varepsilon^{n+1}). \tag{19}$$

Having substituted expression (18) for W in (19), and expanded

$$\frac{\partial}{\partial y}\left[\frac{\partial\left(\sum\limits_{i=0}^{n}\varepsilon^i W_i\right)}{\partial y}\right]^p,\quad F(x,y,\sum_{i=0}^{n}\varepsilon^i W_i)$$

in powers of ε, and comparing the term with the same degrees of ε for determining W_i; $i=0,1,...,n$ we get the following equations:

$$\frac{\partial W_0}{\partial x}+\frac{\partial W_0}{\partial y}+F(x,y,W_0)=0, \tag{20}$$

$$\frac{\partial W_i}{\partial x}+\frac{\partial W_i}{\partial y}+b(t,x)W_i=f_i(t,x), \tag{21}$$

where $b(t,x)=\dfrac{\partial F(x,y,W_0)}{\partial W_0}$, the functions $f_i(t,x)=H_i(W_0,W_1,...,W_{i-1})$ depend on $W_0,W_1,...,W_{i-1}$; $i=1,2,...,n$, and on their derivatives. We can obviously write the formula for the functions f_i, but they are of bulky form. The represent the obvious form for f_1 and f_2:

$$f_1=\triangle W_0,\ f_2=\triangle W_1-\frac{\partial}{\partial x}(\triangle W_0)-\frac{1}{2!}\frac{\partial^2 F(x,y,W_0)}{\partial W_0^2}W_1^2. \tag{22}$$

We'll solve equations (20), (21) under the following boundary conditions:

$$W_0\left|_{x=0}\right.=0,\qquad W_0\left|_{y=0}\right.=0, \tag{23}$$

$$W_i\left|_{x=0}\right.=0,\qquad W_i\left|_{y=0}\right.=0,\ i=1,2,...,n. \tag{24}$$

Suppose that $F(x,y,W_0)$ satisfies the conditions of lemma 1 for $m=2n+3$. Then by the same lemma, problem (20), (23) has a unique solution, moreover $W_0 \in C^{2n+3}(D)$ and the function $W_0(x,y)$ vanishes together with all the derivatives for $x = y$.

Problems (21), (24) where from the functions W_i; $i = 1,2,...,n$ are determined successively, are linear. The solutions of these problems are written in the obvious form by the formula

$$W_i(x,y) = \begin{cases} \int\limits_0^x f_i(\tau, y-x+\tau)\exp\left[-\int\limits_\tau^x b(\xi, y-x+\xi)d\xi\right]d\tau \quad \text{as} \quad y > x, \\ \int\limits_0^y f_i(x-y+\tau,\tau)\exp\left[-\int\limits_\tau^y b(x-y+\xi,\xi)d\xi\right]d\tau \quad \text{as} \quad y < x. \end{cases} \tag{25}$$

Using the fact that the right side of equation (21) vanish together with the derivatives for $x = y$, (see (22)), it is easy to show that the functions $W_i(x,y)$ determined from formula (25) will be smooth functions in D, moreover $W_i \in C^{2(m-i)+3}(D)$. Hence and from (18) it follows that $W \in C^3(D)$. Furthermore, the constructed functions $W_i(x,y)$ satisfy the condition

$$\frac{\partial^j W_i(x,y)}{\partial x^{j_1}\partial y^{j_2}}\big|_{x=y} = 0; \ j = j_1 + j_2; j = 0,1,...,2(n-i)+3; i = 0,1,..,n. \tag{26}$$

The constructed approximate solution of W satisfies equation (19) and the following boundary conditions (see (18), (23), (24)):

$$W\big|_{x=0} = 0, \ W\big|_{y=0} = 0. \tag{27}$$

But, generally speaking, the function doesn't satisfy boundary conditions from (2), (3) for $x = 1$ and $y = 1$. Therefore, we have to construct boundary layer functions near the boundaries $x = 1$ and $y = 1$.

In order to remove discrepancy for $x = 1$, to the function W add the function

$$V = V_0 + \varepsilon V_1 + ... + \varepsilon^{n+1}V_{n+1}, \tag{28}$$

of boundary layer near the boundary $x = 1$ so that the obtained sum $W + V$ coned satisfy the boundary conditions

$$(W+V)\big|_{x=1} = 0, \ \ \frac{\partial}{\partial x}(W+V)\big|_{x=1} = 0. \tag{29}$$

Furthermore, on choosing V, the equality

$$L_{\varepsilon,1}(W+V) - L_{\varepsilon,1}W = 0(\varepsilon^{n+1}), \tag{30}$$

should be fulfilled. Here $L_{\varepsilon,1}$ is a new decomposition of the operator L_ε near the boundary $x = 1$.

In order to write a new decomposition of the operator L_ε. We make change of variables: $1 - x = \varepsilon\tau$, $y = y$. Consider an auxiliary function $r = \sum\limits_{j=0}^{n+1}\varepsilon^j r_j(\tau,y)$, where $r_j(\tau,y)$ are some smooth functions determined near the boundary $x = 1$. The expansion of $L_\varepsilon(r)$ in powers of ε in the coordinates (τ,y) has the form

$$L_{\varepsilon,1}r \equiv -\varepsilon^{-1}\left\{\frac{\partial^3 r_0}{\partial\tau^3} + \frac{\partial^2 r_0}{\partial\tau^2} + \frac{\partial r_0}{\partial\tau} + \right.$$

$$+\sum_{j=1}^{n+1}\left[\frac{\partial^3 r_j}{\partial\tau^3}+\frac{\partial^2 r_j}{\partial\tau^2}+\frac{\partial r_j}{\partial\tau}+Q_j\left(\tau,y,r_0,r_1,...,r_{j-1}\right)\right]+0(\varepsilon^{n+2})\Big\}, \tag{31}$$

where Q_j are the known function dependent on $\tau, y, r_0, r_1, ..., r_{j-1}$ and their derivatives. Here we give formula only for Q_1 and Q_2:

$$Q_1=-\frac{\partial r_0}{\partial y}-F(1,y,r_0), \tag{32}$$

$$Q_2=-\frac{\partial r_1}{\partial y}-\frac{\partial F(1,y,r_0)}{\partial r_0}r_1-\frac{\partial^3 r_0}{\partial\tau\partial y^2}+\frac{\partial^2 r_0}{\partial y^2}+\frac{\partial F(1,y,r_0)}{\partial x}\tau. \tag{33}$$

By expanding each function $W_i(1-\varepsilon\tau,y)$ in Taylor's formula at the point $(1,y)$, we get a new expansion in powers of ε in the coordinates (τ,y) of the function W in the following form:

$$W=\sum_{j=0}^{n+1}\varepsilon^j\omega_j(\tau,y)+0(\varepsilon^{n+2}), \tag{34}$$

where $\omega_0=W_0(1,y)$ is independent of τ, the remaining functions ω_k are determine from the formula

$$\omega_k=\sum_{i+j=k}(-1)^i\frac{\partial^i W_j(1,y)}{\partial x^i}\tau^i;\quad k=1,2,...,n+1. \tag{35}$$

Having substituted expressions (28), (34) for the functions V, W in (30), and taking into account (31), we get the following equations for defining $V_0, V_1, ..., V_{n+1}$:

$$\frac{\partial^3 V_0}{\partial\tau^3}+\frac{\partial^2 V_0}{\partial\tau^2}+\frac{\partial V_0}{\partial\tau}=0, \tag{36}$$

$$\frac{\partial^3 V_j}{\partial\tau^3}+\frac{\partial^2 V_j}{\partial\tau^2}+\frac{\partial V_j}{\partial\tau}+Q_j(\tau,y,\omega_0+V_0,\omega_1+V_1,...,\omega_{j-1}+V_{j-1})-$$
$$-Q_j(\tau,y,\omega_0,\omega_1,...,\omega_{j-1})=0. \tag{37}$$

The boundary conditions for equations (36), (37) are found by substitution of the expressions for W and V from (18) and (28), in (29) respectively, and equating the terms with the same powers with respect to ε. They are of the form

$$V_j\left|_{\tau=0}\right.=-W_j\left|_{x=1}\right.;\ j=0,1,...,n;\ V_{n+1}\left|_{\tau=0}\right.=0, \tag{38}$$

$$\frac{\partial V_0}{\partial\tau}\left|_{\tau=0}\right.=0,\ \frac{\partial V_j}{\partial\tau}\left|_{\tau=0}\right.=\frac{\partial W_{j-1}}{\partial x}\left|_{x=1}\right.;\ j=1,2,...,n+1. \tag{39}$$

Thus, V_0 is the solution of the boundary layer type function (36) satisfying boundary condition (38) for $j=0$ and the first boundary condition from (39).

In addition to the zero root, the characteristical equation corresponding to ordinary differential equation (36) has two roots with negative real parts denoted by $\lambda_{1,2}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$. It should be noted that the amount of the lost boundary conditions on $x=1$ is also two. Therefore, problem (1)-(3) regularly degenerates on $x=1$ (see [4]).

Obviously, the solution of boundary layer type problem (36), (38) for $j = 0$ is of the form

$$V_0(\tau, y) = \frac{W_0(1, y)}{\lambda_1 - \lambda_2}(\lambda_2 e^{\lambda_1 \tau} - \lambda_1 e^{\lambda_2 \tau}). \tag{40}$$

Since the function V_0 is known, we can define the function V_1. Taking into account (32) and (37)-(39) for $j = 1$ we get that the problem for V_1, has the form

$$\frac{\partial^3 V_1}{\partial \tau^3} + \frac{\partial^2 V_1}{\partial \tau^2} + \frac{\partial V_1}{\partial \tau} = H_1(\tau, y), \tag{41}$$

$$V_1 \mid_{\tau=0} = W_1 \mid_{x=1}, \ \frac{\partial V_1}{\partial \tau} \mid_{\tau=0} = \frac{\partial W_1}{\partial x} \mid_{x=1}, \tag{42}$$

where H_1, is determined by the formula

$$H_1(\tau, y) = \frac{\partial V_0}{\partial y} + F_1(1, y, \omega_0 + V_0) - F(1, y, \omega_0). \tag{43}$$

Using (40) and the Lagrange formula, we can write the right side of (43) as follows:

$$H_1(\tau, y) = \frac{1}{\lambda_1 - \lambda_2}\left[\frac{\partial W_0(1, y)}{\partial y} + W_0(1, y)\frac{\partial F(1, y, \omega_0 + \theta V_0)}{\partial U}\right]\left(\lambda_2 e^{\lambda_1 \tau} - \lambda_2 e^{\lambda_2 \tau}\right), \tag{44}$$

where $0 < \theta < 1$.

As it is seen from (44) the right side of equation (41) is a boundary layer type function. Therefore, there exists a solution of this boundary layer type equation, satisfying the boundary conditions (42).

Without details, we note that following (33) and (37) for $j = 2$, we can write the right side of the equation for V_2 in the form

$$H_2 = -\frac{\partial V_1}{\partial y} + \frac{\partial^2 F_1(1, y, \omega_0 + \theta_1 V_0)}{\partial U^2}\omega_1 V_0 + \frac{\partial F(1, y, \omega_0 + V_0)}{\partial U} V_1 -$$

$$-\frac{\partial^3 V_0}{\partial \tau \partial y^2} - \frac{\partial^2 V_0}{\partial y^2} - \frac{\partial^2 F(1, y, \omega_0 + \theta_2 V_0)}{\partial x \partial U}\tau V_0, \tag{45}$$

where $0 < \theta_1 < 1, 0 < \theta_2 < 1$. Since V_0 and V_1 are boundary layer functions, form (45) it follows that H_2 is also a boundary layer function. Therefore, we can define the function V_2 as solution of boundary layer type equation (37) for $j = 2$, satisfying boundary conditions (38), (39) for $j = 2$.

Continuing this process we constructed the remaining function $V_3, V_4, ..., V_{n+1}$ contained in the right side of (28). Multiply all the functions V_j by the smoothing functions (see [4]) and again denote by V_j; $j = 0, 1, ..., n + 1$ the obtained new functions.

A the function V vanishes for $x = 0$ at the expense of the smoothing function, then from the first boundary condition of (27) it follows that along with the conditions (29) the sum $W + V$ satisfies also the following boundary condition

$$(W + V) \mid_{x=0} = 0. \tag{46}$$

Following the second boundary condition from (27) and (28) we have that if all the functions V_j will vanish for $y = 0$, i.e.

$$V_j \mid_{y=0} = 0; \ j = 0, 1, ..., n + 1, \tag{47}$$

then the sum $W+V$ will satisfy the boundary condition

$$(W+V)\,|_{y=0}=0. \qquad (48)$$

For fulfilled condition (47) the functions W_i should satisfy the following conditions:

$$\frac{\partial^k W_i(1,0)}{\partial x^{k_1}\partial y^{k_2}}=0; k=k_1+k_2; i=0,1,...,n; k_1+k_2+i=0,1,...,n+3. \qquad (49)$$

Suppose that the function $F(x,y,U)$ satisfies the condition

$$\frac{\partial^k f(1,0)}{\partial x^{k_1}\partial y^{k_2}}=0; k=k_1+k_2; k=0,1,....,2n+2, \qquad (50)$$

in the case of linear dependence of F on U (see (5)), and the condition

$$\frac{\partial^k F(1,0,0)}{\partial x^{k_1}\partial y^{k_2}}=0; k=k_1+k_2; k=0,1,...,2n+2, \qquad (51)$$

in the case of nonlinear dependence of F on U. Then using formula (25), we can show that conditions (49), and consequently (47), (48) will be satisfied.

Thus, the constructed sum $W+V$ satisfies equation (30) and boundary conditions (46), (48), (29). But this sum may not satisfy the second boundary condition from (3) on $y=1$. Therefore we should construct the function

$$\eta=\eta_0+\varepsilon\eta_1+...+\varepsilon^{n+1}\eta_{n+1}, \qquad (52)$$

of boundary layer type near the boundary $y=1$, that must provide fulfill must of the boundary condition

$$(W+V+\eta)\,|_{y=1}=0. \qquad (53)$$

Therewith, the equations wherefrom the functions $\eta_j; j=0,1,...,n+1$ will be determined, are obtained from the equality

$$L_{\varepsilon,2}(W+V+\eta)-L_{\varepsilon,2}(W+V)=0(\varepsilon^{n+1}), \qquad (54)$$

where $L_{\varepsilon,2}$ is another decomposition of the operator L_ε near the boundary $y=1$. Here, the change of variables near the boundary $y=1$ are made by the formula: $x=x,\ 1-y=\varepsilon\xi$. The expansion of $L_{\varepsilon,2}\left(\sum\limits_{j=0}^{n+1}\varepsilon^j h_j(x,\xi)\right)$ in powers of ε in the coordinates (x,ξ) is of the form

$$L_{\varepsilon,2}\left(\sum_{j=0}^{n+1}\varepsilon^j h_j(x,\xi)\right)\equiv\varepsilon^{-1}\left\{\frac{\partial}{\partial\xi}\left(\frac{\partial h_0}{\partial\xi}\right)^{2k+1}+\frac{\partial^2 h_0}{\partial\xi^2}+\frac{\partial h_0}{\partial\xi}+\right.$$

$$\left.+\sum_{j=1}^{n+1}\varepsilon^j\left[\left(\frac{\partial}{\partial\xi}\left(\frac{\partial h_0}{\partial\xi}\right)^{2k}+1\right)\right]\frac{\partial h_j}{\partial\xi}+\frac{\partial h_j}{\partial\xi}+R_j(h_0,h_1,...,h_{j-1})\Big]+o(\varepsilon^{n+2})\right\}, \qquad (55)$$

where R_j are the known function dependent $h_0,h_1,...,h_{j-1}$ and their derivatives.

Expanding each function $W_i(x, 1-\varepsilon\xi)$ and $V_j(\tau, 1-\varepsilon\xi)$ in the Taylor formula at the points $(x, 1)$ and $(\tau, 1)$, respectively, from (54), (55) we get the following equations:

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\eta_0}{\partial\xi}\right)^{2k+1} + \frac{\partial^2\eta_0}{\partial\xi^2} + \frac{\partial\eta_0}{\partial\xi} = 0, \tag{56}$$

$$\frac{\partial}{\partial\xi}\left\{\left[\left(\frac{\partial\eta_0}{\partial\xi}\right)^{2k+1} + 1\right]\frac{\partial\eta_j}{\partial\xi}\right\} + \frac{\partial\eta_j}{\partial\xi} = G_j,\ j = 1, 2, ..., n+1, \tag{57}$$

where $G_j(\eta_0, \eta_1, ..., \eta_{j-1})$ are the known functions. For example, G_1 is of the form

$$G_1 = \frac{\partial^3\eta_0}{\partial x\partial\xi^2} + \frac{\partial\eta_0}{\partial x} + F(x, 1, W_0(x, 1) + V_0(\tau, 1) + \eta_0(x, \xi)) -$$

$$-F(x, 1, W_0(x, 1) + V_0(\tau, 1)). \tag{58}$$

The boundary conditions for equations (56), (57) are found by substituting the expressions for W, V, η from (18), (28), (52), respectively, in (53) and equality the terms with the same powers in ε. They are of the following form

$$\eta_j\,|_{\xi=0} = -(W_j + V_i)_{y=1}; j = 0, 1, ..., n;\ \eta_{n+1}\,|_{\xi=0} = -V_{n+1}\,|_{y=1}. \tag{59}$$

Therewith we find V_j as a boundary layer functions therefore, the relation

$$\lim_{\xi\to+\infty}\eta_j = 0;\quad j = 0, 1, ..., n+1 \tag{60}$$

replaces the second boundary condition.

For $j = 0$, lemma 1 answers the question on the existence and uniqueness of the solution of problem (56), (59), (60) for $\varphi(x) = -(W_0 + V_0)\,|_{y=1}$. By the same lemma, the function $\eta_0(x, \xi)$ being the solution of (56), (59), (60) for $j = 0$ satisfies condition (13) for $k = 2n+3$.

Using the Lagrange theorem, we can represent the right side of (58) in the form

$$G_1 = \frac{\partial^3\eta_0}{\partial x\partial\xi^2} + \frac{\partial\eta_0}{\partial x} + \frac{\partial F(x, 1, W_0(x, 1) + V_0(\tau, 1) + \theta\eta_0(x, \xi))}{\partial U}\eta_0. \tag{61}$$

Then following (13), from (61) we get that the right side of equation (57) for $j = 1$ satisfies condition (14) for $i = 1$. Therefore, by lemma 3 for $\psi(x) = -(W_1 + V_1)\,|_{y=1}$ and $h = G_1$ we can affirm that the function η_1 being the solution of (57), (59), (60) for $j = 1$ satisfies condition (17) for $i = 1$.

The remaining functions $\eta_2, \eta_3, ..., \eta_{n+1}$ are constructed sequentially by the same reasonings, only at each time the obvious form of the functions $G_2, G_3, ...,$ G_{n+1} should be taken into account.

Multiply all the functions $\eta_j;\ j = 0, 1, ..., n+1$ by the smoothing multiplier and leave the previous denotation for the obtained new functions. Since the functions $\eta_j;\ j = 0, 1, ..., n+1$ vanish for $y = 0$ at the expense of smoothing multipliers, it follows from (48) that the constructed sum $W + V + \eta$ along with condition (53) satisfies the following boundary condition

$$(W + V + \eta)\,|_{y=0} = 0. \tag{62}$$

From (46) we get that if η will vanish for $x=0$, the sum $W+V+\eta$ will also satisfy the following boundary condition

$$(W+V+\eta)\left|_{x=0}\right. =0. \tag{63}$$

Since $W_0\left|_{x=0}\right. =0,\quad V_0\left|_{x=0}\right. =0$, the function $\varphi(x)$ determined by the formula $\varphi(x)=-(W_0+V_0)\left|_{y=1}\right.$ vanishes for $x=0$, i.e. $\varphi(0)=0$. Therefore problem (11), (12) that contains a variable x as a parameter, for $x=0$ has a trivial solution, i.e. $\eta_0\left|_{x=0}\right. =0$. Suppose that the function $F(t,x,U)$ satisfies the condition

$$\frac{\partial^k f(0,1)}{\partial x^{k_1}\partial y^{k_2}}=0;\;\; k=k_1+k_2;\quad k=0,1,...,2n+2, \tag{64}$$

when F is linearly dependent on U and condition

$$\frac{\partial^k F(0,1,0)}{\partial x^{k_1}\partial y^{k_2}}=0;\;\; k=k_1+k_2;\quad k=0,1,...,2n+2, \tag{65}$$

when F it nonlinearly depends on U. Then the remaining functions $\eta_1,\eta_2,...,\eta_{n+1}$ contained in the right side of (52) will also vanish for $x=0$. Consequently, condition (63) will be satisfied.

As all the functions $W_i;\;\; i=0,1,...,n$ together with their derivatives vanish for $x=y$ in D, then without any additional conditions on $F(x,y,U)$ at the angular point $x=1,\; y=1$ all the functions $\eta_j;j=0,1,...,n+1$ consequently the function η determined by formula (52) vanish together with the first derivatives for $x=1$. Hence and from (29) we get that the sum $W+V+\eta$ satisfies also the following boundary conditions:

$$(W+V+\eta)\left|_{x=1}\right. =0,\quad \frac{\partial}{\partial x}(W+V+\eta)\left|_{x=1}\right. =0. \tag{66}$$

Introduce the denotation: $\widetilde{U}=W+V+\eta,\; \widetilde{U}-U=z$, where U is the solution of problem (1)-(3). Hence and from (18), (28), (52) we get the following asymptotic expansion in small parameter of the solution of problem (1)-(3):

$$U=\sum_{i=0}^{n}\varepsilon^i W_i+\sum_{j=0}^{n+1}\varepsilon^j W_j+\sum_{j=0}^{n+1}\varepsilon^j\eta_j+z, \tag{67}$$

where z is a residual number.

It holds

Lemma 4. *For the residual term z in (67) it is valid the estimation*

$$\varepsilon^2\int_0^1\left(\frac{\partial z}{\partial x}\left|_{x=1}\right.\right)^2 dy+\varepsilon^{2k+1}\iint_D\left(\frac{\partial z}{\partial y}\right)^{2k+2}dxdy+$$

$$+\varepsilon\iint_D\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]dxdy+C_1\iint_D z^2dxdy\le C_2\varepsilon^{2(n+1)}, \tag{68}$$

where $C_1>0,\; C_2>0$ are the constants independent of ε.

Proof. Summing up (19), (30), (54) we see that $\widetilde{U}$ satisfies the equation

$$L_\varepsilon\widetilde{U}=0(\varepsilon^{n+1}). \tag{69}$$

Subtracting (69) from (1), we get

$$\varepsilon^2 \frac{\partial}{\partial x}(\triangle z) - \varepsilon^p \frac{\partial}{\partial y}\left[\left(\frac{\partial U}{\partial y}\right)^p - \left(\frac{\partial \widetilde{U}}{\partial y}\right)^p\right] - \varepsilon \triangle z + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} +$$

$$+F(x,y,U) - F(x,y,\widetilde{U}) = 0(\varepsilon^{n+1}). \tag{70}$$

From (2), (3), (63), (66), (62), (53) it follows that the residual term z satisfies the boundary conditions

$$z\,|_{x=0} = 0,\ z\,|_{x=1} = 0,\ \frac{\partial z}{\partial x}\ |_{x=1} = 0,\ z\,|_{y=0} = 0,\ z\,|_{y=1} = 0. \tag{71}$$

Having multiplied the both parts of (70) by z and integrating the obtained expressions in domain D allowing with boundary conditions (71), after definite transformations we get estimation (68).

Lemma 4 is proved.

Combining the obtained results, we get the following statement.

Theorem. *Assume that* $F(x,y,U) \in C^{2n+3}(D(-\infty,+\infty))$, *conditions (4), (6) and condition (7) are fulfilled in the case of linear dependence of* F *on* U, *conditions (8) for* $m = 2n+2$, *(51), (65) in the case of nonlinear dependence of* F *on* U. *Then for the solution of problem (1)-(3) it is valid asymptotic representation (67), where the functions* W_i, *are determined by the first iterative process,* V_j, η_j *are the boundary layer type functions near the boundaries* $x = 1$ *and* $y = 1$ *that are determined by the appropriate iterative processes,* z *is a residual term, and estimation (68) is valid for it.*

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