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A MEAN VALUE THEOREM APPROACH TO THE REMOVABLE SETS OF PARABOLIC EQUATIONS

Abstract

In this paper, we prove Landis-Gerver's type mean value theorem adopted to the parabolic equations. As an application, Carlson type theorem on removable sets for $H^{\alpha,\frac{\alpha}{2}}$–Holder continuous solutions is investigated for the divergence structure linear parabolic equations. In partial, when α is sufficiently small, we show that the compact set E is removable iff the anisotropic Hausdorff measure $\Lambda^{\frac{(n+\alpha)}{n+2}}(E) = 0$.

1.Introduction

In this paper, we consider the problem of a removable singularity for the class of Hölder continuous solutions of linear parabolic equations of divergent form

$$\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}(x,t)\frac{\partial u}{\partial x_k}\right) - \frac{\partial u}{\partial t} = 0. \tag{1}$$

The elements of the matrix $\|a_{ik}(x,t)\|_{i,k=1,2,...n}$ are assumed to be the measurable functions in domain $D \subset \mathbb{R}^{n+1}$, which satisfy the condition of uniform parabolicity: there exists a $\lambda \in (0,1]$ such that the inequality

$$\lambda\,|\xi|^2 \le \sum_{i,k=1}^{n} a_{ik}(x,t)\xi_i\xi_k \le \lambda^{-1}\,|\xi|^2 \tag{2}$$

is satisfied for any $\xi \in \mathbb{R}^n$ and almost all $x, t \in D$.

Let $D \subset \mathbb{R}^{n+1}$. Points from $\mathbb{R}^{n+1}$will be denoted by $z = (x,t)$, where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n; t \in \mathbb{R}$. We denote by $H^{\alpha,\frac{\alpha}{2}}(D)$ $(0 < \alpha \le 1)$, the class of continuous functions $f : D \longrightarrow \mathbb{R}$, which satisfy the condition

$$\left|f(x,t) - f(x',y')\right| \le K\left(\left|x - x'\right|^2 + \left|t - t'\right|\right)^{\frac{\alpha}{2}},$$

where $K > 0$ does not depend on the points (x,t), $(x',t') \in D$. Put $H^{\alpha}(D) := H^{\alpha,\alpha}(D)$. Let $\Omega \in \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. We denote by $W^{1,2}(\Omega)$ the Sobolev space of functions $u \in L^2(\Omega)$ defined in Ω, for which derivatives of first order u_{x_i}; $i = 1, 2, ..., n$, belong, in the sense of the distribution theory, to $L_2(\Omega)$ $(u_{x_i} \in L_{2,loc}(\Omega))$. The norm of space $L_2(\Omega)$ is given as $||f||_{L_2(\Omega)} = \left(\int\limits_{\Omega} |f|^2\ dx\right)^{\frac{1}{2}}$. We denote by $\dot{W}^{1,2}(\Omega)$ a subspace of the space $W^{1,2}(\Omega)$, where the class of functions $C_0^{\infty}(\Omega)$ is an everywhere dense set. We also recall that $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ is an n -dimensional Euclidean ball with center at x_0 and radius $r > 0$, and $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$ is the surface of the ball $B(x_0, r)$.

Let the domain D be a cylinder $Q_{t_1,t_2} = \Omega \times [t_1, t_2]$. Then the notation $f(x,t) \in L^s[t_1, t_2; L^r(\Omega)]$ means that function f has finite norm

$$\|f\|_{L^2[t_1,t_2;L^s(\Omega)]} = \left(\int_{t_1}^{t_2} \|f(.,t)\|^s_{L^r(\Omega)}\, dt \right)^{\frac{1}{s}} ; 1 \le r \le \infty, 0 < s < \infty.$$

The notation $f(x,t) \in L^2\left[t_1, t_2; W^{1,2}(\Omega)\right]$ means that the f has the finite norm

$$\|f\|_{L^2[t_1,t_2;W^{1,2}(\Omega)]} = \left(\int_{t_1}^{t_2} \|f(\cdot,t)\|^2_{W^{1,2}(\Omega)}\, dt \right)^{\frac{1}{2}} .$$

For a differentiable function of $n+1$ variables $f(x,t)$ denote by $\nabla_x f$ the vector $\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \frac{\partial f}{\partial x_n}\right)$. Denote by $kC(z_0, r)$ the cylinder

$$\left\{(x,t) \in \mathbb{R}^{n+1} : |x - x_0| < kr;\, t_0 - k^2 r^2 < t < t_0\right\}.$$

We denote by ∂D the boundary of the domain $D \subset \mathbb{R}^{n+1}$, and by $\Gamma(D)$ the proper boundary of this domain D, that is the set of points $(y, \tau) \in \partial D$ for which there exists a cylinder $\left\{(x,t) : |x - y| < r, \tau - r^2 < t < \tau\right\}$ wholly embedded into D for some $r > 0$.

Definition 1.1. *A function u(x,t) is called a weak solution, in the domain D, of the nonhomogeneous equation* $Lu = h$; $h \in L^1\left[t_1, t_2, L^1(\Omega)\right]$ *if for any cylinder* $Q_{t_1,t_2} = \Omega \times (t_1, t_2) \subset D$ *we have* $u \in L^2\left[t_1, t_2, W^{1,2}(\Omega)\right] \cap L^\infty\left[t_1, t_2, L^2(\Omega)\right]$ *and the integral identity*

$$\int_\Omega u(x,t_2)\varphi(x,t_2)dx + \int_{t_1}^{t_2} dt \int_\Omega \left(\sum_{i,k=1}^n a_{ik}(x,t) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_k} - u \frac{\partial \varphi}{\partial t} \right) dx$$

$$= \int_\Omega u(x,t_1)\varphi(x,t_1)dx + \int_{t_1}^{t_2} dt \int_\Omega h(x,t)\varphi(x,t)dx \tag{3}$$

is fulfilled for any $\varphi \in C(\Omega \times (t_1, t_2))$ *with a compact support in* $\Omega \times (t_1, t_2)$.

Assuming in Definition 1.1 $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$, instead of the identity (3), we will have

$$\int_{t_1}^{t_2} dt \int_\Omega \left(\sum_{i,k=1}^n a_{ik}(x,t) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_k} - u \frac{\partial \varphi}{\partial t} \right) dx = \int_{t_1}^{t_2} dt \int_\Omega h(x,t)\varphi(x,t)dx, \tag{4}$$

where is assumed $u \in L^2\left[t_1, t_2; W^{1,2}(\Omega)\right]$.

By the existence and uniqueness of a solution in the sense of both definitions, for $h \in L^s[t_1, t_2, L^r(\Omega)]$; $\frac{n}{2r} + \frac{1}{s} < 1$ we have, they are equivalent (see, f.e.

[1],[2],[17],[19]). Moreover, if $u(x,t_1)=0$ and $u(x,t_2)=0$ for $(x,t)\in\partial\Omega\times(t_1,t_2)$, then for a solution we have the representation

$$u(x,t)=-\int\limits_{t1}^{t_2}d\tau\int\limits_{\Omega}g(y,\tau)h(y,\tau)dy,$$

where $g_{y,\tau}(x,t)$ is Green's function of the Dirichlet problem for the equation (1) with a singularity at the point (y,τ). According to [1], Green function has estimates $g_{y,\tau}$, $\nabla_x g_{y,\tau}\in L^{s'}\left[t_1,t_2,L^{r'}(\Omega)\right]$, where the numbers r,s are as above. Moreover, Green function is unique and $\|g(x,t)\|_{L^{s'}\left[t_1,t_2,L^{r'}(\Omega)\right]}\le C$ in Q_{t_1,t_2}. For Green function the following estimate holds in the cylinder Q_{t_1,t_2}

$$C_1\left(t-\tau\right)^{-\frac{n}{2}}\exp\left(-\frac{\beta_1\left|x-y\right|^2}{(t-\tau)}\right)\le g_{y,\tau}(x,t)\le C_2\left(t-\tau\right)^{-\frac{n}{2}}\exp\left(-\frac{\beta_2\left|x-y\right|^2}{(t-\tau)}\right)$$

if the points x,y lie at a positive distance δ from the boundary $\partial\Omega$, where C_i,β_i; $i=1,2$ are positive constants not depending on (x,t), $(y,\tau)\in Q_{t_1,t_2}$ (see [1], [3]).

Definition 1.2. *Let $E\subset\subset D$ be a compact subset of the bounded domain $D\subset\mathbb{R}^{n+1}$. We say, the set E is removable for the class $H^{\alpha,\frac{\alpha}{2}}(D)$ of solutions of the equation (1) if every weak local solution of the equation (1) in $D\backslash E$ ((in the sense that u solves (1) with $L^2\left[t_1,t_2,W^{1,2}(\Omega)\right]\cap L^{\infty}\left[t_1,t_2,L^2(\Omega)\right]$ for every cylinder $Q_{t_1,t_2}\subset D\backslash E$) belonging to the space $H^{\alpha,\frac{\alpha}{2}}(D)$ throughout the domain D is extendable into the compact set E as a solution.*

Definition 1.3. *Let $s\in\mathbb{R}_+$ be some number, $E\subset\mathbb{R}^n$ be a bounded closed subset.Let the finite system of cylinders $\{C_\nu=C(z_\nu,r_\nu\}_{\upsilon=1,2,...N}$ be a finite system of cylinders such that $r_\nu\le\delta$ cover the set E, i.e. $E\subset\bigcup\limits_{\nu}C_\nu$. Put $\Lambda^{s,\delta}(E)=\inf\left\{\sum\limits_{\nu}|C_\nu|^s\right\}$. Here $|C_\nu|$ stands for an $n+1$ dimensional Lebesgue measure of a cylinder C_ν, i.e. r_ν^{n+2}, and the lower bound is taken with respect to all the cylinders. Put*

$$\Lambda^s(E)=\lim_{\delta\to 0}\Lambda^{s,\delta}(E).$$

This characteristic is a parabolic generalization of Hausdorff measure.

According to Carlson theorem [6], a necessary and sufficient condition for the compact set E to be removable in the class of harmonic functions outside E and belonging to class $H^\alpha(D)$ is expressed in terms of a Hausdorff measure of order $n-2+\alpha$ of the form

$$mes_{n-2+\alpha}(E)=0,\ \ 0<\alpha<1$$

(For the case $\alpha=1$ this result is obtained in [23], [28]). R. Harley and J. Polking [11] proved the corresponding result for general elliptic equations with variable coefficients (see also, E. Dolzhenko [7] and A. Pokrovski [25] dealing with questions of removable sets for the class of solutions of elliptic equations with derivatives from Hölder classes). For the case $p\ge 2$, the authors [14] proved a sufficient condition of removability in the class $H^{\alpha}(D)$ $(0<\alpha\leq 1)$ of solutions of $p-$Laplace equation $\left(A=|\nabla u|^{p-2}\nabla u\right)$ of the form

$$mes_{n-p+(p-1)\alpha}(E)=0;\ \ 0<\alpha\leq 1. \tag{5}$$

Furthermore, T.Kilpelainen and X.Zong [12] proved a complete analogue of Carlson's result for p -Laplace equation. The mentioned authors proved the necessity of the condition (5) and also gave an alternative proof of sufficiency that included the range of an exponent $1<p<2$. The same approach was applied in [21] to the case of a metric space. In [22], for the $H^{\alpha}(D)$ class of solutions of degenerated quasilinear elliptic equations a necessary and sufficient condition for removability sets was proved .

A necessary and sufficient condition for the compact set $E\subset D$ to be removable for the class $H^{\alpha,\frac{\alpha}{2}}(D)$ of solutions of the heat equation $\triangle u-u_t=0$ was obtained in J. Kral [16] (see also, [20]). There the sufficiency follows from the general results for semi-elliptic equations with constant complex coefficients (for the bibliography see [20], while the necessity follows from the work [16], where Hölder continuity of heat potential has been proved). According to the results of, the compact set $E\subset D$ is removable for the class $H^{\alpha,\frac{\alpha}{2}}(D)$ $(0<\alpha<1)$ of solutions of a heat equation if and only if

$$\Lambda^{\frac{(n+\alpha)}{(n+2)}}(E)=0. \tag{6}$$

We also refer to the works [4], [8], [15], [13], which deal with the problems of removable singularities for other classes of solutions. For example, in [13] the removability of the t -axis is considered for the class of solutions of heat equation having logarithmic growth when approaching the axis. The approaches similar to [6] was considered in [29] for heat equation.

In the present paper, we will prove a mean value Theorem of Landis-Gerver type adopted to the parabolic equations. We will show that the condition (6) is necessary and sufficient for a compact set to be removable. The sufficiency is proved by application of remainded mean value theorem. To prove the necessity, we use a parabolic version of the Frostman's lemma. By means of this lemma we construct a special measure with the needed estimate with respect to all cylinders and then find a solution of the corresponding nonhomogeneous problem. For this solution we prove the L_{∞} -estimate, where essential use is made of the upper bound of the fundamental solution of (1) from [3]. The obtained result enables us to claim that the found solution belongs to a certain anisotropic Campanato class which enables Hölder continuity of the solution order α with respect to the x -axis and, of order $\frac{\alpha}{2}$ with respect to the t -axis.

Lemma 1.1. *Assume that $D\subset\mathbb{R}^{n+1}$ is a bounded domain and a function $u\in L^1(D)$ satisfies the inequality*

$$\int\limits_{C(z,r)}\left|u-(u)_{z,r}\right|dx\leq Mr^{n+2+\alpha};\alpha\in(0,1]$$

for any cylinder. $C(z,r)\subset D$. Then $u\in H^{\alpha,\frac{\alpha}{2}}(D)$ and for any $D'\subset\subset D$ the *estimate*

$$\sup_{D'}|u| + \sup_{z_1,z_2\in D'}\frac{|u(z_1)-u(z_2)|}{\left(|x_1-x_2|^2+|t_1-t_2|\right)^{\frac{\alpha}{2}}} \le C\left(M+\|u\|_{L^1(D)}\right)$$

is satisfied, where $C = C(n,\alpha,D',D)$, $(u)_{z,r} = \frac{1}{|C(z,r)|}\iint\limits_{C(z,r)} u(x,t)dxdt.$

The proof of this lemma can be obtained following the method of S. Campanato (see [5], or deduce from the corresponding general results for homogeneous spaces in [26])

Lemma 1.2. *Let* $0<\alpha\le 1$, $E\subset\mathbb{R}^{n+1}$ *be a compact set such that* $\Lambda^{\frac{(n+\alpha)}{(n+2)}}(E)>0$. *Then there exists a measure* μ *with positive mass in* E *and having support in* E *such that for any cylinder* $C(z,r)$ *the inequality holds:*

$$\mu\left(C\left(z,r\right)\right)\le Cr^{n+\alpha},$$

where the constant $C>0$ *does not depends on cylinder.*

The proof of this lemma can be derived by using similar approaches as in [6], or can be directly deduced from the corresponding general results for homogeneous spaces [27].

For the equation (1) we have the well-known local estimate of Hölder's norm for a solution

$$\left|u(z)-u(z')\right|\le C\left(\left|x-x'\right|^2+\left|t-t'\right|\right)^{\frac{\kappa}{2}}, \tag{7}$$

where $z=(x,t)$, $z'=(x',t')$ are arbitrary points of the domain $D\subset\mathbb{R}^{n+1}$ lying at a positive distance δ from the domain boundary, the constant $C>0$ depends only on the norm $\|u\|_{L^2(D)}$ and the constants δ,n,λ. Let $\kappa\in(0,1]$ be a maximal number in the inequality (7), i.e. the best Hölder exponent (in this connection see, for example, [17], [9], [19], [24], [30]).

The main result of the present paper is the following statement

Theorem 1.1. *Let* $0<\alpha<\kappa$, $0<\alpha\le 1$. *Let* $D\subset\mathbb{R}^{n+1}$ *be a bounded domain,* $E\subset\subset D$ *be a compact subset. Then for the compact set* E *to be removable in the class* $H^{\alpha,\frac{\alpha}{2}}(D)$ *of solutions of the equation (1) such that* $u\in L^2\left[t_1,t_2,W^{1,2}(\Omega)\right]\cap L^\infty\left[t_1,t_2,L^2(\Omega)\right]$ *in any cylinder* $Q_{t_1,t_2}\subset D\backslash E$ *, it is necessary and sufficient that is satisfied the condition* (6)

Theorem 1.1 immediately implies the following well-known result-the Carlson-Kral theorem for a heat equation.

Corollary 1.1. *Let* $0<\alpha\le 1$. *Then for the compact set* $E\subset\subset D$ *to be removable in the class* $H^{\alpha,\frac{\alpha}{2}}(D)$ *of solutions of a heat equation such that* $u\in L^2\left[t_1,t_2,W^{1,2}(\Omega)\right]\cap L^\infty\left[t_1,t_2,L^2(\Omega)\right]$ *for any cylinder* $Q_{t_1,t_2}\subset D\backslash E$ *it is necessary and sufficient that the condition* (6) *be satisfied.*

2. Proof of main results

There was proved the following n -dimensional generalization of mean value theorem in the monograph [18].

Theorem 2.1. *Let $D \subset \mathbb{R}^n$ be a domain in the ball layer $B(x_0,2r)\backslash\overline{B}(x_0,r)$ such that the domain D has limit points at surfaces of spheres $S(x_0,2r)$ and $S(x_0,r)$. Suppose the coefficients of square form $\sum\limits_{i,k=1}^{n} a_{ik}(x)\eta_i\eta_k$ are continuously differentiable and defined at some neighborhood of the domain D. Furthermore, there exists a $\lambda \in (0,1)$ such that for any $x \in D$, $\eta \in \mathbb{R}^n$ it follows that*

$$\lambda\left|\eta\right|^2 \leq \sum_{i,k=1}^{n} a_{ik}(x)\eta_i\eta_k \leq \lambda^{-1}\left|\eta\right|^2$$

Assume that $f : D \longrightarrow \mathbb{R}$ is a function which is twice continuously differentiable in the domain D.

Then there exists a piecewise continuous surface Σ such that the surface separates in D the spheres $S(x_0,2r)$ and $S(x_0,r)$. At that, the integral over Σ is estimated as

$$\int\limits_{\Sigma}\left|\frac{\partial f}{\partial \nu}\right| ds \leq K\frac{\left(\underset{D}{oscf}\right) mes_n D}{r^2},$$

where $\dfrac{\partial f}{\partial \nu} = \sum\limits_{i,k=1}^{n} a_{ik}(x)\frac{\partial f}{\partial x_i}N_k$ is a conormal derivative, the constant K depend on n,λ. Here $N = (N_1,N_2,...N_n)$ in a unit vector ordered perpendicular to the surface Σ.

Using this theorem, below we derive own version, adopted for investigation of removable sets of parabolic equations.

Theorem 2.2. *Let $D \subset \mathbb{R}^{n+1}$be a domain contained in the cylindrical layer $C(z_0,2r)\backslash\overline{C}(z_0,r)$ such that the domain D has limit points at surfaces of cylindrical $\partial C(z_0,2r)$ and $\partial C(z_0,r)$. Suppose the coefficients of square form $\sum\limits_{i,k=1}^{n} a_{ik}(x,t)\eta_i\eta_k$ are continuously differentiable and are defined at some neighborhood of the domain D. Furthermore, there exists a $\lambda \in (0,1)$ such that for any $(x,t) \in D$, $\eta \in \mathbb{R}^n$ it follows that*

$$\lambda\left|\eta\right|^2 \leq \sum_{i,k=1}^{n} a_{ik}(x,t)\eta_i\eta_k \leq \lambda^{-1}\left|\eta\right|^2$$

Assume that $f : D \longrightarrow \mathbb{R}$ is a function of two variables, which is twice continuously differentiable in some neighborhood of domain D.

Then there exists a piecewise continuous surface Σ such that the surface separates in D the proper surfies of cylinders $\Gamma(C(z_0,2r))$ and $\Gamma(C(z_0,r))$. At that, the integral over Σ is estimated as

$$\int\limits_{\Sigma}\left|\frac{\partial f}{\partial \mu}\right| d\sigma \leq K\left(\underset{D}{oscf}\right) r^n, \tag{8}$$

where $\frac{\partial f}{\partial \mu} = \sum\limits_{i,k=1}^{n} a_{ik}(x,t)\frac{\partial f}{\partial x_i}N_k$ is a conormal derivative, the constant K depend on n,λ. Here $N = (N_0,N_1,N_2,...N_n)$ is a unit vector ordered perpendicular to the surface Σ and $d\sigma$ is an element of n-dimensional area on surface Σ

Remark 2.1. In the Theorem 2.2, we have considered all cylindrical layer $\Omega = C(z_0, 2r)\backslash C(z_0, r)$. The following estimate holds if one consider in place Ω any domain D contained in Ω , that has a limit points on $\partial C(z_0, 2r)$ and $\partial C(z_0, r)$. In general settings, we come to the estimate

$$\int\limits_{\Sigma} \left| \frac{\partial f}{\partial \mu} \right| d\sigma \leq K \left(\underset{D}{osc} f \right) \frac{(mes_{n+1} D)}{r^2}.$$

in place of 2.2.

Proof of Theorem 2.2. Denote by $\Omega_\tau = \Omega \cap \{t = \tau\}$. It follows from Theorem 2.1 that there exists a piecewise smooth surface $\sum_\tau \subset \Omega_\tau$ such that $\sum_\tau$ separates spheres $S(x_0, 2r)$ and $S(x_0, r)$. At that, the estimate holds:

$$\int\limits_{\Sigma_\tau} \left| \sum_{i,k=1}^{n} a_{ik}(x,t) \frac{\partial f}{\partial x_i} N_k \right| ds \leq K \left(\underset{D}{osc} f \right) \frac{(mes_n \Omega_\tau)}{r^2}. \tag{9}$$

By uniformly continuity of the functions f and a_{ik}, $i, k = 1, 2, ...n$, it follows that the inequality (9) with $2K$ will be hold for any surface $\sum_t'$ which is a small shifting of $\sum_\tau$ along axis t , $t \in (\tau - \Delta\tau, \tau + \Delta\tau)$:

$$\int\limits_{\Sigma_t} \left| \sum_{i,k=1}^{n} a_{ik}(x,t) \frac{\partial f}{\partial x_i} N_k \right| ds \leq 2\omega_{n-1} K \left(\underset{D}{osc} f \right) r^{n-2} \tag{10}$$

Let us construct the desired surface for the inequality (8). In connection with (10), find sufficiently small $\Delta\tau$ such that for a sequence $\{\tau_\gamma\}_{\gamma=1}^m$ with $t_0 - 2r^2 = \tau_0 < \tau_1 < ... < \tau_m = t_0 + 2r^2$ and $\tau_{\gamma+1} - \tau_\gamma < \Delta\tau$; $\gamma = 1, 2, ...m$ the inequality (10) take place.

Then we attain to the inequality

$$\int\limits_{\widetilde{\Sigma}_\gamma} \left| \sum_{i,k=1}^{n} a_{ik}(.,.) \frac{\partial f}{\partial x_i} N_k \right| d\sigma \leq 2\omega_{n-1} K \left(\underset{D}{osc} f \right) r^{n-2} (\tau_{\gamma+1} - \tau_\gamma)$$

multiplying (10) by $\tau_{\gamma+1} - \tau_\gamma$, where $\widetilde{\Sigma}_\gamma = \sum_{\tau_\gamma} \times [\tau_\gamma, \tau_{\gamma+1}]$ is a cylinder, $d\sigma$ is an element of $n-$dimensional area on $\widetilde{\Sigma}_\gamma$, $\gamma = 1, 2, ...m$. We had used that $N_0 = 0$ on $\widetilde{\Sigma}_\gamma$. Evidently, $d\sigma = dsdt$ on $\widetilde{\Sigma}\gamma$. Now sew the cylinders $\widetilde{\Sigma}_\gamma$ and $\widetilde{\Sigma}_{\gamma+1}$ by using piece of surface $\{t = \tau_{\gamma+1}\}$, say they are $\widehat{\Sigma}_\gamma$. Since $\widehat{\Sigma}_\gamma$ is perpendicular to t axis (since on this surface $N_0 = 1, N_1 = 0, ..., N_n = 0$), we have

$$\int\limits_{\widehat{\Sigma}_\gamma} \left| \sum_{i,k=1}^{n} a_{ik}(.,.) \frac{\partial f}{\partial x_i} N_k \right| d\sigma = 0.$$

Now for the desirable surface is union of all surfaces $\widetilde{\Sigma}\gamma, \widehat{\Sigma}\gamma$. Denoting it by $\sum$, we will have

$$\int\limits_{\Sigma}\left|\frac{\partial f}{\partial \mu}\right| d\sigma=\sum_{\gamma=1}^{m}\int\limits_{\widetilde{\Sigma}_\gamma\cup\widehat{\Sigma}_\gamma}\left|\sum_{i,k=1}^{n} a_{ik}(.,.)\frac{\partial f}{\partial x_i}N_k\right| d\sigma$$

$$\le 2\omega_{n-1}K\left(\underset{D}{osc} f\right) r^{n-2}\sum_{\gamma=1}^{m}(\tau_{\gamma+1}-\tau_\gamma)\le K_1\left(\underset{D}{osc} f\right) r^n.$$

This completes the proof of Theorem 2.2.

Proof of sufficiency of Theorem 1.1. We shall derive the proof in several steps.

Approximation. Let $D\subset\mathbb{R}^{n+1}$ be a bounded domain, $u\in W^{1,2}(D\backslash E)$ be a solution of the equation (1.1) and let $\Lambda^{\frac{(n+\alpha)}{(n+2)}}(E)=0$, $u\in C^{\alpha,\frac{\alpha}{2}}(D)$. Let us continue the function u out of D by zero. Denote by u_ε the smooth mollifying of u with kernel ρ, i.e. $u_\varepsilon(x)=\frac{1}{\varepsilon^n}\int\limits_{\mathbb{R}^n}\rho\left(\frac{x-y}{\varepsilon}\right)u(y)dy$; $\rho\in C_0^\infty(\mathbb{R}^n)$, $\int\limits_{\mathbb{R}^n}\rho(y)dy=1, \varepsilon>0$. Evidently, $u\in C_0^\infty(D)$ and its support is an ε neighborhood of support u. Put $u^{(j)}=u_{\frac{1}{j}}$. Then $u^{(j)}\longrightarrow u$ uniformly at any subdomain $\overline{G}\subset D$. We have the same convergence also in the norm of space $W^{1,2}(G\backslash U)$, where $E\subset U$ and U is an open subset of G. Indeed, for sufficiently large $j\ge j_0(U)$ we have the inequalities $u^{(j)}\le C\left(u\chi_{D\backslash U}\right)^*$ and $\left|\nabla u^{(j)}\right|\le C\left(\left|\nabla u^{(j)}\right|\chi_{D\backslash U}\right)^*$, where the upper symbol (*) denotes the maximal function, $\chi_{D\backslash U}$ is characteristics function of the set $D\backslash U$. Using the boundedness of the domain D and Holder inequality, we have $u\in W^{1,1}_{loc}(D\backslash E)$. Then $\left\|u^{(j)}-u\right\|\longrightarrow 0$, therefore, for some suitable subsequence of $\left\{u^{(j)}\right\}$, denote it again as $\left\{u^{(j)}\right\}$, we have $u^{(j)}\longrightarrow u$ and $\nabla u^{(j)}\longrightarrow\nabla u$ a.e. in $D\backslash U$ (see f.e. [10]). Now it follows from the convergence to zero integrable function a.e. in $D\backslash U$ and existence of majorate functions as $C\left(u\chi_{D\backslash U}\right)^*$ and $C\left(|\nabla u|\chi_{D\backslash U}\right)^*$ respectively, we have the convergence

$$\int\limits_{D\backslash U}\left|D_x^\alpha u^{(j)}-D_x^\alpha u\right|^2 dx\longrightarrow 0,\ |\alpha|\le 1\quad as\ \ j\longrightarrow\infty. \tag{11}$$

Integrability of the functions $C\left(u\chi_{D\backslash U}\right)^*$ and $C\left(|\nabla u|\chi_{D\backslash U}\right)^*$ follows from the boundedness of maximal operator on space $L_2(\mathbb{R}^n)$ and the fact that $u\in W^{1,2}(D\backslash U)$.

Equation for approximations. It follows from the condition $\Lambda^{\frac{(n+\alpha)}{(n+2)}}(E)=0$ that $mes_{n+1}(E)=0$. Let $E'\supset E$ is arbitrary open subset. We can suppose $mes_{n+1}E'<\eta$, where $\eta>0$ is any number.

Let $\varepsilon>0$ be any number. Cover the set E by finite number of cylinders $\{C_\nu=C(z_\nu,r_\nu\}_{\nu=1}^m$ such that $\bigcup\limits_{\nu=1}^{m}C_\nu\supset E$ and $r_\nu<\delta$,

$$\sum_{\nu=1}^{m} r_\nu^{n+\alpha} < \varepsilon. \qquad (12)$$

Suppose the number $\delta = \delta(\varepsilon, \eta)$ is small such that the set $\Gamma'' = \bigcup\limits_{\nu=1}^{m} (4C_\nu)$ lies in E'.

It follows from Theorem 2.2 that there exists peaceful smooth surfies $\gamma_\nu^{(j)}; \nu = 1, 2, ...m$ such that the $\gamma_\nu^{(j)}$separates the surfies $\partial\,(2C_\nu)$ and $\partial(4C_\nu)$ and simultaneously, the estimate holds:

$$\int\limits_{\gamma_\nu^{(j)}} \left| \frac{\partial u^{(j)}}{\partial \mu} \right| d\sigma \leq K \underset{2C_\nu}{osc}\, u^{(j)} r_\nu^n.$$

Denote by $\Gamma_\nu^{(j)}$ the interior of $\gamma_\nu^{(j)}$. Then $\Gamma^{(j)} = \bigcup\limits_\nu \Gamma_\nu^{(j)} \supset \Gamma' = \bigcup\limits_\nu (2C_\nu)$. Put $\sigma_\nu^{(j)} = \Gamma^{(j)} \cap \gamma_\nu$. Let $\sigma_\nu^{(j)} \neq \varnothing$ for some ν.Then it follows from (11) that

$$\int\limits_{\sigma_\nu^{(j)}} \left| \frac{\partial u^{(j)}}{\partial \mu} \right| d\sigma \leq K \underset{2C_\nu}{osc}\, u^{(j)} r_\nu^n.$$

Evidently, $G\backslash\overline{\Gamma}$ is a strictly subset of $D\backslash E$. Therefore, for any $\psi \in C_0^1(D\backslash\Gamma')$ we have the identity

$$\sum_{i,k=1}^{n} \iint\limits_{G\backslash\Gamma'} \left(a_{ik}(x,t) \frac{\partial u}{\partial x_k} \frac{\partial \psi}{\partial x_i} - u \frac{\partial \psi}{\partial t} \right) dxdt = 0. \qquad (13)$$

Therefore and by (11) for $u^{(j)}$ and $\forall\psi \in C_0^1(D\backslash\Gamma')$ we have the identity

$$\sum_{i,k=1}^{n} \iint\limits_{G\backslash\Gamma'} \left(a_{ik}^{(j)}(x,t) \frac{\partial u^{(j)}}{\partial x_k} \frac{\partial \psi}{\partial x_i} - u^{(j)} \frac{\partial \psi}{\partial t} \right) dxdt = \delta_j.$$

in place of the equation (13). This follows from the convergence $u^{(j)} \longrightarrow u$ and $\nabla u^{(j)} \longrightarrow \nabla u$ a.e. in $G\backslash\Gamma'$ and Vitali's theorem. Since the integrated functions are equiintegrable : for any subset $g \subset D\backslash\Gamma'$ do to Young's inequality,

$$\sum_{i,k=1}^{n} \iint\limits_{g} \left| a_{ik}^{(j)} \frac{\partial u^{(j)}}{\partial x_i} \frac{\partial \psi}{\partial x_k} - u^{(j)} \frac{\partial \psi}{\partial t} \right| dxdt$$

$$\leq C \iint\limits_{g} \left(\left| \nabla u^{(j)} \right| |\nabla \psi| + \left| u^{(j)} \right| \left| \frac{\partial \psi}{\partial t} \right| \right) dxdt$$

$$\leq C \left(\iint\limits_{g} \left| \nabla u^{(j)} \right|^2 dxdt + \iint\limits_{g} |\nabla \psi|^2 dxdt + \iint\limits_{g} \left| \frac{\partial \psi}{\partial t} \right|^2 dxdt + \iint\limits_{g} \left| u^{(j)} \right|^2 dxdt \right)$$

$$\leq C\left(\iint\limits_{g}\left|\bigtriangledown u^{(j)}-\bigtriangledown u\right|^{2}dxdt+\iint\limits_{g}\left|\bigtriangledown u\right|^{2}dxdt+\iint\limits_{g}\left|\nabla\psi\right|^{2}dxdt\right)$$

$$+C\left(\iint\limits_{g}\left|u^{(j)}-u\right|^{2}dxdt+\iint\limits_{g}\left|u^{(j)}-u\right|^{2}dxdt+\iint\limits_{g}\left|u\right|^{2}dxdt\right)\longrightarrow 0 \qquad (14)$$

as $mes_{n+1}g \longrightarrow 0$ and $j \longrightarrow \infty$. Here and further by δ_j we denote different sequences that have zero limit as $j \longrightarrow \infty$.

Approximation of Green's formulae. Let the function $\varphi \in C_0^1(D)$. Define the quasimetrix $d(z,z') = \max\left\{|x-x'|,\sqrt{t-t'}\right\}$ in $\mathbb{R}^{n+1}$, where $z=(x,t)$; $z'=(x',t')$. Put $d(z) = dist\left(z,\Gamma^j\right)$ and $\psi = \varphi\xi\left(\frac{d(x)}{\tau}\right)$, where $0 \leq \xi(x) \leq 1$ is an infinitely differentiable function such that $\xi(x)$ is equal to zero if $s \leq 0$ and equals to 1 for $s \geq 1$. $\tau > 0$ is a parameter, $\forall\varphi \in C_0^1(D)$. Evidently, $\psi \in C_0^1\left(D\setminus\Gamma^j\right)$. For $j=1,2,...$ from (14) it follows that

$$\delta_j=\sum_{i,k=1}^{n}\iint\limits_{D\setminus\Gamma'}\left(a_{ik}^{(j)}\frac{\partial u^{(j)}}{\partial x_k}\frac{\partial\varphi}{\partial x_i}\xi-u^{(j)}\frac{\partial\varphi}{\partial t}\right)dxdt$$

$$+\frac{1}{\tau}\sum_{i,k=1}^{n}\iint\limits_{D\setminus\Gamma''}\left(a_{ik}^{(j)}\frac{\partial u^{(j)}}{\partial x_k}\frac{\partial d}{\partial x_i}\xi'\left(\frac{d(z)}{\tau}\right)\varphi-u^{(j)}\xi'\left(\frac{d(z)}{\tau}\right)\frac{\partial d}{\partial t}\varphi\right)dxdt.$$

Here using the Lebesgue theorem, easily can be shown that the first term tends to

$$\sum_{i,k=1}^{n}\iint\limits_{D\setminus\Gamma^{(j)}}\left(a_{ik}^{(j)}u_{x_k}^{(j)}\varphi_{x_i}+u^{(j)}\varphi_t\right)dxdt \quad \text{as} \quad \tau\to 0.$$

For the second term, applying the known Federer's formula (see, f.e. [18]), we have:

$$\frac{1}{\tau}\sum_{i,k=1}^{n}\iint\limits_{D\setminus\Gamma'}\left(\xi'\left(\frac{d(x)}{\tau}\right)a_{ik}\frac{\partial u^{(j)}}{\partial x_k}\frac{\partial\varphi}{\partial x_i}+u^{(j)}\frac{\partial\varphi}{\partial t}\right)dxdt$$

$$=\frac{1}{\tau}\int\limits_0^{\tau}\left(\int\limits_{\{d(x)=p\}\cap(D\setminus\Gamma')}\varphi\frac{\left(\sum\limits_{i,k=1}^{n}a_{ik}^{(j)}\frac{\partial u^{(j)}}{\partial x_k}\frac{\partial d}{\partial x_i}-u^{(j)}\frac{\partial d}{\partial t}\right)}{\left(|\nabla d|^2+\left(\frac{\partial d}{\partial t}\right)^2\right)^{\frac{1}{2}}}ds_p\right)\xi'\left(\frac{p}{\tau}\right)dp$$

Do to the intermediate point theorem, there exists a $p_0 \in (0,\tau)$ such that the last expression is equal to

$$\left(\int\limits_{d(x)=p_0}\varphi\frac{\left(\sum\limits_{i,k=1}^{n}a_{ik}^{(j)}\frac{\partial u^{(j)}}{\partial x_k}\frac{\partial d}{\partial x_i}-u^{(j)}\frac{\partial d}{\partial t}\right)}{\left(|\nabla d|^2+\left(\frac{\partial d}{\partial t}\right)^2\right)^{\frac{1}{2}}}d\sigma_{p_0}\right)\frac{1}{\tau}\int\limits_0^{\tau}\xi'\left(\frac{t}{\tau}\right)dt$$

$$\longrightarrow \sum_{i,k=1}^{n} \int_{d(x)=p_0} \varphi \left(|\nabla d|^2 + \left(\frac{\partial d}{\partial t} \right)^2 \right)^{-\frac{1}{2}} \left(\sum_{i,k=1}^{n} a_{ik}^{(j)} \frac{\partial u^{(j)}}{\partial x_k} \frac{\partial d}{\partial x_i} - u^{(j)} \frac{\partial d}{\partial t} \right) d\sigma_{p_0}$$

$$\longrightarrow \int_{\partial \Gamma^{(j)}} \varphi \left(\frac{\partial u^{(j)}}{\partial \mu} - u^{(j)} N_0 \right) d\sigma \quad \text{as} \quad \tau \to 0.$$

Taking into account these limits and tending τ to zero, we find the following, in place of Green's formulae

$$\sum_{i,k=1}^{n} \iint_{D \setminus \Gamma^{(j)}} \left(\sum_{i,k=1}^{n} a_{ik}^{(j)} \frac{\partial u^{(j)}}{\partial x_k} \frac{\partial \varphi}{\partial x_i} - u^{(j)} \frac{\partial \varphi}{\partial t} \right) dxdt =$$

$$= \int_{\partial \Gamma^{(j)}} \varphi \left(\frac{\partial u^{(j)}}{\partial \mu} - u^{(j)} N_0 \right) ds + \delta_j \tag{15}$$

The belongness $u \in W^{1,2}(D)$. Using the last formulae, prove that $u \in W^{1,2}(D)$. Put $\varphi = u^{(j)} \xi \in C_0^1(D)$, where $\xi \in C_0^1(D)$ is equal to 1 in G.

Then

$$\iint_{D \setminus \Gamma^{(j)}} \left(\sum_{i,k=1}^{n} a_{ik}^{(j)} u_{x_k}^{(j)} \left(u^{(j)} \xi \right)_{x_i} - u^{(j)} \left(u^{(j)} \xi \right)_t \right) dxdt$$

$$= \int_{\partial \Gamma^{(j)}} u^{(j)} \left(\frac{\partial u^{(j)}}{\partial \mu} - u^{(j)} N_0 \right) d\sigma + \delta_j$$

$$\leq \left\| u^{(j)} \right\|_{C(D)} \int_{\partial \Gamma^{(j)}} \left(\left| \frac{\partial u^{(j)}}{\partial \mu} \right| \right) d\sigma - \int_{\partial \Gamma^{(j)}} \left(u^{(j)} \right)^2 N_0 d\sigma + \delta_j.$$

$$\leq CK \left\| u \right\|_{C(D)} \sum_{\nu} r^{n+\alpha} + \delta_j = O\left(\varepsilon\right) + \delta_j.$$

Hence

$$\iint_{D \setminus \Gamma^{(j)}} \left(\sum_{j,k=1}^{n} a_{ik}^{(j)} u_{x_k}^{(j)} u_{x_i}^{(j)} + \left(u^{(j)} \right)^2 \right) \xi dxdt$$

$$= - \sum_{j,k=1}^{n} \iint_{D \setminus \Gamma^{(j)}} \left(\sum_{j,k=1}^{n} a_{ik}^{(j)} u_{x_k}^{(j)} u^{(j)} \xi_{x_i} + \left(u^{(j)} \right)^2 \xi_t \right) dxdt + O\left(\varepsilon\right) + \delta_j \; .$$

Therefore,

$$\iint_{D \setminus \Gamma^{(j)}} \left| \nabla u^{(j)} \right|^2 \xi dxdt \leq \lambda^{-2} \left\| u^{(j)} \right\|_{C(D)} \iint_{D \setminus \Gamma^{(j)}} \left| \nabla u^{(j)} \right| \left| \nabla \xi \right| dxdt + O(\varepsilon) + \delta_j.$$

whence

$$\iint\limits_{D\backslash\Gamma''} \left|\nabla u^{(j)}\right|^2 dxdt \le \iint\limits_{D\backslash\Gamma^{(j)}} \left|\nabla u^{(j)}\right|^2 \xi dxdt \le$$

$$\le C \int\limits_{D\backslash G} \left|\nabla u^{(j)}\right| dx + O(\varepsilon) + \delta_j = O(1). \tag{16}$$

Now by using the convergence $\nabla u^{(j)} \longrightarrow \nabla u$ a.e. in $D\backslash\Gamma''$ and the equiintegrability of function sequence $\left\{\left|\nabla u^{(j)}\right|^2\right\}$, we have

$$\iint\limits_{g} \left|\nabla u^{(j)}\right|^2 dxdt \le C \iint\limits_{g} \left|\nabla u^{(j)} - \nabla u\right|^2 dxdt + C \iint\limits_{g} \left|\nabla u\right|^2 dxdt \to 0$$

as $mes_n \to 0$. Whence tending $j \to \infty$ in () we find

$$\iint\limits_{G\backslash\Gamma''} \left|\nabla u\right|^2 dxdt \le C \iint\limits_{D\backslash G} \left|\nabla u\right| dxdt.$$

The sufficiency of (6)**.** For any $\varphi \in C_0^1(D)$ from the equality () and $\varphi(x) \le \|\varphi\|_{C(D)}$, $x \in D$, we have

$$\iint\limits_{D\backslash\Gamma^{(j)}} \left(\sum_{i,k=1}^{n} a_{ik}^{(j)} u_{x_k}^{(j)} \varphi_{x_i} - u^{(j)}\varphi_t\right) dxdt \le \|\varphi\|_{C(D)} \int\limits_{\partial\Gamma^{(j)}} \left|\frac{\partial u^{(j)}}{\partial\mu}\right| d\sigma + \delta_j$$

$$\le \|\varphi\|_{C(D)} \sum_{\nu} \int\limits_{\gamma_\nu} \left|\frac{\partial u^{(j)}}{\partial\mu}\right| d\sigma + \delta_j.$$

Now using equiintegrability of expression in integral, as above, we obtain

$$\iint\limits_{D\backslash\Gamma^{(j)}} \left(\sum_{i,k=1}^{n} a_{ik} u_{x_k} \varphi_{x_i} - u\varphi_t\right) dxdt \le CK \|\varphi\|_{C(D)} \sum_{\nu} r_\nu^{n+\alpha} + \delta_j, \tag{17}$$

whence

$$\iint\limits_{D\backslash\Gamma^{(j)}} \left(\sum_{i,k=1}^{n} a_{ik} u_{x_k} \varphi_{x_i} - u\varphi_t\right) dxdt \le O(\varepsilon) + \delta_j. \tag{18}$$

Since $C_0^1(D)$ is dense in $\dot{W}^{1,2}(D)$ and $u \in W_{loc}^{1,2}(D\backslash E)$, the inequality is true for all functions $\varphi \in \dot{W}^{1,2}(D)$. Further, from (17), (18) and the belongness $u \in W^{1,2}(D)$ it follows that

$$\iint\limits_{D\backslash E'} \left(\sum_{i,k=1}^{n} a_{ik} u_{x_k} \varphi_{x_i} - u\varphi_t\right) dxdt = O(\varepsilon).$$

Since ε, η are arbitrary, we have

$$\iint_D \left(\sum_{i,k=1}^n a_{ik} u_{x_k} \varphi_{x_i} - u\varphi_t \right) dxdt = 0,$$

i.e. the function $u \in W^{1,2}(D)$ is a solution (1) in the domain D.

This complete the sufficiency of Theorem 2.2.

Proof of necessity of Theorem 1.1. Let $\Lambda^{\frac{(n+\alpha)}{(n+2)}}(E) > 0$ for a compact subset $E \subset D$. We will show that the set E is removable. It follows from the Lemma 1.2 that there exists a measure μ with positive mass and a support on the set E such that

$$\mu\left(C(z,r)\right) \le Cr^{n+\alpha} \tag{19}$$

for any cylinder $C(z,r)$.

Let $u(x,t) = \int\limits_{C(z_0,r)} \Gamma_{y,\tau}(x,t) d\mu(y,\tau)$, where $\Gamma_{y,\tau}(x,t)$ is a fundamental solution of the equation (1). Then $\nabla_x u(x,t) = \int\limits_{C(z_0,r)} \nabla_x \Gamma_{y,\tau}(x,t) d\mu(y,\tau)$. Therefore, by virtue of summability $\Gamma_{y,\tau},\ \nabla_x \Gamma_{y,\tau} \in L^1\left[t_0 - r^2, t_0; L^1(\Omega)\right]$, we have

$$\int\limits_{t_0-r^2}^{t_0} \left(\int\limits_{B(x_0,r)} \left(\sum_{i,k=1}^n a_{ik}(x,t) u_{x_i} \varphi_{x_k} \right) dx \right) dt - \iint\limits_{C(z_0,r)} u\varphi_t dxdt = \int\limits_{C(z_0,r)} \varphi d\mu$$

for any $\varphi \in C_0^\infty(C(z_0,r))$.

Let $u^{(j)}$ be the the above constructed averaging of u with a smooth kernel. Then by virtue of $u, u_x \in L^1(C(z_0,r))$ we will have $u^{(j)} \longrightarrow u$ and $\nabla_x u^{(j)} \longrightarrow \nabla_x u$ as $j \longrightarrow \infty$ in the norm of the space $L^1\left(C(z_0,r)\right)$. Therefore,

$$\int\limits_{t_0-r^2}^{t_0} \left(\int\limits_{B(x_0,r)} \left(\sum_{i,k=1}^n a_{ik}(x,t) u^{(j)}_{x_i} \varphi_{x_k} \right) dx \right) dt - \iint\limits_{C(z_0,r)} u^{(j)}\varphi_t dxdt = \int\limits_{C(z_0,r)} \varphi d\mu + \delta_j.$$

Let $h^{(j)}$ be the solution of the equation (1) which coincides with $u^{(j)}$ on the proper boundary of the cylinder $C(z_0,r)$. For the existence of a solution of this problem see for example [18], [16], [28]. Then

$$\int\limits_{t_0-r^2}^{t_0} \left(\int\limits_{B(x_0,r)} \left(\sum_{i,k=1}^n a_{ik}(x,t) h^{(j)}_{x_i} \varphi_{x_k} \right) dx \right) dt - \iint\limits_{C(z_0,r)} h^{(j)}\varphi_t dxdt = \int\limits_{C(z_0,r)} \varphi d\mu + \delta_j$$

and for the function $\nu^{(j)} = u^{(j)} - h^{(j)}$ we have the identity

$$\int\limits_{t_0-r^2}^{t_0} \left(\int\limits_{B(x_0,r)} \left(\sum_{i,k=1}^n a_{ik}(x,t) \nu^{(j)}_{x_i} \varphi_{x_k} \right) dx \right) dt - \iint\limits_{C(z_0,r)} \nu^{(j)}\varphi_t dxdt = \int\limits_{C(z_0,r)} \varphi d\mu + \delta_j$$

Assuming $\varphi = \nu^{(j)}$ we have

$$\int_{t_0-r^2}^{t_0}\left(\int_{B(x_0,r)}\left(\sum_{i,k=1}^{n} a_{ik}(x,t)\nu^{(j)}_{x_i}\nu^{(j)}_{x_k}\right)dx\right)dt - \iint_{C(z_0,r)} \nu^{(j)}\nu^{(j)}_t dxdt = \int_{C(z_0,r)} \nu^{(j)}d\mu + \delta_j$$

Whence for sufficiently large j we obtain

$$C_1\iint_{C(z_0,r)}\sum_{k=1}^{n}\left(\nu^{(j)}_{x_k}\right)^2 dxdt + \int_{\{x\in B(x_0,r),t=t_0\}}\left(\nu^{(j)}\right)^2 dx$$

$$\leq 2\left(\sup_{C(z_0,r)}\left|\nu^{(j)}\right|\right)\mu\left(C(z_0,r)\right) + \delta_j$$

Further, by virtue of the Sobolev inequality for n -dimensional balls and the fact that $ess\sup\limits_{C(z_0,r)}\left|\nu^{(j)}\right| \leq 2ess\sup\limits_{C(z_0,r)}|\nu|$, for sufficiently large j we obtain

$$C_1\iint_{C(z_0,r)}\left(\nu^{(j)}\right)^2 dxdt \leq 4r^2\mu\left(C\left(z_0,r\right)\right)\left(\sup_{C(z_0,r)}|\nu|\right).$$

Whence by Hölder inequality and the estimate (19) we have

$$\iint_{C(z_0,r)}\left|\nu^{(j)}\right| dxdt \leq 2C_2 r^{n+2+\frac{\alpha}{2}}\left(\sup_{C(z_0,r)}|\nu|\right)^{\frac{1}{2}}. \qquad (20)$$

Now prove the following estimate to finish our arguments.

Estimation of $\left(\sup\limits_{C(z_0,r)}|\nu|\right)$. Let $\forall(x,t)\in C(z_0,r)$ be fixed. According to [1], [3], for ν we have the estimate

$$|\nu(x,t)| = \left|\int_{C(z_0,r)} g_{y,\tau}(x,t)d\mu(y,\tau)\right| \leq \int_{C(z_0,r)} G_{y,\tau}(x,t)d\mu(y,\tau),$$

where the function $G_{y,\tau}(x,t)$ is a fundamental solution of the equation (1) and by [3], $G_{y,\tau}(x,t) \leq \Gamma_{y,\tau}(x,t) = \frac{C_1}{(t-\tau)^{\frac{n}{2}}}e^{-C_2\frac{|x-y|}{(t-\tau)}}$. Thus

$$|\nu(x,t)| \leq C_1\int_0^\infty \mu\left\{(y,\tau) : \Gamma_{y,\tau}(x,t) > a\right\}da$$

$$= C_1\int_0^{r^{-n}}\mu\left\{\Gamma \geq a\right\}da + C_1\int_{r^{-n}}^{\infty}\mu\left\{\Gamma \geq a\right\}da = C_1\left(I_1 + I_2\right)$$

It is not difficult to understand from the geometrical standpoint that the set $\{(y,\tau)\in C(z_0,r):\Gamma_{y,\tau}(x,t)>a\}$ lies inside the cylinder $C\left(z,C_3a^{-\frac{1}{n}}\right)$ for some C_3 and for all $a\in(r^{-n},\infty)$. Therefore by using $\mu\left(\Gamma\geq a\right)\leq Ca^{-\frac{n+\alpha}{n}}$, we have

$$I_2\leq\int\limits_{r^{-n}}^{\infty}a^{-\frac{n+\alpha}{n}}da=C(n,\alpha)r^{\alpha}.$$

For $0<a<r^{-n}$ we apply the estimate

$$\mu\left\{\Gamma\geq a\right\}\leq\mu\left(C(z_0,r)\right)=Cr^{n+\alpha}.$$

Then for the integral I_1 we have

$$I_1\leq\int\limits_{0}^{r^{-n}}a^{n+\alpha}da=Cr^{\alpha}.$$

As a result we obtain the estimate

$$|\nu(x,t)|\leq Cr^{\alpha};\ \ C=C(n,\alpha,\lambda) \tag{21}$$

Necessity of the condition. (6) We obtain the proof of the estimate (21) using there the estimate $\int\limits_{C(z_0,r)}\Gamma_{y,\tau}(x,t)d\mu(y,\tau)\leq Cr^{\alpha}$ for $(x,t)\in C(z_0,r)$, which means the uniform convergence of the integral in the representation of the function

$$u(x,t)=\int\limits_{\mathbb{R}^{n+1}}\Gamma_{y,\tau}(x,t)d\mu(y,\tau).$$

The last means that sequence of approximations $u^{(j)}$ really converges to the function u uniformly on any compact set in $\mathbb{R}^{n+1}$. In view of this fact and the estimates (21), (20), passing to limit as $j\to\infty$, we obtain

$$\iint\limits_{C(z_0,r)}\nu(x,t)dxdt\leq Cr^{n+2+\alpha}.$$

Now,

$$\iint\limits_{C(z_0,r)}\left|u-(u)_{z_0,r}\right|dxdt\leq\iint\limits_{C(z_0,r)}|u-h|\,dxdt+\iint\limits_{C(z_0,r)}\left|h-(h)_{z_0,r}\right|dxdt$$

$$\iint\limits_{C(z_0,r)}\left|(h)_{z_0,r}-(u)_{z_0,r}\right|dxdt\leq2\iint\limits_{C(z_0,r)}|u-h|\,dxdt+\iint\limits_{C(z_0,r)}\left|h-(h)_{z_0,r}\right|dxdt$$

$$\leq2\iint\limits_{C(z_0,r)}\nu(x,t)dxdt+\iint\limits_{C(z_0,r)}\left|h-(h)_{z_0,r}\right|dxdt\leq Cr^{n+2+\alpha}$$

whence by virtue of $\alpha \le \kappa$ we have

$$\frac{1}{|C(z_0,r)|}\iint\limits_{C(z_0,r)} \left|u-(u)_{z_0,r}\right| dxdt \le Cr^{\alpha}.$$

Further, by virtue of Lemma l.1, we obtain

$$u\in H^{\alpha,\frac{\alpha}{2}},$$

i.e. the set E is not removable in the class $H^{\alpha,\frac{\alpha}{2}}$, which proves the necessity of the condition (6). Here, for the function h we have used the estimate of Hölder norm (7) for solutions of the equation (1) in order to have the inequality

$$\iint\limits_{C(z_0,r)} \left|h-(h)_{z_0,r}\right| dxdt \le Cr^{n+2+\kappa}.$$

This complete the proof of Theorem 1.1.

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Received February 15, 2011; Revised April 20, 2011