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ON COMPLETENESS OF THE SYSTEM OF EIGEN AND ADJOINT VECTORS OF QUADRATIC OPERATOR BUNDLES

Abstract

In the paper, sufficient conditions on the coefficients of the quadratic operator bundles of elliptic type are obtained. When these conditions are fulfilled, the system of eigen and adjoint vectors is complete in the trace space of the solutions of appropriate operator-differential equations. The theorems on the completeness of decreasing elementary solutions in the space of regular solutions of a homogeneous equation are also obtained.

Let H be a separable Hilbert space, A be a positive-definite self-adjoint operator in H. Denote by H_r a scale of Hilbert spaces generated by the operator A. Remined that $H_\gamma = D\left(A^\gamma\right)$ is a scalar product in H_γ, and is given by the equality $(x,y)_\gamma = (A^\gamma x,\ A^\gamma y)$ for the elements $x,y \in H_\gamma \quad (\gamma \geq 0)$. For $\gamma = 0$ assume that $H_0 = H,\ (x,y)_0 = (x,y)$.

Denote by $L_2\left(R_+;H\right)$ a Hilbert space of the functions $f(t)$ determined almost everywhere in $R_+ = (0,\infty)$, measurable, quadratically integrable in the Bochner sense, with the values in H with the norm

$$\|f\|_{L_2(R_+;H)} = \left(\int\limits_0^\infty \|f(t)\|^2\,dt\right)^{1/2} < +\infty.$$

In the sequel, $L(X;Y)$ denotes the space of linear bounded operators acting from X to Y. Following the monograph [1] define the space of the functions

$$W_2^2(R_+;H) = \left\{u\colon u'' \in L_2(R_+;H),\quad A^2u \in L_2(R_+;H)\right\}$$

with the norm

$$\|u\|_{W_2^2(R_+;H)} = \left(\left\|A^2u\right\|^2_{L_2(R_+;H)} + \left\|u''\right\|^2_{L_2(R_+;H)}\right)^{1/2}.$$

Here and in the sequel, the derivatives are understood in the sense of distributions theory. Define also the following subspace in $W_2^2(R_+;H)$

$$\overset{0}{W}{}_2^2(R_+;H) = \left\{u{:}u \in W_2^2(R_+;H),\ u'(0) = 0\right\}$$

For $R = (-\infty,\infty)$ the spaces $L_2(R,H)$ and $W_2^2(R,H)$ are determined in the same way.

Consider the following quadratic operator bundle

$$P(\lambda) = \lambda^2 E + \lambda(pA + A_1) + qA^2, \tag{1}$$

where λ is a spectral parameter, and the operator coefficients satisfy the following conditions:

1) $p \in R = (-\infty, \infty)$, $q < 0$;

2) A is a positive-definite self-adjoint operator with a completely continuous inverse A^{-1};

3) The operator $B_1 = A_1 A^{-1}$ is bounded in H.

In the present paper we'll show sufficient conditions on the coefficients of operator bundle (1), that provide completeness of the chain of eigen and adjoint vectors in the space $H_{1/2}$ responding to some boundary value problem. Note that the operator bundle (1) was investigated by many authors for $p = 0$, $q = -1$ (see [2-9], and in the general form in [10].

Combine bundle (1) with the following boundary value problem

$$P\left(\frac{d}{dt}\right)u(t) = 0, \tag{2}$$

$$u'(0) = \varphi, \quad \varphi \in H_{1/2}. \tag{3}$$

Definition 1. *If for any* $\varphi \in H_{1/2}$ *there exists a vector-function* $u \in W_2^2(R_+; H)$ *that satisfies equation (2) almost everywhere in* R_+*, boundary conditions (3) in the sense of the convergence*

$$\lim_{t \to +0} \left\| u'(t) - \varphi \right\|_{1/2} = 0,$$

and it holds the estimation

$$\|u\|_{W_2^2(R_+;H)} \leq const \, \|\varphi\|_{1/2},$$

then problem (2), (3) is said to be regularly solvable, and $u(t)$ *its regular solution.*

Definition 2. *Let* λ_i *be an eigen value of the bundle* $P(\lambda)$*, and* $\varphi_{i,0}^{(l)}$ *be an appropriate eigen vector, i.e.* $\varphi_{i,0}^{(l)} \neq 0$ *and* $P(\lambda_i)\varphi_{i,0}^{(l)} = 0$*. Then the system* $\varphi_{i,0}^{(l)}, \varphi_{i,1}^{(l)}, ..., \varphi_{i,m_{il}}^{(l)}$, $l = \overline{1, q_i}$ *satisfying the relations*

$$P(\lambda_i)\varphi_{i,k}^{(l)} + \frac{\partial P(\lambda_i)}{\partial \lambda}\varphi_{i,k-1}^{(l)} + 2\varphi_{i,k-2}^{(l)} = 0, \quad k = \overline{0, m_{il}}, \quad \left(\varphi_{i,-1}^{(l)}, \ \varphi_{i,-2}^{(l)} = 0\right)$$

is called a system of eigen and adjoint vectors of the bundle $P(\lambda)$ *responding to the eigen values* λ_i.

Definition 3. *Let* $\varphi_{i,0}^{(l)}, \varphi_{i,1}^{(l)}, ..., \varphi_{i,m_{il}}^{(l)}$, $i = \overline{1, q_i}$ *be a chain of eigen and adjoint vectors responding to the eigen value* λ_i*, moreover* $\operatorname{Re}\lambda_i < 0$*. Then the vector functions*

$$u_{i,h}^{l}(t) = e^{\lambda_i t}\left(\varphi_{i,h}^{(l)} + \varphi_{i,h-1}^{(l)}\frac{t}{1!} + ... + \varphi_{i,0}^{(l)}\frac{t^h}{h!}\right), \quad h = \overline{0, m_{il}}, \quad l = \overline{1, q_i}$$

satisfy the equation $P\left(\frac{d}{dt}\right)u(t) = 0$ *and are called the decreasing elementary solutions of the homogeneous equation* $P\left(\frac{d}{dt}\right)u(t) = 0$.

Let $\varphi_{i,h}^{(l,1)} = \dfrac{d}{dt} u_{i,h}^{(l)}(t)\,|_{t=0}$. Obviously, the system $\varphi_{i,h}^{(l,1)}$ responds to boundary value problem (2), (3). Denote

$$K(\Pi_-) = \left\{ \varphi_{i,h}^{(l,1)}, \quad h = \overline{0, m_i},\ l = \overline{1, q_i},\ i = \overline{1, \infty} \right\}.$$

Note that while fulfilling conditions 1)-3), the operator bundle $P(\lambda)$ has only a discrete spectrum with a unique limit point at infinity. Indeed,

$$P(\lambda) = \lambda^2 E + \lambda(qA + A_1) + qA^2 =$$

$$= q(\lambda^2 q^{-1} A^{-2} + \lambda(pq^{-1} E + q^{-1} A_1 A^{-1}) A^{-1} + E) A^2 = q(E + L(\lambda)) A^{-2}, \qquad (4)$$

where

$$L(\lambda) = \lambda^2 q^{-1} A^{-2} + \lambda(q^{-1} p A^{-1} + q^{-1} B_1 A^{-1}).$$

Since A^{-1} is a completely continuous operator, then $L(\lambda)$ is a completely continuous operator for any $\lambda \in \mathbb{C}$. $E + L(0) = E$ is invertible, then by the Keldysh lemma [11] the operator bundle $E + L(\lambda)$ and simultaneously, the bundle $P(\lambda)$ has a discrete spectrum with a unique limit point at infinity. If additionally we require $A^{-1} \in \sigma_\rho$, $(0 < \rho < \infty)$, then it holds

Lemma 1. *Let conditions 1)-3) be fulfilled, and $A^{-1} \in \sigma_\rho$. Then the operator function $A^2 P^{-1}(\lambda)$ is represented in the form of the ratio of two entire functions of order ρ and of minimal type for order ρ.*

Proof. It is seen from equality (4) that $A^2 P^{-1}(\lambda) = \dfrac{1}{q}(E + L(\lambda))^{-1}$, and the coefficients of the operator bundle $L(\lambda)$

$$q^{-1} A^{-2} \in \sigma_{\rho/2},\ q^{-1} p A^{-1} + q^{-1} B_1 A^{-1} \in \sigma_\rho.$$

Then by the Keldysh lemma [11], $(E + L(\lambda))^{-1}$, consequently $A^2 P^{-1}(\lambda)$ satisfy the statement of the lemma.

The lemma is proved.

Lemma 2. *Let conditions 1)-3) be fulfilled. Then while fulfilling the condition $\|B_1\| < \sqrt{p^2 + 4|q|}$, the operator bundle $P(\lambda)$ is invertible in the sectors*

$$S_{\pm(\frac{\pi}{2} \pm \theta)} = \left\{ \lambda : \lambda = r e^{\pm i(\frac{\pi}{2} \pm \alpha)},\ r > 0,\ |\alpha| \le \theta \right\}$$

for small $\theta > 0$, and in these sectors it holds the estimation

$$\left\| \lambda^2 P^{-1}(\lambda) \right\| + \left\| \lambda A P^{-1}(\lambda) \right\| + \left\| \lambda^2 P^{-1}(\lambda) \right\| \le const \qquad (5)$$

Proof. From conditions 1) and 2) it follows that the operator bundle $P_0(\lambda) = \lambda^2 E + \lambda p A + q A^2$ is investible on the imaginary axis, since $P_0(\lambda) =$
$= (\lambda E - \omega_1 A)(\lambda E - \omega_2 A)$, where

$$\omega_1 = -\frac{1}{2}\left(p + \sqrt{p^2 + 4|q|}\right) < 0,\quad \omega_2 = -\frac{1}{2}\left(p - \sqrt{p^2 + 4|q|}\right) > 0.$$

Then from the equality

$$P(\lambda) \equiv P_0(\lambda) + P_1(\lambda) = (E - P_1(\lambda)P^{-1}(\lambda))P_0(\lambda)$$

it follows that if for $\lambda = i\xi$ the inequality $\left\|P_1(\lambda)P_0^{-1}(\lambda)\right\| \leq \alpha_1 < 1$ is fulfilled, then the operator bundle $P(\lambda)$ is also invertible. Obviously,

$$\left\|P_i(i\xi)P_0^{-1}(i\xi)\right\| = \left\|(i\xi A_1)P_0^{-1}(i\xi)\right\| \leq \|B_1\| \left\|\xi A P_0^{-1}(i\xi)\right\|. \tag{6}$$

Further, using the spectral expansions of the operator A, we get

$$\left\|\xi A P_0^{-1}(i\xi)\right\| = \sup_{\mu\in\sigma(A)} \left|\xi\mu(-\xi^2 + i\xi p\mu + q\mu^2)^{-1}\right| \leq$$

$$\leq \sup_{\mu\geq\sigma_0} \left(\left(\xi^2 - q\mu^2\right)^2 + p^2\mu^2\right)^{-1/2} \leq \sup_{\tau\geq 0} \left|\tau\left(\left(\tau^2 - q\right)^2 + q^2\tau^2\right)\right|^{-1/2} =$$

$$= \left(p^2 + 4\,|q|\right)^{-1/2} \quad (q < 0). \tag{7}$$

Taking into account inequality (7) in (6) we get

$$\left\|P_1(i\xi)P_0^{-1}(i\xi)\right\| \leq \|B_1\| \left(p^2 + 4\,|q|\right)^{-1/2} = \alpha_1 < 1,$$

i.e. the operator bundle $P(\lambda)$ is invertible on the imaginary axis. On the other hand, on the imaginary

$$\left\|A^2 P_0^{-1}(i\xi)\right\| = \sup_{\mu\geq\sigma(A)} \left\|\mu^2 P_0^{-1}(i\xi)\right\| \leq$$

$$\leq \sup_{\tau\geq 0} \left(\left(\tau^2 - q\right)^2 + p^2\tau\right)^{-1/2} = \frac{1}{|q|}. \tag{8}$$

Similarly we have $(q < 0)$

$$\left\|\xi^2 P^{-1}(\xi)\right\| \leq \sup_{\mu\geq\sigma} \left\|\xi^2\left(\left(\xi^2 - q\mu^2\right)^2 + p^2\mu^2\right)^{-1/2}\right\| \leq 1. \tag{9}$$

From inequalities (7)-(9) it follows that inequality (5) is valid on the imaginary axis. Further, for $\theta > 0$, $i\xi \in R$ and $\lambda = i\xi e^{i\theta}, |\alpha| \leq \theta$ we have

$$P(\lambda) = (E + (i\xi)^2 P^{-1}(i\xi)(e^{2i\alpha} - 1) + i\xi(p + A_1)P^{-1}(i\xi)(e^{i\alpha} - 1))P(i\xi).$$

Then for rather small θ allowing for inequality (5) for $\lambda = i\xi$ we get that $P(\lambda)$ is invertible and inequality (5) is fulfilled in the sectors $S_{\pm\left(\frac{\pi}{2}\pm\theta\right)}$ for small θ.

The lemma is proved.

Lemma 3. *The problem*

$$P_0 u \equiv P_0\left(\frac{d}{dt}\right)u = \frac{d^2u}{dt^2} + pA\frac{du}{dt} + qA^2u = 0, \tag{10}$$

$$u'(0) = \varphi, \tag{11}$$

is regularly solvable.

Proof. Since the general regular equation $P\left(\frac{d}{dt}\right)u(t)=0$ in $W_2^2(R;H)$ is of the form $u_0(t)=e^{-\frac{1}{2}\left(p+\sqrt{p^2+4|q|}\right)tA}\psi$, where $\psi\in H_{3/2}$, it follows from condition (11) that $-\frac{1}{2}\left(p+\sqrt{p^2+4\left|q\right|}\right)A\psi=\varphi$ or

$$\psi=-2\left(p+\sqrt{p^2+4\left|q\right|}\right)^{-1/2}A^{-1}\varphi\in H_{3/2}$$

and $u_0(t)\in W_2^2(R_+;H)$, $\left\|u_0\left(t\right)\right\|_{W_2^2(R_+;H)}\leq const\left\|\varphi\right\|_{1/2}$.

The lemma is proved.

Remark 1. Using lemma 3, we can reduce the regular solvability of problem (2), (3) to the solution of the equation $P\omega=g$ in the space $\overset{0}{W}{}_2^2(R_+;H)$. Indeed, we'll look for the solution of problem (2), (3) in the form $u(t)=\omega(t)+u_0(t)$, where $u_0(t)$ is a regular solution of problem (10), (11). Then it is obvious that $u'(0)=\omega'(0)+u_0'(0)$, i.e. $\omega'(0)=0$ from the equality

$$Pu=P_0u+P_1u=P_0\left(\omega+u_0\right)+P_1\left(\omega+u_0\right)=$$

$$=\left(P_0+P_1\right)\omega+P_1u_0=P\omega+P_1u_0=0$$

it follows that for ω we get the equation

$$P\omega=-P_1u_0,\quad \omega\in\overset{0}{W}{}_2^2(R_+;H),\quad \left(\omega'(0)=0\right).$$

Obviously, for $g=-P_1u_0$

$$\left\|g\right\|_{L_2(R_+;H)}=\left\|A_1\frac{du_0}{dt}\right\|_{L_2(R_+;H)}\leq\left\|B_1\right\|\left\|A\frac{du_0}{dt}\right\|_{L_2(R_+;H)}\leq$$

$$\leq const\left\|u_0\right\|_{W_2^2(R_+;H)}\leq const\left\|\varphi\right\|_{1/2}.$$

Thus, for determining $\omega\in\overset{0}{W}{}_2^2(R_+;H)$ we get the following boundary value problem

$$P_0\left(\frac{d}{dt}\right)\omega(t)=g(t),\tag{12}$$

$$\omega'(0)=0,\tag{13}$$

It holds

Theorem 1. *The operator P_0 isomorphically maps the space $\overset{0}{W}{}_2^2(R_+;H)$ onto $L_2(R_+;H)$.*

Proof. It follows from lemma 3 that $KerP_0=\{0\}$. Prove that $\operatorname{Im}P_0u=$ $=L_2(R_+;H)$. It is easy to see that for any $g\in L_2(R_+;H)$ the function

$$u_1(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(-\xi^2E+i\xi pA+A^2\right)^{-1}\int_0^{\infty}g(s)e^{i\xi(s-t)}dsd\xi,$$

belongs to the space $W_2^2(R_+;H)$. Indeed, from the Plancherel theorem it follows that

$$\|u\|^2_{W_2^2(R,H)} = \left\|A^2u\right\|^2_{L_2(R_+,H)} + \left\|u''\right\|^2_{L_2(R_+,H)} =$$
$$= \left\|\xi^2\widehat{u}(\xi)\right\|^2_{L_2(R_+,H)} + \left\|A^2\widehat{u}(\xi)\right\|^2_{L_2(R_+,H)} =$$
$$= \left\|\xi^2P_0^{-1}(i\xi)\,\widehat{g}(\xi)\right\|^2_{L_2(R_+,H)} + \left\|A^2P_0^{-1}(i\xi)\,\widehat{g}(\xi)\right\|^2_{L_2(R_+,H)} \le$$
$$\le \sup_{\xi\in R}\left\|\xi^2P_0^{-1}(i\xi)\right\|\|g\|_{L_2(R_+,H)} + \sup_{\xi\in R}\left\|A^2P_0^{-1}(i\xi)\right\|\|g\|_{L_2(R_+,H)}$$

Taking into account inequalities (8) and (9), we get $u(t) \in W_2^2(R_+;H)$. Here $\widehat{u}(\xi)$ and $\widehat{g}(\xi)$ are the Fourier transformations of the functions $u(t)$ and $\widehat{g}(\xi)$, where $\widehat{g}(t) = g(t)$ for $t > 0$ and $\widehat{g}(t) = 0$ for $t < 0$. Further, it is obvious that $u(t)$ satisfies the equation $P\left(\frac{d}{dt}\right)u(t) = g(t)$ for $t \in R_+$ almost everywhere. Denote by $\xi(t)$ the contraction of the function $u(t)$ on $[0,\infty)$. Obviously, [1], $\xi(t) \in W_2^2(R_+;H)$ and $\xi(0) \in H_{3/2}$, $\xi'(0) \in H_{1/2}$. Now, we'll look for the solution of $\omega(t)$ in the form $\omega(t) = \xi(t) + e^{-\frac{1}{2}\left(p+\sqrt{p^2+4|q|}\right)At}\psi$, where ψ is an unknown vector from $H_{3/2}$. Then, for determining ψ from (13), we get $\xi'(0) = -\frac{P}{2} - \frac{\sqrt{p^2+4|q|}}{2}A\psi$ or $\psi = \left(-\frac{p}{2} - \frac{\sqrt{p^2+4|q|}}{2}\right)^{-1} A^{-1}\xi'(0)$. Since $\xi'(0) \in H_{1/2}$, then $\psi \in H_{3/2}$. Thus, $\omega(t) \in \overset{0}{W}{}_2^2(R_+;H)$. Consequently, $\operatorname{Im} P_0 =$
$= L_2(R_+;H)$. Allowing for the Banach theorem on the inverse operator, from the inequality $\left\|P_0\left(\frac{d}{dt}\right)u\right\| \le const\,\|u\|_{W_2^2(R;H)}$ we get that operator P_0 has an inverse operator acting from $\overset{0}{W}{}_2^2(R_+;H)$ to $L_2(R_+;H)$.

The theorem is proved.

From theorem 1 it follows that the norms $\|P_0u\|_{L_2(R_+;H)}$ and $\|u\|_{W_2^2(R_+;H)}$ are equivalent in the space $\overset{0}{W}{}_2^2(R_+;H)$. Therefore, from the theorem on intermediate derivatives it follows that the number

$$N_1 = \sup_{0\ne u\in \overset{0}{W}{}_2^2(R_+;H)} \frac{\|Au'\|_{L_2(R_+;H)}}{\|P_0u\|_{L_2(R_+;H)}}. \tag{14}$$

is finite.

Theorem 2. *The number N_1 is determined in the following way*

$$N_1 = \begin{cases} \frac{1}{2}|q|^{-1/2}, & p \le 0 \\ \left(p^2+4|q|\right)^{-1/2}, \quad , & p \ge 0 \end{cases}.$$

Proof. Following the paper [13], consider the operator bundle for $s \in$
$\in \left[0, p^2+4|q|\right)$ in H_2

$$F_1(\lambda; s; A) = \lambda^2E + c_1(s)\lambda A + c_2(s)A^2,$$

where $c_1(s)=\sqrt{p^2+4|q|-s}$, $c_2(s)=|q|$. It is easily verified that for $s\in$ $\in\left[0,\ p^2+4|q|\right)$ the operator bundle is represented in the form

$$F_1(\lambda;s;A)=(\lambda E-\omega_1(s)A)(\lambda E-\omega_2(s)A),$$

where $\operatorname{Re}\omega_1(s)<0$, $\operatorname{Re}\omega_2(s)<0$ for $s\in\left[0,\ p^2+4|q|\right)$. Further, for $u\in$ $\in\overset{0}{W}{}_2^2(R_+;H)$ we easily verify the validity of the equality [13]

$$\left\|F_1\left(\frac{d}{dt};s;A\right)u\right\|^2_{L_2(R_+;H)}+Q(s;\psi)=\|P_0u\|^2_{L_2(R_+;H)}-s\left\|Au'\right\|^2_{L_2(R_+;H)}, \tag{16}$$

where $\psi\in H_{3/2}$, and

$$Q(s;\psi)=(c_1(s)c_2(s)-pq)\|\psi\|^2=\left(|q|\sqrt{p^2+4|q|-s}-pq\right)\|\psi\|^2.$$

From the results of the paper [13] it follows that $N_1=\left(p^2+4|q|\right)^{-1/2}$ if and onli if $Q(s;\psi)>0$ for $s\in\left(0,p^2+4|q|\right)$. Hence, it is seen that

$$|q|\sqrt{p^2+4|q|-s}+p|q|>0$$

for $p\in(0,\infty)$. If $p\in(-\infty,0]$, the number N_1^{-2} will be a solution of the equation

$$|q|\sqrt{p^2+4|q|-s}=|p||q|,$$

i.e. $N_1^{-2}=4|q|$. Thus, $N_1=2^{-1}|q|^{-1/2}$ for $p\le 0$.

The theorem is proved. From theorem 2 we get:

Theorem 3. *Let conditions 1)-3) be fulfilled, and*

$$\|B_1\|<N_1^{-1}=\begin{cases}2|q|^{1/2}, & p\le 0\\ \sqrt{p^2+4|q|}, & p\ge 0\end{cases} \tag{17}$$

Then the operator P isomorphically maps the space $\overset{0}{W}{}_2^2(R_+;H)$ onto $L_2(R_+;H)$.

Proof. By theorem 1, $P_0^{-1}\colon L_2(R_+;H)\to\overset{0}{W}{}_2^2(R_+;H)$ is an isomorphishm. Denote $\upsilon=P_0\omega$. Then for υ we get the equation $\upsilon+P_1P_0^{-1}\upsilon=f$ in the space $L_2(R_+;H)$. Since

$$\left\|P_1P_0^{-1}\upsilon\right\|_{L_2(R_+;H)}=\|P_1\omega\|_{L_2(R_+;H)}\le\|B_1\|\left\|A\omega'\right\|_{L_2(R_+;H)}\le$$

$$\le\|B_1\|N_1\|P_0\omega\|_{L_2(R_+;H)}=N_1\|B_1\|\|\upsilon\|=\alpha\|\upsilon\|_{L_2(R_+;H)},\quad \alpha<1,$$

the operator $E+P_1P_0^{-1}$ is invertible in the space $L_2(R_+;H)$ and $\upsilon=\left(E+P_1P_0^{-1}\right)^{-1}f$. Hence $u=P_0^{-1}\left(E+P_1P_0^{-1}\right)^{-1}f$.

The theorem is proved. The following theorem follows from theorem 2 and remark 1:

Theorem 4. *Let the conditions of theorem 3 be fulfilled. Then problem (2)-(3) is regularly solvable.*

Now prove a theorem on the completeness of the system $K(\Pi_-)$.

Theorem 5. *Let conditions 1)-3) and inequality (17) be fulfilled. Then when one of the following conditions is fulfilled*

a) $A^{-1} \in \sigma_\rho \quad (0 < \rho \le 1)$;

b) $A^{-1} \in \sigma_\rho \quad (0 < \rho < \infty), \quad B_1 \in \sigma_\infty(H)$.

Then system $K(\Pi_-)$ is complete in $H_{1/2}$.

Proof. By theorem 4, problem (2), (3) is regularly solvable. Then for any $\varphi \in H_{1/2}$, there exists a vector function $u(t) \in \overset{0}{W}{}_2^2(R_+; H)$ that satisfies equation (2) almost everywhere in R_+. Denote by $\widetilde{u}(\lambda)$ its Laplace transform. Then

$$P(\lambda)\widetilde{u}(\lambda) = Q(\lambda), \; Q(\lambda) = \varphi + \lambda u(0) + (pA + A_1)\,u(0).$$

Hence we have

$$u(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \widetilde{u}(\lambda)\, e^{\lambda t} d\lambda = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} P^{-1}(\lambda) Q(\lambda) d\lambda.$$

From estimations (5) it follows that on the sectors $S_{\pm\left(\frac{\pi}{2}+\theta\right)}$ for small $\theta > 0$ it holds the estimation $\left\|P^{-1}(\lambda)\right\| \le c\,|\lambda|^{-2}$, for large $|\lambda|$. Therefore,

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} P^{-1}(\lambda)\, Q_1(\lambda)\, e^{\lambda t} d\lambda,$$

where $\Gamma_{\pm\theta} = \left\{\lambda : \lambda e^{\pm i(\pi/2+\theta)}, \; r > 0\right\}$.

Suppose that the system $K(\Pi_-)$ is not a complete system in the space $H_{1/2}$. Then there exists a vector $\varphi \in H_{1/2}, \; \varphi \ne 0$ and $\left(\varphi, \varphi_{i,h}^{(l,1)}\right)_{1/2} = 0 \quad i = \overline{1,\infty}, \; l = \\ = \overline{1, q_i}$. Then from the Keldysh lemma [11] on the expansion of the resolvent in the vicinity of the eigen values, it follows that the vector-function is

$$R(\lambda) = \left(A^{1/2} P^{-1}\left(\overline{\lambda}\right)\right)^* \lambda A^{1/2}\varphi$$

holomorphic in the left half-plane. Since $\widetilde{u}(\lambda)$ is a Laplace transform of the functions from the space $W_2^2(R_+; H)$, it is holomorphic in the right half-plane as well, and has boundary value almost everywhere on the imaginary axis. Thus, for $t > 0$

$$\left(u'(t), \varphi\right)_{1/2} = \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} \left(A^{1/2} P^{-1}(\lambda)\,\lambda Q(\lambda), A^{1/2}\varphi\right) e^{\lambda t} d\lambda =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} \left(Q(\lambda), (\overline{\lambda} A^{1/2} P^{-1}(\lambda)\right)^* A^{1/2}\varphi) e^{\lambda t} d\lambda = \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} S(\lambda) e^{\lambda t} d\lambda, \qquad (18)$$

where $S(\lambda) = \left(Q(\lambda), R\left(\overline{\lambda}\right)\right)$, and $S(\lambda)$ is an entire function. And it follows from lemma 1 that $S(\lambda)$ is an entire function of order ρ and of minimal type for order ρ. Then from the Fragmen-Lindeloff theorem, allowing for the condition a) or b) it follows that $S(\lambda) = b_0 + \lambda b_1,\ b_0, b_1 \in H$. (Here we take into account estimation (5) as well). Then from equality (18) it follows that for $t > 0$

$$(u'(t), \varphi)_{1/2} = \frac{1}{2\pi i} \int\limits_{\Gamma_{\pm\theta}} (b_0 + \lambda b_1)\, e^{\lambda t} d\lambda \equiv 0,$$

i.e. $(u'(t), \varphi)_{1/2} = 0$. Passing to limit $t \to 0$, we get $\|\varphi\|^2_{1/2} = 0$ i.e. $\varphi = 0$.

The theorem is proved.

The following theorem on the completenesss of decreasing elementary solutions is valid.

Theorem 6. *Let the conditions of theorem 5 be fulfilled. Then the system of elementary solutions is complete in the space of regular solutions of problem (2), (3).*

Proof. Let $\wp$ be a space of regular solutions of problem (2), (3). Obviously, $\wp$ is a closed space in $W_2^2(R_+; H)$. Let $\varphi \in H_{1/2}$. By theorem 4, for any $\varepsilon > 0$ one can find the number $N(\varepsilon)$ and the numbers $a^{(l)}_{i,h,N_\varepsilon}$ such that

$$\left\| \varphi - \sum_{i=1}^{N_\varepsilon} \sum_{(l)} \sum_{(h)} a^{(l)}_{i,h,N_\varepsilon} \varphi^{(l,1)}_{i,h} \right\|_{1/2} < \varepsilon.$$

Then taking into account $\varphi = u'(0)$, and $\varphi^{(l,1)}_{i,h} = \frac{d}{dt} u^{(l)}_{i,h}(t)\,|_{t=0}$, from the regular solvability of problem (1), (2) it follows that

$$\left\| u(t) - \sum_{i=1} \sum_{(l)} \sum_{(h)} a^{(l)}_{i,h,N_\varepsilon} \varphi^{(l)}_{i,h} \right\| \le const\ \varepsilon < \varepsilon_1.$$

The theorem is proved.

References

[1]. Lions J.L., Majenes E. *Inhomogeneous boundary value problems and their applications* // Moscow, Mir, 1971, 371p. (Russian)

[2]. Krein M.G., Langer G.K. *On some mathematical principles of linear theory of damped vibrations of continua* // Proc. of the International Symposium on application of theory of functions in continuum mechanics // 1966, vol. 2, pp.283-322. (Russian)

[3]. Gasymov M.G. *To theory of polynomial operator bundles* // DAN SSSR, 1971, vol. 199, No4, pp. 747-750. (Russian)

[4]. Gasymov M.G., Mirzoyev S.S. *On solvability of boundary value problems for elliptic type operator-differential equations of second order* // Differents. Uravn, 1992, vol.28, No4, pp. 651-661. (Russian)

[5]. Kostynchenko A.G., Shkalikov A.A. *To theory of self_adjoint quadratical bundles* // Vestnik MGU, ser. mat. i mekh., 1988, No6, pp. 40-51. (Russian)

[6]. Goryuk I.V. *A theorem on completeness of a system of eigen and adjoint vectors of the operator bundle* $L(\lambda) = \lambda^2 C + \lambda B + E$ // Vestnik MGU, ser. mat. i mekh., 1970, No1, pp. 55-60. (Russian)

[7]. Mirzoyev S.S., Veliyev S.G. *On the solutions of a class of second order operator-diffrential equations in the class of holomorphic vector-functions* // Ukrainsky matematichesky zhurnal, 2010, vol. 62, pp. 801-813. (Russian)

[8]. Shkalikov A.A. *On basicity of eigen vectors of quadratic operator bundles* // *Matematicheskiye zametki*, 1981, vol. 30, No3, pp. 371-385. (Russian)

[9]. Radzhievsky G.E. *Quadratic operator pensils* // Preprint 76.24, Kiev, Institute of Mathematics of AS Ukr SSR, 1976, 48 p. (Russian)

[10]. Mirzoyev S.S., Guliyeva F.A. *On completeness of elementary solutions of a class of second order differential equations with operator coefficients* // Matematicheskiye zametki, 2009, vol. 86, No5, pp. 797-800. (Russian)

[11]. Keldysh M.V. *On completeness of eigen functions of some classes of not self-adjoint linear operators* // UMN, 1971, vol. 26, No4(160), pp. 15-41. (Russian)

[12]. Shkalikov A.A. *Elliptic equations in Hilbert space and related spectral problems* // Proc. of I.G. Petrovsky seminar, 1989, No14, pp. 140-224. (Russian)

[13]. Mirzoev S.S. *On the norms of operators of intermediate derivatives* // Transactions of NAS Azerbaijan, series of phis. techn., math. Sciences, 2005, vol. XXV, No4, pp. 35-100.

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