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SOME PROPERTIES OF THE 4-AND 5-MAJORITY ALGEBRAS

Abstract

By using congruence schemes we present a new properties of 4-majority algebras and 5-majority algebras (in other terminology, for algebras having 4-ary (respectively, 5-ary) near unanimity operation among its term operations).

Congruence scheme is a very important idea in the study of congruence lattices of algebras (see [1]–[5]); in particular in [6] we find some properties of varieties with a finite number of congruence schemes and relationships with first order definable principal congruences and in [7] we find some properties of congruence schemes and compatible relations of algebras. In the present paper by using congruence schemes we find some new properties of 4-majority algebras, in other terminology, algebras having a 4-ary near unanimity term operation, and 5-majority algebras. General information on algebras and terms may be found in [8].

A term function $m(x_1,...,x_k)$ of an algebra $A=(A,F)$ is called a *k-majority term* (or sometimes *k-ary near unanimity term*; see [9], p.34) if

$$m(y,x,...,x)=m(x,y,x,...,x)=...=m(x,...,x,y)=x$$

holds for all $x,y\in A$. Algebras with a *k-majority term* are called *k-majority algebras.* Clearly, 3-majority term is exactly the majority term; for instance, any lattice admits a majority term. We mention that various characterizations of majority algebras are given in [5]. A *quasiorder* of an algebra $A=(A,F)$ is a reflexive, transitive, binary relation $\zeta\subseteq A\times A$ which is compatible with the operations of A. Let Quard (A) stand for the set of all quasiorders of A.

1^0. 4-majority algebras. Let us consider the following conditions.

(I) Algebra $A=(A,F)$ is an algebra with some 4-majority term function $m(t,u,v,w)$.

(II) For every $a,b,c,e\in A$ and for every compatible reflexive relations $\alpha,\beta,\delta,\gamma$ on A the following SCHEME-4 is satisfied:

if $(a,b)\in\alpha$, $(a,c)\in\beta$, $(c,e)\in\delta$ and $(e,b)\in\gamma$ then there are $x_1,...,x_6\in A$ such that $(a,x_1)\in\alpha\cap\beta$, $(x_1,c)\in\beta$, $(x_1,x_2)\in\beta$, $(x_2,c)\in\beta$, $(x_1,x_3)\in\delta$, $(e,x_4)\in\gamma$, $(e,x_6)\in\gamma$, $(x_4,x_6)\in\gamma$, $(x_5,x_6)\in\delta$, $(x_6,b)\in a\cap\gamma$; next $(x_1,b)\in\alpha$, $(x_3,b)\in\alpha$, $(x_5,b)\in\alpha$, $(a,x_3)\in\alpha$, $(a,x_5)\in\alpha$, $(a,x_6)\in\alpha$; moreover there is $x\in A$ such that $(x_2,x)\in\delta\&(x_3,x)\in\beta$ and $(x,x_4)\in\delta\&(x,x_5)\in\gamma$.

(III) All compatible reflexive binary relations $\alpha,\beta,\delta,\gamma\subseteq A\times A$ satisfy

$$\alpha\cap(\beta\circ\delta\circ\gamma)\subseteq[((\alpha\cap\beta)\circ(\beta\circ\delta\cap\delta\circ\beta)\circ(\delta\circ\gamma\cap\gamma\circ\delta))\cap\alpha]\circ(\alpha\cap\gamma)\cap$$

$$\cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (a \cap \gamma)) \cap \alpha] \quad (1)$$

(IV) For every quasiorders $\alpha, \beta, \delta, \gamma$ of A we have:

$$\alpha \cap (\beta \circ \delta \circ \gamma) = [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$

$$\cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (a \cap \gamma)) \cap \alpha] \quad (2)$$

Proposition1. Let $A = (A, F)$ be an algebra; then (I) $\rightarrow$ (II) $\rightarrow$ (III) $\rightarrow$ (IV).

Proof. (I) $\rightarrow$ (II). Let $m(t, u, v, w) : A^4 \rightarrow A$ be a 4-majority term function of algebra $A = (A, F)$. Suppose $(a, b) \in \alpha$, $(a, c) \in \beta$, $(c, e) \in \delta$ and $(e, b) \in \gamma$, where $a, b, c, e \in A$ and $\alpha, \beta, \delta, \gamma \subseteq A \times A$ are compatible reflexive relations of A. Take $x := m(a, c, e, b)$, $x_1 := m(a, a, c, b)$, $x_2 := m(a, c, c, b)$, $x_3 := m(a, a, e, b)$, $x_4 := m(a, e, e, b)$, $x_5 := m(a, c, b, b)$ and $x_6 := m(a, e, b, b)$. Then we obtain:

$$(a, x_1) = (m(a, a, a, b), m(a, a, c, b)) \in \beta.$$

And simultaneously

$$(a, x_1) = (m(a, c, a, a), m(a, a, c, b)) \in \alpha.$$

So, $(a, x_1) \in \alpha \cap \beta$.

Similarly, we get: $(x_6, b) = (m(a, e, b, b), m(b, e, b, b)) \in \alpha$ and $(x_6, b) = (m(a, e, b, b), m(a, b, b, b)) \in \gamma$, whence $(x_6, b) \in \alpha \cap \gamma$. Also, it is clear that

$$(x_1, c) = (m(a, a, c, b), m(c, c, c, b)) \in \beta,$$

$$(x_1, x_2) = (m(a, a, c, b), m(a, c, c, b)) \in \beta,$$

$$(x_2, c) = (m(a, c, c, b), m(c, c, c, b)) \in \beta,$$

$$(x_1, x_3) = (m(a, a, c, b), m(a, a, e, b)) \in \delta,$$

$$(e, x_4) = (m(a, e, e, e), m(a, e, e, b)) \in \gamma,$$

$$(e, x_6) = (m(a, e, e, e), m(a, e, b, b)) \in \gamma,$$

$$(x_4, x_6) = (m(a, e, e, b), m(a, e, b, b)) \in \gamma,$$

$$(x_5, x_6) = (m(a, c, b, b), m(a, e, b, b)) \in \delta,$$

$$(x_1, b) = (m(a, a, c, b), m(b, b, c, b)) \in \alpha,$$

$$(x_3, b) = (m(a, a, e, b), m(b, b, e, b)) \in \alpha,$$

$$(x_5, b) = (m(a, c, b, b), m(b, c, b, b)) \in \alpha,$$

$$(a, x_3) = (m(a, a, e, a), m(a, a, e, a)) \in \alpha,$$

$$(a, x_5) = (m(a, c, a, a), m(a, c, b, b)) \in \alpha,$$

$$(a, x_6) = (m(a, e, a, a), m(a, e, b, b)) \in \alpha,$$

$$(x_2, x) = (m(a, c, c, b), m(a, c, e, b)) \in \delta,$$

$$(x_3, x) = (m(a, a, e, b), m(a, c, e, b)) \in \beta,$$

$$(x, x_4) = (m(a, c, e, b,), m(a, e, e, b)) \in \delta,$$

$$(x, x_5) = (m(a, c, e, b), m(a, c, b, b)) \in \gamma.$$

(II) →(III). Let $(a, b) \in \alpha \cap (\beta \circ \delta \circ \gamma)$. Then there are an elements $c, e \in A$ such that $(a, c) \in \beta$, $(c, e) \in \delta$ and $(e, b) \in \gamma$. Also, $(a, b) \in \alpha$. By applying SCHEME-4 we obtain elements $x, x_1, ..., x_6 \in A$ such that $(a, x_1) \in \alpha \cap \beta$, $(x_1, c) \in \beta$, $(x_1, x_2) \in \beta$, $(x_2, c) \in \beta$, $(x_1, x_3) \in \delta$, $(e, x_4) \in \gamma$, $(e, x_6) \in \gamma$, $(x_4, x_6) \in \gamma$, $(x_5, x_6) \in \delta$, $(x_6, b) \in \alpha \cap \gamma$; next $(x_1, b) \in \alpha$, $(x_3, b) \in \alpha$, $(x_5, b) \in \alpha$, $(a, x_3) \in \alpha$, $(a, x_5) \in \alpha$, $(a, x_6) \in \alpha$; moreover, $(x_2, x) \in \delta \& (x_3, x) \in \beta$, $(x, x_4) \in \delta \& (x, x_5) \in \gamma$. So, $(a, b) \in [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma)$.

Similarly, we get:

$$(a, b) \in (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha].$$

Thus

$$(a, b) \in [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$

$$\cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha].$$

Hence, the inclusion (1) is proved.

(III) → (IV). If $\alpha, \beta, \delta, \gamma \in Quord(A)$, then

$$[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$

$$\cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha] \subseteq$$

$$\subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \subseteq \alpha \circ \alpha \subseteq \alpha$$

and

$$[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$

$$\cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha] \subseteq$$

$$\subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \subseteq$$

$$\subseteq \beta \circ \beta \circ \delta \circ \delta \circ \gamma \circ \gamma \subseteq \beta \circ \delta \circ \gamma.$$

Thus

$$[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$

$$\cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha] \subseteq \alpha \cap (\beta \circ \delta \circ \gamma).$$

As the converse inclusion holds by the assumption, we obtain the relation (2).

Let $\theta(a, b)$ stand for the principal congruence of an algebra $A = (A, F)$ generated by the pair $(a, b) \in A^2$. It is well-known that if $\varphi : A \to B$ is a homomorphism of the algebra A into the algebra B then $(u,v) \in \theta(a,b)$ implies $(\varphi(u),\varphi(v)) \in \theta(\varphi(a),\varphi(b))$ (see [8,Chapter II, section 6]).

Theorem 1. *Let V be a variety of algebras. Then the following assertions are equivalent.*

(a) V has a 4-majority term.

(b) Any algebra $A=(A,F)\in V$ satisfies SCHEME-4.

(c) For any algebra $A=(A,F)\in V$ and any compatible reflexive relations $\alpha,\beta,\delta,\gamma \subseteq A\times A$ we have

$$\alpha\cap(\beta\circ\delta\circ\gamma)\subseteq[((\alpha\cap\beta)\circ(\beta\circ\delta\cap\delta\circ\beta)\circ(\delta\circ\gamma\cap\gamma\circ\delta))\cap\alpha]\circ(\alpha\cap\gamma)\cap$$

$$\cap(\alpha\cap\beta)\circ[((\beta\circ\delta\cap\delta\circ\beta)\circ(\delta\circ\gamma\cap\gamma\circ\delta)\circ(\alpha\cap\gamma))\cap\alpha].$$

(d) For any algebra $A\in V$, every $\alpha,\beta,\delta,\gamma\in ConA$ satisfy the equality

$$\alpha\cap(\beta\circ\delta\circ\gamma)=[((\alpha\cap\beta)\circ(\beta\circ\delta\cap\delta\circ\beta)\circ(\delta\circ\gamma\cap\gamma\circ\delta))\cap\alpha]\circ(\alpha\cap\gamma)\cap$$

$$\cap(\alpha\cap\beta)\circ[((\beta\circ\delta\cap\delta\circ\beta)\circ(\delta\circ\gamma\cap\gamma\circ\delta)\circ(\alpha\cap\gamma))\cap\alpha].$$

Proof. In view of Proposition 1, (a) implies (b) and (b) implies (c). Proposition 1 also gets that (c) implies (d), as $ConA\subseteq QuordA$.

(d) implies (a). Consider now the free algebra $F_V(x,y,z,t)\in V$. As $(x,t)\in\theta(x,t)\cap(\theta(x,y)\circ\theta(y,z)\circ\theta(z,t))$, the assumption of (d) implies

$$(x,t)\in[(\theta(x,t)\cap\theta(x,y))\circ((\theta(x,y)\circ\theta(y,z)\cap\theta(y,z)\circ\theta(x,y))\circ$$

$$\circ(\theta(y,z)\circ\theta(z,t)\cap\theta(z,t)\circ\theta(y,z))\cap\theta(x,t)]\circ$$

$$\circ(\theta(x,t)\cap\theta(z,t))\cap(\theta(x,t)\cap\theta(x,y))\circ$$

$$\circ[((\theta(x,y)\circ\theta(y,z)\cap\theta(y,z)\circ\theta(x,y))\circ$$

$$\circ(\theta(y,z)\circ\theta(z,t)\cap\theta(z,t)\circ\theta(y,z))\circ(\theta(x,t)\cap\theta(z,t)))\cap\theta(x,t)].$$

Hence there is a term $m(x,y,z,t)\in F_V(x,y,z,t)$ such that

$$x\,(\theta(x,t)\cap\theta(x,y))\,m(x,x,y,t),$$

$$m(x,x,y,t)\,\theta(x,y)\,m(x,y,y,t)\,\theta(y,z)\,m(x,y,z,t)\quad\&$$

$$\&\quad m(x,x,y,t)\,\theta(y,z)\,m(x,x,z,t)\,\theta(x,y)\,m(x,y,z,t)$$

and

$$m(x,y,z,t)\,\theta(y,z)\,m(x,z,z,t)\,\theta(z,t)\,m(x,z,t,t)\quad\&$$

$$\&\ m(x,y,z,t)\,\theta(z,t)\,m(x,y,t,t)\,\theta(y,z)\,m(x,z,t,t);$$

next $m(x,z,t,t)\,(\theta(x,t)\cap\theta(z,t))\,t$;

moreover, $m(x,x,y,t)\,\theta(x,t)\,t\ \ \&\ \ x\theta(x,t)\,m(x,z,t,t)$.

Now, using a homomorphism $\varphi : F_V(x,y,z,t) \to F_V(x,y,z,t)$ with $\varphi(x)=\varphi(y)=x$, $\varphi(z)=z$ and $\varphi(t)=t$ from $(x, m(x,x,y,t)) \in \theta(x,y)$ we obtain:

$$(x, m(x,,x,x,t)) = (\varphi(x), m(\varphi(x),\varphi(x),\varphi(y),\varphi(t))) =$$

$$= (\varphi(x), \varphi(m(x,x,y,t))) \in \theta(\varphi(x),\varphi(y)) = \theta(x,x) = \Delta.$$

Thus $x = m(x,x,x,t)$. The identities $x = m(x,x,y,x) = m(x,y,x,x) = m(y,x,x,x)$ can be proved in a similar way.

2^0. On 5-majority algebras. Let us consider the following conditions.

(V) Algebra $A = (A,F)$ is an algebra with some 5-majority term function $m(s,t,u,v,w)$.

(VI) For every $a,b,c,e,d \in A$ and for every compatible reflexive relations α, β, δ, ρ, γ on A the following SCHEME-5 is satisfied: if $(a,b)\in\alpha$, $(a,c)\in\beta$, $(c,e)\in\delta$, $(e,d)\in\rho$ and $(d,b)\in\gamma$, then there are $x, x_1, ..., x_{12} \in A$ such that $(a,x_1)\in\alpha\cap\beta$, $x_1\beta c$, $x_2\beta c$, $(x_1,x_2)\in\beta$, $(x_1,x_3)\in\delta$, $(x_2,x_4)\in\delta$, $(x_3,x_4)\in\beta$, $(x_4,x_5)\in\delta$, $(x_4,x_6)\in\rho$, $(x_5,x)\in\rho$, $(x_6,x)\in\delta$; next $x\rho x_7\delta x_9$ & $x\delta x_8\rho x_9$ and $x_9\gamma x_{10}\rho x_{12}$ & $x_9\rho x_{11}\gamma x_{12}$; next $(x_{12},b)\in\alpha\cap\gamma$, $d\gamma x_{11}$ & $d\gamma x_{12}$; moreover $x_1\alpha b$, $x_3\alpha b$, $a\alpha x_{10}$, $a\alpha x_{12}$.

(VII) All compatible reflexive binary relations $\alpha,\beta,\delta,\rho,\gamma \subseteq A\times A$ satisfy

$$\alpha\cap(\beta\circ\delta\circ\rho\circ\gamma) \subseteq [((\alpha\cap\beta)\circ(\beta\circ\delta\cap\delta\circ\beta)\circ(\rho\circ\delta\cap\delta\circ\rho)\circ(\rho\circ\delta\cap\delta\circ\rho)\circ$$
$$\circ(\gamma\circ\rho\cap\rho\circ\gamma)\cap\alpha]\circ(\alpha\cap\gamma)\cap(\alpha\cap\beta)\circ[\alpha\cap((\beta\circ\delta\cap\delta\circ\beta)\circ$$
$$\circ(\rho\circ\delta\cap\delta\circ\rho)\circ(\delta\circ\rho\cap\rho\circ\delta)\circ(\gamma\circ\rho\cap\rho\circ\gamma)\circ(\alpha\cap\gamma))] \quad (3)$$

(VIII) For every quasiorders $\alpha,\beta,\delta,\gamma,\rho$ of algebra A we have:

$$\alpha\cap(\beta\circ\delta\circ\rho\circ\gamma) = [((\alpha\cap\beta)\circ(\beta\circ\delta\cap\delta\circ\beta)\circ(\rho\circ\delta\cap\delta\circ\rho)\circ(\rho\circ\delta\cap\delta\circ\rho)\circ$$
$$\circ(\gamma\circ\rho\cap\rho\circ\gamma)\cap\alpha]\circ(\alpha\cap\gamma)\cap(\alpha\cap\beta)\circ[\alpha\cap((\beta\circ\delta\cap\delta\circ\beta)\circ$$
$$\circ(\rho\circ\delta\cap\delta\circ\rho)\circ(\delta\circ\rho\cap\rho\circ\delta)\circ(\gamma\circ\rho\cap\rho\circ\gamma)\circ(\alpha\cap\gamma))] \quad (4)$$

Proposition 2. Let $A = (A,F)$ be an algebra; then (V) $\to$ (VI) $\to$ (VII) $\to$ (VIII).

Proof. (V) $\to$ (VI). Let $m(s,t,u,v,w) : A^5 \to A$ be a 5-majority term function of algebra $A = (A,F)$. Suppose $(a,b)\in\alpha$, $(a,c)\in\beta$, $(c,e)\in\delta$, $(e,d)\in\rho$ and $(d,b)\in\gamma$ for the elements $a,b,c,e,d\in A$ and $\alpha,\beta,\delta,\rho,\gamma\subseteq A\times A$ are compatible reflexive relations of A. Take $x := m(a,c,e,d,b)$, $x_1 := m(a,a,a,c,b)$, $x_2 := m(a,c,c,c,b)$, $x_3 := m(a,a,a,e,b)$, $x_4 := m(a,c,c,e,b)$, $x_5 := m(a,c,e,e,b)$, $x_6 := m(a,c,c,d,b)$, $x_7 := m(a,c,d,d,b)$, $x_8 := m(a,e,e,d,b)$, $x_9 := m(a,e,d,d,b)$, $x_{10} := m(a,e,b,b,b)$, $x_{11} := m(a,d,d,d,b)$ and $x_{12} := m(a,d,b,b,b)$.

Then we obtain: $(a,x_1) = (m(a,a,a,a,b), m(a,a,a,c,b)) \in \beta$. And simultaneously $(a,x_1) = (m(a,a,a,c,a), m(a,a,a,c,b)) \in \alpha$. So, $(a,x_1) \in \alpha \cap \beta$. Similarly, we get: $(x_{12},b) = (m(a,d,b,b,b), m(a,b,b,b,b)) \in \gamma$ and $(x_{12},b) = (m(a,d,b,b,b), m(b,d,b,b,b)) \in \alpha$, whence $(x_{12},b) \in \alpha \cap \gamma$. Similarly, it is clear that

$$(x_1,c) = (m(a,a,a,c,b), m(c,c,c,c,b)) \in \beta,$$

$$(x_2,c) = (m(a,c,c,c,b), m(c,c,c,c,b)) \in \beta,$$

$$(x_1,x_2) = (m(a,a,a,c,b), m(a,c,c,c,b)) \in \beta,$$

$$(x_2,x_4) = (m(a,c,c,c,b), m(a,c,c,e,b)) \in \delta,$$

$$(x_1,x_3) = (m(a,a,a,c,b), m(a,a,a,e,b)) \in \delta,$$

$$(x_3,x_4) = (m(a,a,a,e,b), m(a,c,c,e,b)) \in \beta,$$

$$(x_4,x_5) = (m(a,c,c,e,b), m(a,c,e,e,b)) \in \delta,$$

$$(x_4,x_6) = (m(a,c,c,e,b), m(a,c,c,d,b)) \in \rho,$$

$$(x_5,x) = (m(a,c,e,e,b), m(a,c,e,d,b)) \in \rho,$$

$$(x_6,x) = (m(a,c,c,d,b), m(a,c,e,d,b)) \in \delta,$$

$$(x,x_7) = (m(a,c,e,d,b), m(a,c,d,d,b)) \in \rho,$$

$$(x,x_8) = (m(a,c,e,d,b), m(a,e,e,d,b)) \in \delta,$$

$$(x_8,x_9) = (m(a,e,e,d,b), m(a,e,d,d,b)) \in \rho,$$

$$(x_7,x_9) = (m(a,c,d,d,b), m(a,e,d,d,b)) \in \delta,$$

$$(x_9,x_{10}) = (m(a,e,d,d,b), m(a,e,b,b,b)) \in \gamma,$$

$$(x_9,x_{11}) = (m(a,e,d,d,b), m(a,d,d,d,b)) \in \rho,$$

$$(x_{10},x_{12}) = (m(a,e,b,b,b), m(a,d,b,b,b)) \in \rho,$$

$$(x_{11},x_{12}) = (m(a,d,d,d,b), m(a,d,b,b,b)) \in \gamma,$$

$$(a,x_{12}) = (m(a,d,a,a,a), m(a,d,b,b,b)) \in \alpha,$$

$$(x_1,b) = (m(a,a,a,c,b), m(b,b,b,c,b)) \in \alpha,$$

$$(a,x_{10}) = (m(a,e,a,a,a), m(a,e,b,b,b)) \in \alpha,$$

$$(x_2,b) = (m(a,a,a,e,b), m(b,b,b,e,b)) \in \alpha.$$

(VI) $\to$ (VII). Let $(a,b) \in \alpha \cap (\beta \circ \delta \circ \rho \circ \gamma)$. Then there are elements $c,e,d \in A$ such that $(a,c) \in \beta$, $(c,e) \in \delta$, $(e,d) \in \rho$ and $(d,b) \in \gamma$. Also, $(a,b) \in \alpha$. By applying SCHEME-5 we obtain elements $x, x_1, ..., x_{12} \in A$ such that $(a,x_1) \in \alpha \cap \beta$, $(x_1,x_2) \in \beta$, $(x_1,x_3) \in \delta$, $(x_2,x_4) \in \delta$, $(x_3,x_4) \in \beta$, $(x_4,x_5) \in \delta$, $(x_4,x_6) \in \rho$, $(x_5,x) \in \rho$, $(x_6,x) \in \delta$, $(x,x_7) \in \rho$, $(x,x_8) \in \delta$, $(x_8,x_9) \in \rho$, $(x_7,x_9) \in \delta$, $(x_9,x_{10}) \in$

γ, $(x_9, x_{11}) \in \rho$, $(x_{10}, x_{12}) \in \rho$, $(x_{11}, x_{12}) \in \gamma$, $(x_{12}, b) \in \alpha \cap \gamma$ and $(x_1, b) \in \alpha$, $(a, x_{12}) \in \alpha$. So,

$$(a,b) \in [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \cap \alpha] \circ (\alpha \cap \gamma)$$

Similarly, we get:

$$(a,b) \in (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\delta \circ \rho \cap \rho \circ \delta) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))]$$

Thus

$$(a,b) \in [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$
$$\cap (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\delta \circ \rho \cap \rho \circ \delta) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))] .$$

Hence, the inclusion (3) is proved.

(VII) $\to$ (VIII). If $\alpha, \beta, \delta, \rho, \gamma \in Quord\,(A)$, then

$$[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\gamma \circ \rho \cap \rho \circ \gamma) \cap \alpha] \circ (\alpha \cap \gamma) \cap (\alpha \cap \beta) \circ$$
$$\circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\delta \circ \rho \cap \rho \circ \delta) \circ$$
$$\circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))] \subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \cap \alpha] \circ (\alpha \cap \gamma) \subseteq \alpha \circ \alpha \subseteq \alpha$$

and

$$[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\gamma \circ \rho \cap \rho \circ \gamma) \cap \alpha] \circ (\alpha \cap \gamma) \cap (\alpha \cap \beta) \circ$$
$$\circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\delta \circ \rho \cap \rho \circ \delta) \circ$$
$$\circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))] \subseteq (\alpha \cap \beta) \circ \beta \circ \delta \circ \delta \circ \rho \circ \rho \circ \gamma \circ \gamma \subseteq$$
$$\subseteq \beta \circ \beta \circ \delta \circ \rho \circ \gamma \subseteq \beta \circ \delta \circ \rho \circ \gamma.$$

Thus

$$[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\gamma \circ \rho \cap \rho \circ \gamma) \cap \alpha] \circ (\alpha \cap \gamma) \cap (\alpha \cap \beta) \circ$$
$$\circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\delta \circ \rho \cap \rho \circ \delta) \circ$$
$$\circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))] \subseteq \alpha \cap (\beta \circ \delta \circ \rho \circ \gamma) .$$

As the converse inclusion holds by the assumption, we obtain the relation (4).

Theorem 2. *Let V be a variety of algebras. Then the following assertions are equivalent.*

(a) V has a 5-majority term.

(b) Any algebra $A=(A,F)\in V$ satisfies SCHEME-5.

(c) For any algebra $A=(A,F)\in V$ and any compatible reflexive relations $\alpha,\beta,\delta,\rho,\gamma\subseteq A\times A$ we have

$$\alpha\cap(\beta\circ\delta\circ\rho\circ\gamma)\subseteq[((\alpha\cap\beta)\circ(\beta\circ\delta\cap\delta\circ\beta)\circ(\rho\circ\delta\cap\delta\circ\rho)\circ$$

$$\circ(\delta\circ\rho\cap\rho\circ\delta)\circ(\gamma\circ\rho\cap\rho\circ\gamma))\cap\alpha]\circ(\alpha\cap\gamma)\cap$$

$$\cap(\alpha\cap\beta)\circ[\alpha\cap((\beta\circ\delta\cap\delta\circ\beta)\circ(\rho\circ\delta\cap\delta\circ\rho)\circ$$

$$\circ(\delta\circ\rho\cap\rho\circ\delta)\circ(\gamma\circ\rho\cap\rho\circ\gamma)\circ(\alpha\cap\gamma))].$$

(d) For any algebra $A\in B$, every $\alpha,\beta,\delta,\rho,\gamma\in ConA$ satisfy the equality

$$\alpha\cap(\beta\circ\delta\circ\rho\circ\gamma)\subseteq[((\alpha\cap\beta)\circ(\beta\circ\delta\cap\delta\circ\beta)\circ(\rho\circ\delta\cap\delta\circ\rho)\circ$$

$$\circ(\gamma\circ\rho\cap\rho\circ\gamma))\cap\alpha]\circ(\alpha\cap\gamma)\cap(\alpha\cap\beta)\circ$$

$$\circ[\alpha\cap((\beta\circ\delta\cap\delta\circ\beta)\circ(\delta\circ\rho\cap\rho\circ\delta)\circ(\gamma\circ\rho\cap\rho\circ\gamma)\circ(\alpha\cap\gamma))].$$

Proof. In view of Proposition 1, (a) implies (b) and (b) implies (c). Proposition 1 also gets that (c) implies (d), as $ConA\subseteq QuordA$.

(d) implies (a). Consider now the free algebra $F_V(x,y,z,u,t)\in V$. As $(x,t)\in\theta(x,t)\cap(\theta(x,y)\circ\theta(y,z)\circ\theta(z,u)\circ\theta(u,t))$, the assumption of (d) implies

$$(x,t)\in[(\theta(x,t)\cap\theta(x,y))\circ(((\theta(x,y)\circ\theta(y,z)\cap\theta(y,z)\circ\theta(x,y))\circ$$

$$\circ(\theta(y,z)\circ\theta(z,u)\cap\theta(z,u)\circ\theta(y,z))\circ$$

$$\circ(\theta(z,u)\circ\theta(u,t)\cap\theta(u,t)\circ\theta(z,u)))\cap\theta(x,t)]\circ$$

$$\circ(\theta(x,t)\cap\theta(u,t))\cap(\theta(x,t)\cap\theta(x,y))\circ$$

$$\circ[((\theta(x,y)\circ\theta(y,z)\cap\theta(y,z)\circ\theta(x,y))\circ$$

$$\circ(\theta(y,z)\circ\theta(z,u)\cap\theta(z,u)\circ\theta(y,z))\circ(\theta(z,u)\circ\theta(u,t)\cap\theta(u,t)\circ\theta(z,u))\circ$$

$$\circ(\theta(x,t)\circ\theta(u,t)))\cap\theta(x,t)].$$

Hence there is a term $m(x,y,z,u,t)\in F_V(x,y,z,u,t)$ such that

$$x\,(\theta(x,t)\cap\theta(x,y))\,m(x,x,x,y,t)\,\theta(x,y)\,m(x,y,y,y,t)\,\theta(y,z)\,(x,y,y,z,t)\,\&$$

$$\&\ m(x,x,x,y,t)\,\theta(y,z)\,m(x,x,x,z,t)\,\theta(x,y))\,m(x,y,y,z,t)$$

and

$$m(x,y,y,z,t)\,\theta(y,z)\,m(x,y,z,z,t)\,\theta(z,u)\,m(x,y,z,u,t)\quad\&$$

$$\& \; m(x,y,y,z,t)\,\theta(z,u)\,m(x,y,y,u,t)\,\theta(y,z)\,m(x,y,z,u,t)\,.$$

Next,

$$m(x,y,z,u,t)\,\theta(y,z)\,m(x,z,z,u,t)\,\theta(z,u)\,m(x,z,u,u,t)\,\theta(z,u)$$

$$m(x,u,u,u,t)\,\theta(u,t)\,m(x,u,t,t,t) \quad \& \quad m(x,y,z,u,t)\,\theta(z,u)\,m(x,y,u,u,t)\,\theta(y,z)$$

$$m(x,z,u,u,t)\,\theta(u,t)\,m(x,z,t,t,t)\,\theta(z,u)\,m(x,u,t,t,t)\,,$$

and $m(x,u,t,t,t)\,(\theta(x,t)\cap\theta(u,t))\,t$.

Now, using the homomorphism $\varphi : F_V(x,y,z,u,t) \to F_V(x,y,z,u,t)$ with $\varphi(x)=\varphi(y)=x$, $\varphi(z)=z$, $\varphi(u)=u$, $\varphi(t)=t$ and from $(x,m(x,x,x,y,t))\in\theta(x,y)$ we obtain:

$$(x,m(x,x,x,x,t)) = (\varphi(x)\,,m(\varphi(x)\,,\varphi(x)\,,\varphi(y)\,,\varphi(t))) =$$

$$=(\varphi(x)\,,\varphi(m(x,x,x,y,t)))\in\theta(\varphi(x)\,,\varphi(y))=\theta(x,x)=\Delta.$$

Thus $x=m(x,x,x,x,t)$. The identities $x=m(x,x,x,y,x)=m(x,x,y,x,x)=m(x,y,x,x,x)=m(y,x,x,x,x)$ can be proved in a similar way.

3^0. Remarks. It is easy to see that under the conditions of Proposition 2(V)→(VI) we have the following relations also: $x_2\delta x_{13}$, $x_4\delta x_{13}$, $x_5\delta x_{13}$ and $x_{13}\rho x_8$, $x_{13}\rho x_9$, $x_{13}\rho x_{11}$; here $x_{13}:=m(a,e,e,e,b)$.

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