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RENEWAL OF A MULTI-PARAMETRIC SPECTRAL PROBLEM WITH GIVEN EIGEN VALUES AND EIGEN ELEMENTS

Abstract

In the paper, the inverse multi-parametric problem is investigated in the following form: for the given sequence of eigen values $\{(\lambda_{1,n}, \lambda_{2,n}, ..., \lambda_{m,n})\}_{n=1,2,...} \subset R_m$ *with real coordinates and the sequences of appropriate given eigen elements*

$$\{\Phi_n\}_{n=1,2,...} = \left\{\varphi_{1,n} \otimes \varphi_{2,n} \otimes ... \otimes \varphi_{m,n}\right\}_{n=1,2,...} \subset H = H_1 \otimes H_2 \otimes ... \otimes H_m$$

(where $\otimes$ *is a tensor product sign) we look for a family of compact self-adjoint permutation operators* $K_{i,1}, K_{i,2}, ..., K_{i,m}$ *in the Hilbert space* $H_i, i = 1; 2; ...; m$ *for which the given sequences* $\{(\lambda_{1,n}, \lambda_{2,n}, ..., \lambda_{m,n})\}_{n=1,2,...}$ *and* $\{\Phi_n\}_{n=1,2,...}$ *are the complete sequences of eigen values and appropriate eigen elements of the problem*

$$\left\{\begin{array}{l} \sum\limits_{j=1}^{m} \lambda_j K_{i,j} \varphi_i = \varphi_i, \quad \varphi_i \in H \\ i = 1; 2; ...; m \end{array}\right. .$$

In the paper, the inverse multi-parametric problem is investigated in the following form: for the given sequence of eigen values $\{(\lambda_{1,n}, \lambda_{2,n}, ..., \lambda_{m,n})\}_{n=1,2,...} \subset R_m$ with real coordinates and the sequences of appropriate given eigen elements

$$\{\Phi_n\}_{n=1,2,...} = \left\{\varphi_{1,n} \otimes \varphi_{2,n} \otimes ... \otimes \varphi_{m,n}\right\}_{n=1,2,...} \subset H = H_1 \otimes H_2 \otimes ... \otimes H_m \quad (1)$$

(where $\otimes$ is a tensor product sign) we look for a family of compact self-adjoint permutation operators $K_{i,1}, K_{i,2}, ..., K_{i,m}$ in the Hilbert space $H_i, i = 1; 2; ...; m$ for which the given sequences $\{(\lambda_{1,n}, \lambda_{2,n}, ..., \lambda_{m,n})\}_{n=1,2,...}$ and (1) are the complete sequences of eigen values and appropriate eigen elements of the problem

$$\left\{\begin{array}{l} \sum\limits_{j=1}^{m} \lambda_j K_{i,j} \varphi_i = \varphi_i, \quad \varphi_i \in H \\ i = 1; 2; ...; m \end{array}\right. . \quad (2)$$

Denote the linear span of the set of the first n elements of the sequence $\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}, ...,$ i.e. of the set $\left\{\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}\right\} \subset H_i$, $i = 1; 2; ...; n$ by $L\left(\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}\right)$, the closure of the linear sub space $L\left(\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}, ...\right)$ by $\overline{L}\left(\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}, ...,\right)$, $i = 1; 2; ...; m$. Introduce the denotation

$$(\Delta_0 \Phi, \Phi) = \det\left((Ker_{i,j}\varphi_i, \varphi_i)\right)^{\otimes}_{\substack{i=1,2,...,m \\ j=1,2,...,m}},$$

$$(\Delta_i \Phi, \Phi) =$$

$$= \det \begin{pmatrix} (K_{1,1}\varphi_1\varphi_1) & ... & (K_{1,i-1}\varphi_1\varphi_1) & (\varphi_1\varphi_1) & (K_{1,i+1}\varphi_1\varphi_1) & ... & (K_{1,m}\varphi_1\varphi_1) \\ ... & ... & ... & ... & ... & ... & ... \\ (K_{m,1}\varphi_m\varphi_m) & ... & (K_{m,i-1}\varphi_m\varphi_m) & (\varphi_m\varphi_m) & (K_{m,i+1}\varphi_m\varphi_m) & ... & (K_{m,m}\varphi_m\varphi_m) \end{pmatrix}^{\otimes},$$

where $\Phi = \varphi_1 \otimes \varphi_2 \otimes ... \otimes \varphi_m \in H = H_1 \otimes H_2 \otimes ... \otimes H_m$. $\Delta_0, \Delta_1, ..., \Delta_m$ are the linear operators determined on the Hilbert space $H = H_1 \otimes H_2 \otimes ... \otimes H_n$.

Let $\varphi_i(\lambda_1, \lambda_2, ..., \lambda_m) \in H_i, i = 1; 2; ...; m$ be a solution of the i-th equation of system (2), that analytically depends on the variables $(\lambda_1, \lambda_2, ..., \lambda_m)$. The following theorem is known (see [2]).

Theorem 1. *In order all the eigen elements $\varphi_i(\lambda_1, \lambda_2, ..., \lambda_m) \in H_i, i = 1; 2; ...; m$ analytically dependent on the parameters $(\lambda_1, \lambda_2, ..., \lambda_m)$ were constants with respect to the variables $(\lambda_1, \lambda_2, ..., \lambda_m)$ it is necessary and sufficient that in problem (2) the operators $K_{i,1}, K_{i,2}, ..., K_{i,m}, i = 1; 2; ...; m$ be permutational. Therewith, the spectral set consists of the totality of hyperplanes.*

Theorem 2. *If the operators $K_{i,1}, K_{i,2}, ..., K_{i,m}$ are compact self-adjoint, permutational operators in the Hilbert space $H_i, i = 1; 2; ...; m$, and elements $\varphi_i(\lambda_1, \lambda_2, ..., \lambda_m) \in H_i, i = 1; 2; ...; m$ analytically dependent on the paremeters $(\lambda_1, \lambda_2, ..., \lambda_m)$ are the eigen elements of the i-th equation of system (2), then $\varphi_i(\lambda_1, \lambda_2, ..., \lambda_m) \in H_i, i = 1; 2; ...; m$ (a constant with respect to $(\lambda_1, \lambda_2, ..., \lambda_m)$ is a joint eigen element of each of the operators $K_{i1}, K_{i2}, ..., K_{im}$, and the spectral hypersurface of the i-th equation of system (2), that corresponds to the eigen element $\varphi_i \in H_i$ is of the form $\sum_{j=1}^{m} \lambda_j \alpha_{i,j} = 1$, $i = 1; 2; ...; m$, where the non-zero values of the numbers $\alpha_{i,1}, \alpha_{i,2}, ..., \alpha_{i,m}$ are the eigen values of the operators $K_{i,1}, K_{i,2}, ..., K_{i,m}, i = 1; 2; ...; m$ respectively, and the zero values of the numbers $\alpha_{i,j}; i, j = 1; 2; ...; m$, means that for the appropriate operator $K_{i,j}\varphi_i = 0$.*

Proof. Let for the fixed values of the index $i = 1; 2; ...; m$ the element $\varphi_i \in H_i$ is an eigen element of the i-th equation of system (2), and according to theorem 2 some spectral hypersurface in the form

$$\sum_{r=1}^{m} \lambda_r c_{i,r} = 1, \quad i = 1; 2; ...; m. \tag{3}$$

corresponds to this eigen element.

Taking into account equalities (3) in the system of equation (2), i. e. excluding parameter λ_j we get

$$c_{ij}\varphi_i - K_{i,j}\varphi_i = \sum_{\substack{r=1 \\ r \neq j}}^{m} \lambda_r \left(c_{ij} K_{i,r}\varphi_i - c_{ir} K_{i,j}\varphi_i\right), \quad \varphi_i \in H_i \quad i = 1; 2; ...; m. \tag{4}$$

For each fixed value of the index i, the appropriate equality in system (4) is true for any value of the parameters $\lambda_r, r = 1, ..., m; r \neq j$. Here the elements φ_j are independent on the parameters $\lambda_r, r = 1, ..., m$, consequently, for any collection of parameters $\lambda_r, r = 1, ..., m$, equalities (4) are true iff the following equalities are fulfilled simultaneously:

$$c_{i,j}\varphi_i - K_{i,j}\varphi_i = 0, \sum_{\substack{r=1 \\ r \neq j}}^{m} \lambda_r \left(c_{i,j} K_{i,r}\varphi_i - c_{i,r} K_{i,j}\varphi_i\right) = 0, \quad i = 1; 2; ...; m.$$

Hence we get the equalities $K_{i,j}\varphi_i = c_{i,j}\varphi_i$, $i = 1; 2; ...; m$. Thus, it is proved that the coefficients $c_{i,1}, c_{i,2}, ..., c_{i,m}$ $i = 1; 2; ...; m$ in equation (3) are the eigen values of the operators $K_{i,1}, K_{i,2}, ..., K_{i,m}$ corresponding to the joint eigen element φ_i, $i = 1; 2; ...; m$ or $K_{i,j}\varphi_i = 0$, (in the case $c_{i,j} = 0$), respectively. The theorem is proved.

Theorem 3. *Let set (1) consist of all the eigen elements of problem (2) where $K_{i,1}, K_{i,2}, ..., K_{i,m}$, $i = 1; 2; ...; m$ are the compact, self-adjoint, permutational operators in the Hilbert space H_i, $i = 1; 2; ...; m$, and in problem (2) the right determinacy condition be fulfilled in the form*

$$(\Delta_0 \Phi, \Phi) = \det\left((K_{i,j}\varphi_i, \varphi_i)\right)^{\otimes}_{\substack{i=1,2,...,m\\ j=1,2,...,m}} > 0, \tag{5}$$

then the closure of the linear subspaces $L\left(\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}, ...\right)$, $i = 1; 2; ...; m$ coincide with the spaces $H_i, i = 1; 2; ...; m$, respertively, i.e. $L\left(\varphi_{i1}, \varphi_{i2}, ..., \varphi_{in}, ...\right) = H_i$, $i = 1; 2; ...; m$.

Proof. Let, vice versa, even if for one value of the index i the equality $L\left(\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}, ...\right) = H_i$ be not fulfilled, i.e. for this value of the index i there exists a subspace $E_i \neq \varnothing$ for which the following equality holds:

$$L\left(\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}, ...\right) \otimes E_i = H_i.$$

The elements of the sequence $\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}, ...$ are the eigen elements of the i-th equation of problem (1). It is known that the Hilbert space H_i may be represented in the form $H_i = \operatorname{Im} K_{i,j} \oplus KerK_{i,j}$, $i, j = 1; 2; ...; m$. Taking into account this relation, we can write $\varphi_{i,n} \perp \bigcap\limits_{j=1}^{m} KerK_{i,j}$. The last relation means

$$L\left(\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}, ...\right) \perp \bigcap_{j=1}^{m} KerK_{i,j}.$$

Consequently, $E_i \supset \bigcap\limits_{j=1}^{m} KerK_{i,j}$. By theorem 2, each element of the sequence $\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}...$ is a joint eigen element of the operators $K_{i,1}, K_{i,2}, ..., K_{i,m}, i = 1; 2; ...; m$ (consequently, is an element of all the subspaces $\operatorname{Im} K_{i,j} \subset H_i$) or $K_{i,j}\varphi_{i,n} = 0$ in some values of the index j (in the cases when $c_{i,j} = 0$ in equality (3)) (consequently, in the cases when $c_{i,j} = 0$ is an element of the appropriate subspace $KerK_{i,j} \subset H_i$, and in the cases when $c_{i,j} \neq 0$ it is an element of the appropriate subspace $\operatorname{Im} K_{i,j} \subset H_i$). From relation (3) It follows that the equalities $c_{i,1} = 0$, $c_{i,2} = 0$,...,$c_{i,m} = 0$ may not be true simultaneously. Consequently, the equalities $K_{i,1}\varphi_{in} = 0$, $K_{i,2}\varphi_{in} = 0, ..., K_{i,m}\varphi_{in} = 0$ may not be true simultaneously. Therefore, $\bigcap\limits_{j=1}^{m} KerK_{i,j} = 0$, $i = 1; 2; ..., m$. By the permutation property of the operators $K_{i,1}, K_{i,2}, ..., K_{i,m}$ the subspace E_i is an invariant subspace for all the operators $K_{i,1}, K_{i,2}, ..., K_{i,m}$. Therefore, the equality i is fulfilled for the appropriate value of the index $K_{i,1}E_i = K_{i,2}E_i = ... = K_{i,m}E_i = E_i$. Consequently, there exists an element $\varphi_0 \in E_i$ for which the equalities $K_{i,j}\varphi_0 \in \eta_j\varphi_0$ are fulfilled, where even if one of the numbers $\eta_1, \eta_2, ..., \eta_m$ doesn't equal zero, i.e. $\eta_1^2 + \eta_2^2 + ... + \eta_m^2 \neq 0$. According to theorem 1, the spectral hypersurface in the form $\sum\limits_{r=1}^{m} \lambda_r c_{i,r}$, where

$c_{i,r} = (K_{i,r}\varphi_0, \varphi_0) = \eta_r(\varphi_0, \varphi_0)$, corresponds to the element $\varphi_0 \in E_i$. Consider the weakly connected (i.e. connected only with the parameters $\lambda_r, r = 1, ..., m$) system of equations

$$\begin{cases} \sum\limits_{r=1}^{m} \lambda_r K_{i,r}\varphi_{i,n} = \varphi_{i,n}, \\ i = 1; 2; ...; j-1; j+1j...jm \\ \sum\limits_{r=1}^{m} \lambda_r K_{j,r}\varphi_0 = \varphi_0. \end{cases} \tag{6}$$

This system of equations may not have a solution, since otherwise the element $\Phi_{n,0} = \varphi_{1,n} \otimes \varphi_{2,n} \otimes ... \otimes \varphi_{j-1,n} \otimes \varphi_0 \otimes \varphi_{j+1,n} \otimes ... \otimes \varphi_{m,n} \subset H = H_1 \otimes H_2 \otimes ... \otimes H_m$ is an eigen element of problem (2). This means that $\varphi_0 \in L(\varphi_{i1}, \varphi_{i2}, ..., \varphi_{in}, ...)$ since the set (1) consists of all the eigen elements of problem (2). This contradicts the condition $\varphi_0 \in E_i$. Therefore, system of equations (6), and consequently the system of equations

$$\begin{cases} \sum\limits_{r=1}^{m} \lambda_r \left(K_{i,r}\varphi_{i,n}, \varphi_{i,n}\right) = \left(\varphi_{i,n}, \varphi_{i,n}\right), \\ i = 1; 2; ...; j-1; j+1; ...; m \\ \sum\limits_{r=1}^{m} \lambda_r \left(K_{j,r}\varphi_0, \varphi_0\right) = (\varphi_0, \varphi_0) \end{cases}$$

has no solution. Then the principal determinant of the last system of equations equals zero, i.e. $(\Delta_0\Phi_{n,0}, \Phi_{n,0}) = 0$. This contradicts condition (5). The obtained contradiction shows that the subspace E_i, may not contain a non-zero element, consequently, $L\left(\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,n}, ...\right) = H_i$. The theorem is proved.

Corollary. *If in problem (2) the right determinacy condition in the form (5) is fulfilled, then the closure of the linear subspace $L(\Phi_1, \Phi_2, ..., \Phi_n, ...)$ coincides with the space $H = H_1 \otimes H_2 \otimes ... \otimes H_m$ i.e. $\overline{L}(\Phi_1, \Phi_2, ..., \Phi_n, ...) = H$.*

If the set $\{e_{i,1}, e_{i,2}, ..., e_{i,n}, ...\} \subset H_i$ is an orthonormed basis of the space $H_i, i = 1; 2; ...; m$ consisting of all the joint eigen elements of the family of compact, self-adjoint, permutational operators $K_{i,1}, K_{i,2}, ..., K_{i,m}$, corresponding to the eigen values $\alpha_{i,1,n_i}, \alpha_{i,2,n_i}, ..., \alpha_{i,m,n_i}$ (i.e. $K_{i,j}e_{m_i} = \alpha_{i,j,n_i}e_{i,n_i}$) then the all possible expandible tensors of the form $E_{n_1,n_2,...,n_m} = e_{1,n_1} \otimes e_{2,n_2} \otimes ... \otimes e_{m,n_m} \in H = H_1 \otimes H_2 \otimes ... \otimes H_m$, $n_1, n_2, ..., n_m = 1, 2, ...$ are the eigen elements of problem (2) that correspond to the eigen values $(\lambda_{1,n_1,n_2,...,n_m}, \lambda_{2,n_1,n_2,...,n_m}, ..., \lambda_{m,n_1,n_2,...,n_m})$. Here the coordinates $\lambda_{i,n_1,n_2,...,n_m}$ are the solutions of the system of equations

$$\begin{cases} \sum\limits_{j=1}^{m} \lambda_j \alpha_{i,j,n_i} = 1, \\ i = 1; 2; ...; m \end{cases}$$

whose solutions are found by the Kramer method in the form

$$\lambda_{i,n_1,n_2,...,n_m} = \frac{D_{i,n_1,n_2,...,n_m}}{D_{0,n_1,n_2,...,n_m}}, \tag{7}$$

$$D_{0,n_1,n_2,...,n_m} = \det\left(\alpha_{i,j,n_i}\right)_{\substack{i=1,2,...,m \\ j=1,2,...,m}},$$

$$D_{i,n_1,n_2,...,n_m} = \det \begin{pmatrix} \alpha_{i,1,n_i} & ... & \alpha_{i,i-1,n_i} & 1 & \alpha_{i,i+1,n_i} & ... & \alpha_{i,m,n_i} \\ ... & ... & ... & ... & ... & ... & ... \\ \alpha_{i,1,n_i} & ... & \alpha_{i,i-1,n_i} & 1 & \alpha_{i,i+1,n_i} & ... & \alpha_{i,m,n_i} \end{pmatrix}$$
$$i = 1, 2, ..., m$$

And vice versa, an arbitrary eigen element $\Phi \in H = H_1 \otimes H_2 \otimes ... \otimes H_m$ of problem (2) is an expandible tensor $\Phi = \varphi_1 \otimes \varphi_2 \otimes ... \otimes \varphi_m$, where $\varphi_i \in H_i$ is a joint eigen element of the operators $K_{i,1}, K_{i,2}, ..., K_{i,m} i = 1; 2; ...; m$, i.e. if all the eigen values of the permutational operators $K_{i,1}, K_{i,2}, ..., K_{i,m} i = 1; 2; ...; m$, are known, then the eigen values $(\lambda_{1,n_1,n_2,...,n_m}, \lambda_{2,n_1,n_2,...,n_m}, ..., \lambda_{m,n_1,n_2,...,n_m})$ of problem (2) are found by means of equalities (7).

Using the above motioned facts, we formulate the inverse problem. Let the following four conditions be fulfilled:

1) $\{(\lambda_{1,n}, \lambda_{2,n}, ..., \lambda_{m,n})\}_{n=1,2,...} \subset R^m$ is a sequence of collections of real numbers.

2) $\{\Phi_n\}_{n=1,2,...} = \{\varphi_{1,n} \otimes \varphi_{2,n} \otimes ... \otimes \varphi_{m,n}\}_{n=1,2,...} \subset H = H_1 \otimes H_2 \otimes ... \otimes H_m$ is some sequence corresponding to the elements of the sequence $\{(\lambda_{1,n}, \lambda_{2,n}, ..., \lambda_{m,n})\}_{n=1,2,...}$,

3) the following equality holds: $(\Delta_0 \Phi_n, \Phi_n) = \det \left((K_{i,j}\varphi_{i,n}, \varphi_{i,n}) \right)^{\otimes}_{\substack{i=1,2,...,m \\ j=1,2,...,m}} > 0$,

4)

$$\left\{ \varphi_{1,p} \otimes \varphi_{2,p} \otimes ... \otimes \varphi_{m,p} \in \{\Phi_n\}_{n=1,2,...,} \wedge \varphi_{1,q} \otimes \varphi_{2,q} \otimes ... \otimes \varphi_{m,q} \in \{\Phi_n\}_{n=1,2,...,} \right\} \Rightarrow$$

$$\Rightarrow \varphi_{1,r_1} \otimes \varphi_{2,r_2} \otimes ... \otimes \varphi_{2,r_m} \in \{\Phi_n\}_{n=1,2,...,}$$

where $r_i = p$ or $r_i = q$, $i = 1; 2; ...; m$.

By condition 4), the elements of the sequence of eigen elements $\{\Phi_n\}_{n=1,2,...} = \{\varphi_{1,n} \otimes \varphi_{2,n} \otimes ... \otimes \varphi_{m,n}\}_{n=1,2,...}$ of problem (2) consist of all possible tensor products of the form $\Phi_{k_1,k_2,...,k_m} = \varphi_{1,n_{k_1}} \otimes \varphi_{2,n_{k_2}} \otimes ... \otimes \varphi_{m,n_{k_m}}$ where the sequence $\left\{\varphi_{i,n_{k_2}}\right\}$, $i = 1, 2, ..., m$ consists of all linearly independent eigen elements of the i-th equation of problem (2). From the sequence $\{\Phi_n\} = \{\varphi_{1,n} \otimes \varphi_{2,n} \otimes ... \otimes \varphi_{m,n}\}$ (consequently from the sequence

$$\Phi_{k_1,k_2,...,k_m} = \varphi_{1,n_{k_1}} \otimes \varphi_{2,n_{k_2}} \otimes ... \otimes \varphi_{m,n_{k_m}} \subset H = H_1 \otimes H_2 \otimes ... \otimes H_m)$$

we choose the subsequence

$$\left\{\Phi_{(2^m-1)k,(2^m-1)k,...,(2^m-1)k}\right\} = \left\{\varphi_{1,n_{k_1}} \otimes \varphi_{2,n_{k_2}} \otimes ... \otimes \varphi_{m,n_{k_m}}\right\}_{k=1,2,...,}$$

in the following way:

a) $\Phi_{(2^m-1),(2^m-1),...,(2^m-1)} = \varphi_{1,1} \otimes \varphi_{2,1} \otimes ... \otimes \varphi_{m,1}$ i.e. $\varphi_{i,n_1} = \varphi_{i,1}$, $i = 1; 2; ...; m$

b) $\Phi_{(2^m-1)(k+1),(2^m-1)(k+1),...,(2^m-1)(k+1)} = \varphi_{1,n_{k+1}} \otimes \varphi_{2,n_{k+1}} \otimes ... \otimes \varphi_{m,n_{k+1}}$ where $\varphi_{in_{k+1}} \notin L\left(\varphi_{i,n_1}, \varphi_{i,n_2}, ..., \varphi_{i,n_k}\right)$ $i = 1; 2; ...; m$.

The eigen values corresponding to the eigen elements

$$\Phi_{(2^m-1)k,(2^m-1)k,...,(2^m-1)k} = \varphi_{1,n_k} \otimes \varphi_{2,n_k} \otimes ... \otimes \varphi_{m,n_k}$$

denote by $\left(\lambda_{1,n_{(2^m-1)k}}, \lambda_{2,n_{(2^m-1)k}}, ..., \lambda_{m,n_{(2^m-1)k}}\right)$.

The eigen values corresponding to the eigen elements

$$\Phi_{(2^m-1)k+1,(2^m-1)k,...,(2^m-1)k} = \varphi_{1,n_{k+1}} \otimes \varphi_{2,n_k} \otimes ... \otimes \varphi_{m,n_k}$$

$$\Phi_{(2^m-1)k,(2^m-1)k+1,...,(2^m-1)k} = \varphi_{1,n_k} \otimes \varphi_{2,n_{k+1}} \otimes ... \otimes \varphi_{m,n_k}, ...,$$

$$\Phi_{(2^m-1)k,(2^m-1)k,...,(2^m-1)k+1} = \varphi_{1,n_k} \otimes \varphi_{2,n_k} \otimes ... \otimes \varphi_{m,n_{k+1}}$$

denote by

$$\left(\lambda_{1,n_{(2^m-1)k+1}}, \lambda_{2,n_{(2^m-1)k+1}}, ..., \lambda_{m,n_{(2^m-1)k+1}}\right),$$

$$\left(\lambda_{1,n_{(2^m-1)k+2}}, \lambda_{2,n_{(2^m-1)k+2}}, ..., \lambda_{m,n_{(2^m-1)k+2}}\right), ...,$$

$$\left(\lambda_{1,n_{(2^m-1)k+m}}, \lambda_{2,n_{(2^m-1)k+m}}, ..., \lambda_{m,n_{(2^m-1)k+m}}\right),$$

respectively.

Now, we can state a theorem that answers to the question on the inverse problem.

Theorem 3. Let conditions 1)-4) be fulfilled, then there exists such a subsequence $\{n_k\}_{k=1,2,...,} \subset N$ that $\{\varphi_{i,n_k}\} \subset H_i$, $i = 1;2;...;m$ is a complete system of joint eigen elements of permutational compact operators

$$K_{i,1}, K_{i,2}, ..., K_{i,2}i = 1;2;...; \text{ where } K_{i,j} = \sum_{k=1}^{\infty} \alpha_{i,j,n_k} P_{i,k},$$

$$\alpha_{i,j,n_k} = \frac{D_{i,j,n_k}}{D_{i,0,n_k}}, \tag{8}$$

here

$$D_{i,0,n_k} =$$

$$= \det \begin{pmatrix} \lambda_{1,n_{(2^m-1)k+1}} & ... & \lambda_{j-1,n_{(2^m-1)k+1}} & \lambda_{j,n_{(2^m-1)k+1}} & \lambda_{j+1,n_{(2^m-1)k+1}} & ... & \lambda_{m,n_{(2^m-1)k+1}} \\ ... & ... & ... & ... & ... & ... & ... \\ \lambda_{1,n_{(2^m-1)k+i-1}} & ... & \lambda_{j-1,n_{(2^m-1)k+i-1}} & \lambda_{j,n_{(2^m-1)k+i-1}} & \lambda_{j+1,n_{(2^m-1)k+i-1}} & ... & \lambda_{m,n_{(2^m-1)k+i-1}} \\ \lambda_{1,n_{(2^m-1)k}} & ... & \lambda_{j-1,n_{(2^m-1)k}} & \lambda_{j,n_{(2^m-1)k}} & \lambda_{j+1,n_{(2^m-1)k}} & ... & \lambda_{m,n_{(2^m-1)k}} \\ \lambda_{1,n_{(2^m-1)k+i+1}} & ... & \lambda_{j-1,n_{(2^m-1)k+i+1}} & \lambda_{j,n_{(2^m-1)k+i+1}} & \lambda_{j+1,n_{(2^m-1)k+i+1}} & ... & \lambda_{m,n_{(2^m-1)k+i+1}} \\ ... & ... & ... & ... & ... & ... & ... \\ \lambda_{1,n_{(2^m-1)k+m}} & ... & \lambda_{j-1,n_{(2^m-1)k+m}} & \lambda_{j,n_{(2^m-1)k+m}} & \lambda_{j+1,n_{(2^m-1)k+m}} & ... & \lambda_{m,n_{(2^m-1)k+m}} \end{pmatrix}$$

$$D_{i,j,n_k} =$$

$$= \det \begin{pmatrix} \lambda_{1,n_{(2^m-1)k+1}} & ... & \lambda_{j-1,n_{(2^m-1)k+1}} & 1 & \lambda_{j+1,n_{(2^m-1)k+1}} & ... & \lambda_{m,n_{(2^m-1)k+1}} \\ ... & ... & ... & ... & ... & ... & ... \\ \lambda_{1,n_{(2^m-1)k+i-1}} & ... & \lambda_{j-1,n_{(2^m-1)k+i-1}} & 1 & \lambda_{j+1,n_{(2^m-1)k+i-1}} & ... & \lambda_{m,n_{(2^m-1)k+i-1}} \\ \lambda_{1,n_{(2^m-1)k}} & ... & \lambda_{j-1,n_{(2^m-1)k}} & 1 & \lambda_{j+1,n_{(2^m-1)k}} & ... & \lambda_{m,n_{(2^m-1)k}} \\ \lambda_{1,n_{(2^m-1)k+i+1}} & ... & \lambda_{j-1,n_{(2^m-1)k+i+1}} & 1 & \lambda_{j+1,n_{(2^m-1)k+i+1}} & ... & \lambda_{m,n_{(2^m-1)k+i+1}} \\ ... & ... & ... & 1 & ... & ... & ... \\ \lambda_{1,n_{(2^m-1)k+m}} & ... & \lambda_{j-1,n_{(2^m-1)k+m}} & 1 & \lambda_{j+1,n_{(2^m-1)k+m}} & ... & \lambda_{m,n_{(2^m-1)k+m}} \end{pmatrix}$$

and $\left(\lambda_{1,n_{(2^m-1)k}}, \lambda_{2,n_{(2^m-1)k}}, ..., \lambda_{m,n_{(2^m-1)k}}\right)$, $\left(\lambda_{1,n_{(2^m-1)k+1}}, \lambda_{2,n_{(2^m-1)k+1}}, ..., \lambda_{m,n_{(2^m-1)k+1}}\right)$, $\left(\lambda_{1,n_{(2^m-1)k+2}}, \lambda_{2,n_{(2^m-1)k+2}}, ..., \lambda_{m,n_{(2^m-1)k+2}}\right)$,..., $\left(\lambda_{1,n_{(2^m-1)k+m}}, \lambda_{2,n_{(2^m-1)k+m}}, ..., \lambda_{m,n_{(2^m-1)k+m}}\right)$ are the eigen values of problem (2) that correspond to the eigen elements

$$\Phi_{(2^m-1)k,(2^m-1)k,...,(2^m-1)k} = \varphi_{1,n_k} \otimes \varphi_{2,n_k} \otimes ... \otimes \varphi_{m,n_k},$$

$$\Phi_{(2^m-1)k+1,(2^m-1)k,...,(2^m-1)k} = \varphi_{1,n_{k+1}} \otimes \varphi_{2,n_k} \otimes ... \otimes \varphi_{m,n_k},$$
$$\Phi_{(2^m-1)k,(2^m-1)k+1,...,(2^m-1)k} = \varphi_{1,n_k} \otimes \varphi_{2,n_{k+1}} \otimes ... \otimes \varphi_{m,n_k}, ...,$$
$$\Phi_{(2^m-1)k,(2^m-1)k,...,(2^m-1)k+1} = \varphi_{1,n_k} \otimes \varphi_{2,n_k} \otimes ... \otimes \varphi_{m,n_{k+1}}$$

respectively. $P_{i,k}$ is an operator of projection onto one-dimensional space $L\left\{\varphi_{i,n_k}\right\} \subset H_i$, $i = 1; 2; ...; m$.

Proof. The sequence of eigen elements $\{\Phi_n\} = \left\{\varphi_{1,n} \otimes \varphi_{2,n} \otimes ... \otimes \varphi_{m,n}\right\}$ of problem (2) consists of all possible tensor products of the form $\Phi_{k_1,k_2,...,k_m} = \varphi_{1,n_{k_1}} \otimes \varphi_{2,n_{k_2}} \otimes ... \otimes \varphi_{m,n_{k_m}}$, where the sequence $\left\{\varphi_{i,n_{k_2}}\right\}$, $i = 1, 2, ..., m$ consists of linearly independent eigen elements of the i-th equation of problem (2). From the sequence $\{\Phi_n\} = \left\{\varphi_{1,n} \otimes \varphi_{2,n} \otimes ... \otimes \varphi_{m,n}\right\}$ (consequently from the sequence

$$\Phi_{k_1,k_2,...,k_m} = \varphi_{1,n_{k_1}} \otimes \varphi_{2,n_{k_2}} \otimes ... \otimes \varphi_{m,n_{k_m}} \subset H = H_1 \otimes H_2 \otimes ... \otimes H_m\Big)$$

we choose the subsequence

$$\left\{\Phi_{(2^m-1)k,(2^m-1)k,...,(2^m-1)k}\right\} = \left\{\varphi_{1,n_k} \otimes \varphi_{2,n_k} \otimes ... \otimes \varphi_{m,n_k}\right\}_{k=1,2,...}$$

in the following way:

a) $\Phi_{(2^m-1),(2^m-1),...,(2^m-1)} = \varphi_{1,1} \otimes \varphi_{2,1} \otimes ... \otimes \varphi_{m,1}$ i.e. $\varphi_{i,n_1} = \varphi_{i,1}$, $i = 1; 2; ...; m$

b) $\Phi_{(2^m-1)(k+1),(2^m-1)(k+1),...,(2^m-1)(k+1)} = \varphi_{1,n_{k+1}} \otimes \varphi_{2,n_{k+1}} \otimes ... \otimes \varphi_{m,n_{k+1}}$ where $\varphi_{in_{k+1}} \notin L\left(\varphi_{i,n_1}, \varphi_{i,n_2}, ..., \varphi_{i,n_k}\right)$ $i = 1; 2; ...; m$.

The eigen values corresponding to the eigen elements

$$\Phi_{(2^m-1)k,(2^m-1)k,...,(2^m-1)k} = \varphi_{1,n_k} \otimes \varphi_{2,n_k} \otimes ... \otimes \varphi_{m,n_k}$$

denote by $\left(\lambda_{1,n_{(2^m-1)k}}, \lambda_{2,n_{(2^m-1)k}}, ..., \lambda_{m,n_{(2^m-1)k}}\right)$. The eigen values corresponding to the eigen elements

$$\Phi_{(2^m-1)k+1,(2^m-1)k,...,(2^m-1)k} = \varphi_{1,n_{k+1}} \otimes \varphi_{2,n_k} \otimes ... \otimes \varphi_{m,n_k}$$
$$\Phi_{(2^m-1)k,(2^m-1)k+1,...,(2^m-1)k} = \varphi_{1,n_k} \otimes \varphi_{2,n_{k+1}} \otimes ... \otimes \varphi_{m,n_k}, ...,$$
$$\Phi_{(2^m-1)k,(2^m-1)k,...,(2^m-1)k+1} = \varphi_{1,n_k} \otimes \varphi_{2,n_k} \otimes ... \otimes \varphi_{m,n_{k+1}}$$

denote by

$$\left(\lambda_{1,n_{(2^m-1)k+1}}, \lambda_{2,n_{(2^m-1)k+1}}, ..., \lambda_{m,n_{(2^m-1)k+1}}\right),$$
$$\left(\lambda_{1,n_{(2^m-1)k+2}}, \lambda_{2,n_{(2^m-1)k+2}}, ..., \lambda_{m,n_{(2^m-1)k+2}}\right), ...,$$
$$\left(\lambda_{1,n_{(2^m-1)k+m}}, \lambda_{2,n_{(2^m-1)k+m}}, ..., \lambda_{m,n_{(2^m-1)k+m}}\right)$$

respectively.

One spectral hypersurface π_{i,n_k} of the equation of the form $\sum_{j=1}^{m} \alpha_{i,j,n_k}\lambda_j = 1$, $i = 1; 2; ...; m$ corresponds to each eigen element φ_{i,n_k} of the i-th equation of system (2). Consequently, to each eigen element

$$\Phi_{(2^m-1)k,(2^m-1)k,...,(2^m-1)k} = \varphi_{1,n_k} \otimes \varphi_{2,n_k} \otimes ... \otimes \varphi_{m,n_k}$$

of system (2) there corresponds a family of spectral hypersurfaces $\pi_{1,k}, \pi_{2,k}, ..., \pi_{m,k}$. The intersection of the hypersurfaces $\pi_{1,k}, \pi_{2,k}, ..., \pi_{i-1,k}, \pi_{i,k+1}, \pi_{i+1,k}, ..., \pi_{m,k}$ will be $\left(\lambda_{1,n_{(2^m-1)k+i}}, \lambda_{2,n_{(2^m-1)k+i}}, ..., \lambda_{m,n_{(2^m-1)k+i}}\right)$, $i = 1; 2; ...; m$.

Then the follosing family of the system of m equations hold:

$$\begin{cases} \sum\limits_{j=1}^{m} \alpha_{i,j,n_k} \lambda_{j,n_{(2^m-1)k}} = 1, \\ \sum\limits_{j=1}^{m} \alpha_{i,j,n_k} \lambda_{j,n_{(2^m-1)k+r}} = 1, \quad \text{for } r = 1; 2; ...; r \neq i \end{cases}$$

From the last system of equations we get $\alpha_{i,j,n_k} = \dfrac{D_{i,j,n_k}}{D_{i,0,n_k}}$, where D_{i,j,n_k} and $D_{i,0,n_k}$ are found in the form (8).

Consequently, for the compact self-adjoint operators $K_{i,1}, K_{i,2}, ..., K_{i,m}$, $i = 1; 2; ...; m$ the following expansions hold:

$$K_{i,j} = \sum_{k=1}^{\infty} \alpha_{i,j,n_k} P_{i,k},$$

where $P_{i,k}$ is an operator of projection onto one-dimensional subspace $L\left\{\varphi_{in_k}\right\} \subset H_i, i = 1; 2; ...; m$.

The completeness of eigen elements $\left\{\varphi_{i,n_k}\right\} \subset H_i, i = 1; 2; ...; m$ is easily obtained from the condition $\overline{L}\left(\Phi_1, \Phi_2, ..., \Phi_n, ...\right) = H$. The theorem is proved.

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