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COMPLETENESS OF PART OF EIGEN AND ADJOINT VECTORS OF A CLASS OF FOURTH ORDER POLYNOMIAL OPERATOR PENCILS

Abstract

In the paper, the completeness of a part of the system of eigen and adjoint vectors of a class of fourth order polynomial operator pencils responding to a boundary value problem on a semi-axis, is proved. The distinctive feature of the pencil under investigation is the availability of such kind multiple characteristic in its principal part that has not been studied earlier.

The first basic results in spectral theory of abstract not self-adjoint operators were obtained by M.V.Keldysh in the paper [1] (see in detail [2]) on n - fold completeness of eigen and adjoint vectors. Some of the further developments of the paper [1] were the results on the completeness of eigen and adjoint vectors responding to boundary value problems on a semi-axis obtained in the papers of M.G. Gasymov [3-5], wherein the original investigation method was suggested. This method is connected with obtaining different tests for solvability of boundary value problems whence completeness of a part of eigen and adjoint vectors is concluded. Later on these results were developed in the papers [6, 7]. The fundamental works of M.G. Krein, G.K. Langer, G.V. Radzievskii and other authors on different problems of completeness of the system of eigen and adjoint vectors should also be noted. The results of these works have found reflection in the survey paper [8].

It is well known that theories occur while considering specific model problems. Many problems of mechanics urge on the matter of the completeness of elementary solutions of boundary value problems for operator-differential equations. But often this matter requires to study the completeness of a part of eigen and adjoinr vectors of polynomial operator pencils generated by these boundary value problems. The present paper is devoted to the matter of the completeness of the system of eigen and adjoint vectors of a class fourth order polynomial operator pencils responding to a boundary value problem on a semi-axis and that arise in solving the problems on stability of plates made of plastic material.

Consider in a separable Hilbert space H a polynomial operator pencil

$$P(\lambda) = (\lambda E - A)^3 (\lambda E + A) + \sum_{j=1}^{3} \lambda^{4-j} A_j, \qquad (1)$$

where E is a unit operator, A, A_j, $j = 1, 2, 3$ are linear operators, A is a self-adjoint positive-definite operator with domain of definition $D(A)$, the operators $A_j A^{-j}$, $j = 1, 2, 3$ are bounded in H.

Denote by H_γ the scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $\gamma \geq 0$, $(x, y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in D(A^\gamma)$.

Following the book [9], determine the space of vector-functions

$$W_2^4\left(R_+;H\right)=\left\{u\left(t\right):\frac{d^4u\left(t\right)}{dt^4}\in L_2\left(R_+;H\right),A^4u\left(t\right)\in L_2\left(R_+;H\right)\right\}$$

with the norm

$$\|u\|_{W_2^4(R_+;H)}=\left(\left\|\frac{d^4u}{dt^4}\right\|^2_{L_2(R_+;H)}+\left\|A^4u\right\|^2_{L_2(R_+;H)}\right)^{1/2}$$

where $L_2\left(R_+;H\right)$ denotes the Hilbert space of all vector-functions $f\left(t\right)$ determined in $R_+=[0,+\infty)$ with values from H that have the finite norm

$$\|f\|_{L_2(R_+;H)}=\left(\int_0^{+\infty}\|f\left(t\right)\|^2\,dt\right)^{1/2}.$$

Here and in the sequel, the derivatives are understood in the sense of distributions theory [9].

Associate pencil (1) with the following boundary value problem

$$P\left(d/dt\right)u\left(t\right)=0,\quad t\in R_+, \tag{2}$$

$$u\left(0\right)=\varphi,\quad \varphi\in H_{7/2}, \tag{3}$$

where $u\left(t\right)\in W_2^4\left(R_+;H\right)$.

Definition 1. *If for any $\varphi\in H_{7/2}$ there exists a vector-function $u\left(t\right)\in W_2^4\left(R_+;H\right)$ that satisfies equation (2) almost everywhere in R_+, boundary condition (3) in the sense of convergence of the space $H_{7/2}$:*

$$\lim_{t\to0}\|u\left(t\right)-\varphi\|_{7/2}=0,$$

and it holds the inequality

$$\|u\|_{W_2^4(R_+;H)}\le const\,\|\varphi\|_{7/2}\,,$$

problem (2), (3) is said to be regularly solvable, and $u\left(t\right)$ a regular solution of problem (2), (3).

Below we give the definition and denotation for clearness of the further statement of the paper.

Definition 2. *If the equation $P\left(\lambda_0\right)\psi=0$ has the non-trivial solution ψ_0, then λ_0 is said to be an eigen value of the operator pencil $P\left(\lambda\right)$, and ψ_0 an eigen vector of the operator pencil $P\left(\lambda\right)$ responding to the eigen value λ_0.*

Definition 3. *Let λ_0 be an eigen value, ψ_0 be one of the eigen vectors responding to be value λ_0. The system of vectors $\psi_1,\psi_2,...,\psi_m$ is called a chain of vectors adjoined to the eigen vector ψ_0 if the vectors of this chain satisfy the following equations:*

$$P\left(\lambda_0\right)\psi_k+\frac{P'\left(\lambda_0\right)}{1!}\psi_{k-1}+\frac{P''\left(\lambda_0\right)}{2!}\psi_{k-2}+\frac{P'''\left(\lambda_0\right)}{3!}\psi_{k-3}+\frac{P''''\left(\lambda_0\right)}{4!}\psi_{k-4}=0,$$

$$k = 0, 1, 2, ..., m, \quad \psi_{-1} = \psi_{-2} = \psi_{-3} = \psi_{-4} = 0,$$

where

$$P'(\lambda_0) = 4\lambda_0^3 E - 6\lambda_0^2 A + 2A^3 + 3\lambda_0^2 A_1 + 2\lambda_0 A_2 + A_3,$$

$$P''(\lambda_0) = 12\lambda_0^2 E - 12\lambda_0 A + 6\lambda_0 A_1 + 2A_2,$$

$$P'''(\lambda_0) = 24\lambda_0 E - 12A + 6A_1, \quad P''''(\lambda_0) = 24E.$$

Under $\sigma_\infty(H)$we understand the set of completely continuous operators acting in H, under $L(H)$ the set of linear bounded operators acting in H.

It we assume that the self-adjoint positive-definite operator A has $A^{-1} \in \sigma_\infty(H)$, we can show that under the condition $A_j A^{-j} \in L(H)$, $j = 1, 2, 3$ operator pencil (1) has a discrete spectrum with a unique limit point at infinity. Indeed,

$$P(\lambda) = \lambda^4 E - 2\lambda^3 A + 2\lambda A^3 - A^4 + \sum_{j=1}^{3} \lambda^{4-j} A_j = (K(\lambda) - E) A^4,$$

where

$$K(\lambda) = \lambda^4 A^{-4} - 2\lambda^3 A^{-3} + 2\lambda A^{-1} + \sum_{j=1}^{3} \lambda^{4-j} A_j A^{-j} A^{-4+j}.$$

Since the operators $A_j A^{-j} \in L(H)$, $j = 1, 2, 3$, $A^{-1} \in \sigma_\infty(H)$, then $K(\lambda) \in \sigma_\infty(H)$ for any $\lambda \in C$ (C is a complex plane) and $K(0) - E = -E$ is invertible. Consequently, $K(\lambda) - E$ by the M.V. Keldysh lemma [2] is dense everywhere except the isolated points that are the eigen values of the pencil $K(\lambda) - E$ and have a limit point only at infinity. And it follows from the representation $P(\lambda) = (K(\lambda) - E) A^4$ that operator pencil (1) also has this property.

Further, in the course of the paper we assume $A^{-1} \in \sigma_\infty.(H)$.

If λ_n $(\mathrm{Re}\,\lambda_n < 0)$ is an eigen value of the pencil (1), and $\{\psi_{0,n}, \psi_{1,n}, ..., \psi_{m,n}\}$ is a system of eigen and adjoint vectors responding to the eigen value λ_n, the vector-functions

$$u_{h,n}(t) = e^{\lambda_n t}\left(\psi_{h,n} + \frac{t}{1!}\psi_{h-1,n} + ... + \frac{t^h}{h!}\psi_{0,n}\right),$$

$$n = 0, 1, 2, ...; h = 0, 1, ..., m,$$

belong to $W_2^4(R_+; H)$, satisfy the equation $P(d/dt)\,u(t) = 0$ and are called elementary solutions of the homogeneous equation. Obviously, these elementary solutions $u_{h,n}(t)$ satisfy the boundary condition:

$$u_{h,n}(0) = \psi_{h,n}, h = 0, 1, ..., m.$$

Here, these arises a problem: under what conditions on the coefficients of pencil (1) the system $\{\psi_{h,n}\}_{n=0}^{\infty}$ is complete in $H_{7/2}$?

Before considering the stated problem investigate the resolvent of pencil (1). The estimations of the resolvent of the pencil $P(\lambda)$ on a imaginary axis and on the adherent sectors are given the two theorems given below.

Theorem 1. *Let A be self-adjoint positive-definite operator, the operators $A_jA^{-j} \in L(H)$, $j = 1, 2, 3$, and the following inequality be fulfilled*

$$\sum_{j=1}^{3} a_{4-j} \left\| A_j A^{-j} \right\| < 1,$$

where

$$a_1 = a_3 = \frac{3\sqrt{3}}{16}, \quad a_2 = \frac{1}{4}.$$

Then on the imaginary axis the resolvent of the pencil $P(\lambda)$ exists and the following estimates hold:

$$\sum_{s=0}^{3} \left\| \lambda^{4-s} A^s P^{-1}(\lambda) \right\| \le const, \tag{4}$$

$$\left\| A^q P^{-1}(\lambda) \right\| \le const\, |\lambda|^{q-4}, \quad 0 < q < 4, \quad \lambda \neq 0. \tag{5}$$

Theorem 2. *Let the conditions of theorem 1 be fulfilled. Then for sufficiently small $\gamma > 0$ on the sectors*

$$S_{-\frac{\pi}{2}\pm\gamma} = \left\{ \lambda : \lambda = re^{i\left(-\frac{\pi}{2}\pm\gamma\right)}, r > 0 \right\},$$

$$S_{\frac{\pi}{2}\pm\gamma} = \left\{ \lambda : \lambda = re^{i\left(\frac{\pi}{2}\pm\gamma\right)}, r > 0 \right\}$$

the operator pencil $P(\lambda)$ is invertible and the estimates of the form (4) and (5) hold.

The proof of theorems 1 and 2 are similar to be appropriate theorems from [10].

Further, proceeding from the ideas of the paper [4], it is necessary to study regular solvability of boundary value problem (2), (3).

Theorem 3. *Let A be a self-adjoint positive-definite operator, the operators $A_jA^{-j} \in L(H)$, $j = 1, 2, 3$, and the following inequality be fulfilled:*

$$\sum_{j=1}^{3} n_{4-j} \left\| A_j A^{-j} \right\| < 1,$$

where the numbers n_j, $j = 1, 2, 3$ are determined as follows:

$$n_1 = \alpha^{-1/2}, n_2 = \beta_0^{-1/2}, n_3 = \frac{3\sqrt{3}}{16},$$

and

$$\alpha = \frac{4}{3^{4/3}\left(9 + \sqrt{57}\right)} \left[2 \cdot 3^{7/3} + \left(9 + \sqrt{57}\right)^{5/3} + 4 \cdot 3^{2/3} \cdot \left(9 + \sqrt{57}\right)^{1/3} \right],$$

β_0 is the solution of the equation $\beta^3 + 2\beta^2 - 39\beta - 140 = 0$ from the interval (0,16). Then boundary value problem (2), (3) is regularly solvable.

Proof. It is easily established that boundary value problem (2), (3) for $A_j = 0$, $j = 1, 2, 3$, i.e.

$$\left(\frac{d}{dt} - A\right)^3 \left(\frac{d}{dt} + A\right) u(t) = 0, \ t \in R_+, \tag{6}$$

$$u(0) = \varphi, \qquad \varphi \in H_{7/2}, \tag{7}$$

is regularly solvable. Further, for proving regular solvability of boundary value problem (2), (3) under assumption that even if one of A_j, $j = 1, 2, 3$ is not zero, the regular solution of this problem should be sought in the form $u(t) = u_0(t) + \nu(t)$, where $u_0(t)$ is a regular solution of problem (6), (7) and $\nu(t) \in W_2^4(R_+; H)$. Therefore, boundary value problem (2), (3) may be reduced to the following problem with respect to $\nu(t)$:

$$P(d/dt)\,\nu(t) = f(t), \tag{8}$$

$$\nu(0) = 0, \tag{9}$$

where $f(t) \in L_2(R_+; H)$. Subject to the conditions of the given theorem the regular solvability of problem (8), (9) is established in [11] that completes its proof. The theorem is proved.

It is known that if $B \in \sigma_\infty(H)$, then $(B^*B)^{1/2}$ is a completely continuous self-adjoint operator in H. The eigen values of the operator $(B^*B)^{1/2}$ will be called s - numbers of the operator B. We'll enumerate the non-zero s - numbers of the operator B in decreasing order with regard to their multiplicity. Denote

$$\sigma_p = \left\{ B : B \in \sigma_\infty(H) ; \sum_{k=1}^{\infty} s_k^p(B) < \infty \right\}, \quad 0 < p < \infty.$$

Now answer the stated main question of the paper.

Theorem 4. *Let the conditions of theorem 3 be fulfilled. If one of the following condition is fulfilled:*

1) $A^{-1} \in \sigma_p \, (0 < p \leq 1)$;

2) $A^{-1} \in \sigma_p \, (0 < p < \infty)$, $A_j A^{-j} \in \sigma_\infty(H)$, $j = 1, 2, 3$,

the system $\{\psi_{h,n}\}_{n=0}^{\infty}$ *is complete in the space* $H_{7/2}$.

Proof. Prove by contradiction. If the system $\{\psi_{h,n}\}_{n=0}^{\infty}$ is not complete in the space $H_{7/2}$ then there exists a non-zero vector $\psi \in H_{7/2}$ such that $(\psi, \psi_{h,n})_{7/2} = 0$, $n = 0, 1, 2, \ldots.$ Then from the expansions of the principal part of the resolvent in the vicinity of eigen values it follows that $\left(A^{7/2} P^{-1}\left(\overline{\lambda}\right)\right)^* A^{7/2}\psi$ will be a holomorphic vector-function in the half-plane $\Pi_- = \{\lambda : \operatorname{Re}\lambda < 0\}$. By the conditions of the theorem, problem (2) (3) is regularly solvable. If $u(t)$ is a regular solution of problem (2), (3), we can represent it in the form

$$u(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \widehat{u}(\lambda)\, e^{\lambda t} d\lambda, \tag{10}$$

where

$$\widehat{u}(\lambda) = P^{-1}(\lambda)\sum_{s=0}^{3} T_s u^{(3-s)}(0),$$

therewith

$$T_0 = E, \quad T_1 = \lambda E + Q, T_2 = \lambda^2 E + \lambda Q + A_2,$$

$$T_3 = \lambda^3 E + \lambda^2 Q + \lambda A_2 + A_3 + 2A^3, Q = -2A + A_1.$$

Taking into attention the paper [4], in (10) we can change the integration contour $\Gamma_{\pm\theta} = \left\{\lambda : \lambda = re^{\pm i\left(\frac{\pi}{2}+\theta\right)}, r > 0\right\}$. As a result we have that for $t > 0$

$$(u(t), \psi)_{7/2} = \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} \left(A^{7/2}P^{-1}(\lambda)\sum_{s=0}^{3} T_s u^{(3-s)}(0), A^{7/2}\psi\right) e^{\lambda t} d\lambda =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} \sum_{s=0}^{3}\left(T_s u^{(3-s)}(0), \left(A^{7/2}P^{-1}\left(\overline{\lambda}\right)\right)^* A^{7/2}\psi\right) e^{\lambda t} d\lambda = \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} q(\lambda) e^{\lambda t} d\lambda,$$

where

$$q(\lambda) = \sum_{s=0}^{3}\left(T_s u^{(3-s)}(0), \left(A^{7/2}P^{-1}\left(\overline{\lambda}\right)\right)^* A^{7/2}\psi\right).$$

Now, considering in the case 1) the estimates of the resolvent of pencil (1) (theorems 1 and 2), in the case 2) using the M.V. Keldysh theorem [2] with applying the Fragmen–Lindeloff theorem, we get that $q(\lambda)$ is a polynomial. Consequently, for $t > 0$

$$\frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} q(\lambda) e^{\lambda t} d\lambda = 0,$$

i.e. for $t > 0$

$$(u(t), \psi)_{7/2} = 0.$$

Passing to limit as $t \to 0$, we have

$$(\varphi, \psi)_{7/2} = 0, \forall\varphi \in H_{7/2}.$$

Consequently, $\psi = 0$. We get contradiction. The theorem is proved.

Note that the three-fold completeness of a part of eigen and adjoint vectors of a fourth order polynomial operator pencil that differs from the case under consideration, but also arises while solving the problems of stability of plates made of plastic material, is proved in the paper [12].

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