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REGULARITY ANALYSIS FOR NONLINEAR TIME OPTIMAL CONTROL PROBLEMS SUBJECT TO STATE CONSTRAINTS

Abstract

In this paper the necessary conditions for optimality are obtained for regular solutions of the time optimal control problems with state and endpoint constraints.

Introduction

For the first time Pontryagin maximum principle for problems with state constraints was obtained by Gamkrelidze R. V. in 1959 [1], [2]. In 1963 another variant of the maximum principle [3] has been received. After that, the matter was the subject of many studies [5], [6]. This list of works is not exhaustive.

In the case, where there is a restriction only on the control function and there are no state constraints, necessary optimality conditions gives Pontryagin's maximum principle [1]. These problems have been well studied because of the absolute continuity and non-triviality of adjoint functions.

The optimal control problems with state constraints are recognized as an important and difficult class of the similar problems, since the maximum principle for such problems [3], [5], [6] contains an unknown infinite-dimensional Lagrange multiplier of the complex nature–bounded regular Borel measure which has a rather complicated relationship with the optimal trajectory. Therefore, the optimal control problem with state constraints are outside the scope of the effective application of the Pontryagin maximum principle [1]. Questions arise: Are there any solutions of an optimal control problem for which the corresponding conjugate function is non-trivial and absolutely continuous, and if so, how to find them?

Applying a similar technique in [14], [15] we try to answer these questions for non-autonomous systems with phase and endpoint constraints.

Analogical question about the structure of the measures appearing in the ratios of the maximum principle for the classical optimal control problem was considered by W. W. Hager [7], K. Malanowski [8], Hoang Xuan Phu [9], H.Maurer [10], A. A. Milutin [11], J. F. Bonnans [13], for the differential inclusions by S. M. Aseev [12]. In [7] -[11], [13] sufficient conditions for the absence of a singular component obtained under the condition that the time optimal control function is continuous and takes values strictly in the interior of U.

In [12], sufficient conditions for the absence of a singular component were obtained under the condition that the set of admissible velocities is strictly convex and the Hamiltonian of the system satisfies certain smoothness conditions.

These and the results, obtained in this study are difficult to compare: they are all proved under different assumptions and have different conditions. Apparently, this issue needs a separate study.

Statement of the problem

Consider the time optimal control problem with state and endpoint constraints for non-autonomous system

$$\begin{gathered} \dot{x}=f\left(x,u,t\right), \\ x\left(0\right)\in C_{\alpha},\ x\left(T\right)\in C_{\beta} \\ u\left(t\right)\in U\left(t\right),\ a.a.\ t\in\left[0,T\right], \\ x\left(t\right)\in X,\quad t\in\left[0,T\right], \\ T\rightarrow\min. \end{gathered} \tag{1}$$

Here, $x\in E^n$–state variable, $u\in E^m$-control parameter. Let $\Omega\left(E^n\right)$ be the set of all nonempty compact and *conv* $\Omega\left(E^n\right)$ the set of all nonempty compact convex subset of E^n. Functions $f\left(x,u,t\right),\ \dfrac{\partial f}{\partial x}$ are continuous in (x,u) and measurable in t. Let the set valued map $U:E^1\rightarrow\Omega\left(E^m\right)$ be measurable and satisfy the estimate $\left|U\left(t\right)\right|\leq k\left(t\right)$ where $k\left(t\right)$ is a scalar function, Lebesgue integrable on any finite time interval $[0,T]$. $f\left(x,U\left(t\right),\ t\right)\in conv\ \Omega\left(E^n\right),\ t\in\left[0,T\right].$ $C_{\alpha},\ C_{\beta}\in conv\ \Omega\left(E^n\right),\ X$ is a closed convex subset of E^n.

Let $H\left(F,\psi\right)=\max\left\{(f,\psi):f\in F\right\}$ be the support function of the set $F\subset E^n$ in the direction of ψ, where (f,ψ) denotes the scalar product of vectors f and ψ. $T\left(A,a\right)$ and $N\left(A,a\right)$ are the tangent and normal cones to a closed, convex set A at a point $a\in A$, respectively. All finite-dimensional vectors are considered column vectors.

Function $u\left(t\right)\in U\left(t\right),\ t\in\left[0,T\right]$ is called an admissible control on the interval $\left[0,T\right]$, if it is measurable and one–valued branch of the multivalued mapping $U\left(t\right)$ so that the corresponding solution $x\left(t\right),\ t\in\left[0,T\right]$ of the given system of differential equations satisfy the initial condition $x\left(0\right)\in C_{\alpha}$ and the inclusion $x\left(t\right)\in X,\quad t\in\left[0,T\right].$

The challenge is in finding an admissible control $u\left(t\right),\ t\in\left[0,\overline{T}\right]$, so that the corresponding trajectory $x\left(t\right),\ t\in\left[0,T\right]$ satisfies condition $x\left(T\right)\in C_{\beta}$ and T is minimal.

This work is dedicated to deriving the maximum principle for which the optimal solution is regular.

Optimal solution is called regular, if the corresponding conjugate function is a nontrivial absolute continuous function.

Let $\left(\overline{x}\left(t\right),\ \overline{u}\left(t\right)\right),\ t\in\left[0,\overline{T}\right]$ be a solution to (1) and $U\left(\overline{x}\left(t\right),t\right)\subset U\left(t\right)$ be a subset, defined as:

$$U\left(\overline{x}\left(t\right),t\right)=\left\{u\in U\left(t\right):f\left(\overline{x}\left(t\right),\ u,t\right)\in T\left(X,\overline{x}\left(t\right)\right)\right\},\quad t\in\left[0,\overline{T}\right].$$

Non emptiness of the subset will be proved below.

Here, $T\left(X,\overline{x}\left(t\right)\right)$ is the tangent cone to X at $\overline{x}\left(t\right)\in X$, i.e.

$$T\left(X,\overline{x}\left(t\right)\right)=cl\left\{\lambda\left(y-\overline{x}\left(t\right)\right),\lambda\geq0,\ y\in X\right\},$$

where clA means the closure of A.

Consider the corresponding problem without phase constraints

$$\begin{cases} \dot{x}\left(t\right)\in f\left(x\left(t\right),\ U\left(t\right),t\right),\ t\in\left[0,T\right], \\ x\left(0\right)\in\widehat{C}_{\alpha},\ x\left(T\right)\in\widehat{C}_{\beta}, \\ T\rightarrow\min. \end{cases} \tag{2}$$

Definition. *If there are sets* $\widehat{C}_\alpha,\ \widehat{C}_\beta \in conv\ \Omega(E^n)$ *and the decision* $(x_0(t),\ u_0(t)),\ t\in[0,\overline{T}]$ *of the problem (2), for which the inclusions*

$$T\left(f(\overline{x}(t), U(\overline{x}(t),t),t), \dot{\overline{x}}(t)\right) \subset T\left(f(x_0(t), U(t),t), \dot{x}_0(t)\right), a.a.\ \ t\in[0,\overline{T}],$$

$$T(C_\alpha\cap X,\ \overline{x}(0)) \subset T\left(\widehat{C}_\alpha,\ x_0(0)\right),$$

$$T\left(C_\beta\cap X,\ \overline{x}(\overline{T})\right) \subset T\left(\widehat{C}_\beta,\ x_0(\overline{T})\right),$$

hold, then $(x_0(t),\ u_0(t)), t\in[0,\overline{T}]$ *is called similar to the solution* $(\overline{x}(t),\ \overline{u}(t)),\ t\in[0,\overline{T}]$ *of (1).*

Note. The main requirement of this definition is the equality of optimal values of quality criteria in the original (problem with state constraints (1)) and auxiliary (the problem without phase constraints (2)) problems.

Theorem. *Suppose, there exists a similar solution* $(x_0(t),\ u_0(t)),\ t\in[0,\overline{T}]$ *of (2) to* $(\overline{x}(t),\ \overline{u}(t)),\ t\in[0,\overline{T}]$. *Then, there exists a nonzero absolutely continuous solution of the adjoint system of differential equations*

$$\dot{\psi}(t) = -\frac{\partial f^*(x_0(t),\ u_0(t),t)}{\partial x}\psi(t),\ a.a.\ t\in[0,\overline{T}],$$

with the transversality conditions

$$H(C_\alpha\cap X,\ \psi(0)) = (\overline{x}(0),\ \psi(0)),\quad H\left(C_\beta\cap X,\ -\psi(\overline{T})\right) = \left(\overline{x}(\overline{T}),\ -\psi(\overline{T})\right),$$

for which, the maximum condition

$$\max\{(f(\overline{x}(t),\ u,t),\ \psi(t)) : u\in U(\overline{x}(t),\ t)\} = \left(\dot{\overline{x}}(t),\ \psi(t)\right), a.a.\ t\in[0,\overline{T}]$$

holds.

If the additional condition

$$\left(\frac{\partial f^*(\overline{x}(t),\ \overline{u}(t),t)}{\partial x} - \frac{\partial f^*(x_0(t),\ u_0(t),t)}{\partial x}\right)\psi(t) = 0,\ a.a.\ t\in[0,\overline{T}]$$

holds, then the conjugate system of differential equations will have the form

$$\dot{\psi}(t) = -\frac{\partial f^*(\overline{x}(t),\ \overline{u}(t),t)}{\partial x}\psi(t),\ a.a.\ t\in[0,\overline{T}],$$

with the transversality conditions

$$H(C_\alpha\cap X,\ \psi(0)) = (\overline{x}(0), \psi(0)),$$

$$H\left(C_\beta\cap X,\ -\psi(\overline{T})\right) = \left(\overline{x}(\overline{T}),\ -\psi(\overline{T})\right).$$

Before proving the theorem, we first prove the following lemma.

Lemma. *Let* $x(t),\ t\in[0,T]$ *be an absolutely continuous function, so that* $x(t)\in X,\ \forall t\in[0,T]$, *where* $X\subset E^n$ *is a closed convex subset. Then the inclusion*

$$\dot{x}(t)\in T(X, x(t)),\ a.a.\ t\in[0,T]$$

is true.

Proof. Assume that, there exists the derivative $\dot{x}(t_0)$ at the point $t_0 \in (0,T)$. By the definition,

$$\dot{x}(t_0) = \lim_{t\to t_0} \frac{x(t) - x(t_0)}{t - t_0}$$

Let $t > t_0$. Then

$$\frac{1}{t - t_0} \cdot (x(t) - x(t_0)) \in \{\lambda (y - x(t_0)), \lambda \geq 0, \ y \in X\},$$

since $x(t) \in X$ and $\dfrac{1}{t - t_0} > 0, \ \forall t \in [0,T], \ t > t_0$. Then

$$\lim_{\substack{t\to t_0 \\ t>t_0}} \frac{x(t) - x(t_0)}{t - t_0} \in cl\{\lambda (y - x(t_0)), \lambda \geq 0, \ y \in X\}.$$

Due to the fact that, there exists $\dot{x}(t_0)$, we will have the equality

$$\dot{x}(t_0) = \lim_{\substack{t\to t_0 \\ t>t_0}} \frac{x(t) - x(t_0)}{t - t_0} = \lim_{\substack{t\to t_0 \\ t<t_0}} \frac{x(t) - x(t_0)}{t - t_0}.$$

At the end points 0 and T we hold a similar argument regarding the left (at T) and the right (at 0) limits in the definition of the derivative.

Consequently, we find that

$$\dot{x}(t_0) \in T(X, x(t_0)).$$

From the arbitrariness of $t_0 \in [0,T]$ and the absolutely continuity of $x(t), \ t \in [0,T]$ we conclude that $\dot{x}(t) \in T(X, x(t)), \ a.a. \ t \in [0,T]$.

Thus, we have proved the lemma.

Corollary. *If* $x(t), \ t \in [0,T]$ *is any admissible solution of (1), then*

$$f(x(t), U(t), t) \cap T(X, x(t)) \neq \varnothing, \quad a.a. \ t \in [0,T].$$

The proof of the corollary follows immediately from the lemma.

Now we give the proof of the theorem. Since $(x_0(t), \ u_0(t)), \ t \in [0, \overline{T}]$ is a solution to problem (2), which is a problem without phase constraints, we can apply the Pontryagin maximum principle for this decision [1]. In other words, there exists a nontrivial absolutely continuous function $\psi(t), \ t \in [0, \overline{T}]$ as a solution of the dual system of equations

$$\dot{\psi}(t) = -\frac{\partial f^*(x_0(t), \ u_0(t), t)}{\partial x} \psi(t), \ a.a. \ t \in [0, \overline{T}],$$

with the transversality conditions

$$H\left(\widehat{C}_\alpha, \ \psi(0)\right) = (x_0(0), \psi(0)),$$

$$H\left(\widehat{C}_\beta, \ -\psi(\overline{T})\right) = \left(x_0(\overline{T}), \ -\psi(\overline{T})\right),$$

for which the maximum condition

$$\max\left\{(f(x_0(t),\ u,t),\ \psi(t)) : u\in U(t)\right\} = \left(\dot{x_0}(t),\ \psi(t)\right),\ \ a.a.\ t\in\left[0,\overline{T}\right]$$

holds.

The latest equalities mean that

$$\psi(0)\in N\left(\widehat{C}_\alpha,\ x_0(0)\right),\quad -\psi\left(\overline{T}\right)\in N\left(\widehat{C}_\beta,\ x_0\left(\overline{T}\right)\right), (3)$$

and

$$\psi(t)\in N\left(f(x_0(t),\ U(t),t), \dot{x_0}(t)\right),\quad a.a.\ t\in\left[0,\overline{T}\right],$$

where $N(A,a)$ is a normal cone of the set $A\in conv\,\Omega(E^n)$ at a point $a\in A$.

By the definition of the similar solutions and conditions of the theorem,

$$T\left(C_\alpha\cap X,\ \overline{x}(0)\right)\subset T\left(\widehat{C}_\alpha,\ x_0(0)\right),\quad T\left(C_\beta\cap X,\ \overline{x}\left(\overline{T}\right)\right)\subset T\left(\widehat{C}_\beta,\ x_0\left(\overline{T}\right)\right)$$

and

$$T\left(f\left(\overline{x}(t),U\left(\overline{x}(t),t\right),t\right),\dot{\overline{x}}(t)\right)\subset T\left(f\left(x_0(t),U(t),t\right),\dot{x_0}(t)\right),\ \ a.a.\ \ t\in\left[0,\overline{T}\right],$$

therefore,

$$N\left(\widehat{C}_\alpha,\ x_0(0)\right)\subset N\left(C_\alpha\cap X,\ \overline{x}(0)\right),\quad N\left(\widehat{C}_\beta,\ x_0\left(\overline{T}\right)\right)\subset N\left(C_\beta\cap X,\ \overline{x}\left(\overline{T}\right)\right)$$

and

$$N\left(f\left(x_0(t),U(t),t\right),\dot{x_0}(t)\right)\subset N\left(f\left(\overline{x}(t),U\left(\overline{x}(t),t\right),t\right),\dot{\overline{x}}(t)\right),\ \ a.a.\ \ t\in\left[0,\overline{T}\right],$$

From the inclusions (3), we conclude that the absolutely continuous nontrivial one valued branch $\psi(t),\ t\in\left[0,\overline{T}\right]$ of the multivalued map $N\left(f\left(x_0(t),U(t),t\right),\dot{x_0}(t)\right)$, with the conditions

$$\psi(0)\in N\left(\widehat{C}_\alpha,\ x_0(0)\right),\quad -\psi\left(\overline{T}\right)\in N\left(\widehat{C}_\beta,\ x_0\left(\overline{T}\right)\right),$$

is the one valued branch of the multivalued mapping

$$N\left(f\left(\overline{x}(t),U\left(\overline{x}(t),t\right),t\right),\dot{\overline{x}}(t)\right),\quad t\in\left[0,\overline{T}\right]$$

with the conditions

$$\psi(0)\in N\left(C_\alpha\cap X,\ \overline{x}(0)\right),\quad -\psi\left(\overline{T}\right)\in N\left(C_\beta\cap X,\ \overline{x}\left(\overline{T}\right)\right),$$

also. This means that

$$\max\left\{(f(\overline{x}(t),\ u,t),\ \psi(t)) : u\in U\left(\overline{x}(t),t\right)\right\} = \left(\dot{\overline{x}}(t),\ \psi(t)\right),\ \ a.a.\ t\in\left[0,\overline{T}\right]$$

and

$$H\left(C_\alpha \cap X,\ \psi(0)\right) = \left(\overline{x}(0), \psi(0)\right), \quad H\left(C_\beta \cap X,\ -\psi\left(\overline{T}\right)\right) = \left(\overline{x}\left(\overline{T}\right),\ -\psi\left(\overline{T}\right)\right).$$

In case of the additional condition of the theorem, it is easy to show that the conjugate function $\psi(t),\ t \in \left[0, \overline{T}\right]$ satisfies the system of the differential equations

$$\dot{\psi}(t) = -\frac{\partial f^*\left(\overline{x}(t),\ \overline{u}(t), t\right)}{\partial x}\psi(t),\ a.a.\ t \in \left[0, \overline{T}\right],$$

with the transversality conditions

$$H\left(C_\alpha \cap X,\ \psi(0)\right) = \left(\overline{x}(0), \psi(0)\right), \quad H\left(C_\beta \cap X,\ -\psi\left(\overline{T}\right)\right) = \left(\overline{x}\left(\overline{T}\right),\ -\psi\left(\overline{T}\right)\right).$$

The theorem is proved.

Consider a linear optimal control problem with state constraint

$$\begin{cases} \dot{x}(t) = A(t)\,x(t) + \eta(u,t), \\ x(0) \in C_\alpha,\ x(T) \in C_\beta \\ u(t) \in U(t),\ a.a.\ t \in [0,T], \\ x(t) \in X, \quad t \in [0,T], \\ T \to \min. \end{cases} \tag{4}$$

and the corresponding problem without state constraint

$$\begin{cases} \dot{x}(t) = A(t)\,x(t) + \eta(u,t), \\ x(0) \in \widehat{C}_\alpha,\ x(T) \in \widehat{C}_\beta \\ u(t) \in U(t),\ a.a.\ t \in [0,T], \\ T \to \min. \end{cases} \tag{5}$$

Here, A is a given $n \times n$ matrix of bounded measurable elements, $\eta(u,t)$ measurable in t and continuous in u, $\eta(U(t), t) \in conv\,\Omega(E^n),\ t \in [0,T]$ and $C_\alpha,\ C_\beta,\ \widehat{C}_\alpha,\ \widehat{C}_\beta \in conv\,\Omega(E^n)$.

Consequence. *Let there exist a similar solution $(x_0(t),\ u_0(t)),\ t \in \left[0, \overline{T}\right]$ of (5) to the solution $(\overline{x}(t),\ \overline{u}(t)),\ t \in \left[0, \overline{T}\right]$ of (4). Then there exists a nontrivial solution of the adjoint system of differential equations*

$$\dot{\psi}(t) = -A^*(t)\,\psi(t),\ a.a.\ t \in \left[0, \overline{T}\right],$$

with the transversality conditions

$$H\left(C_\alpha \cap X,\ \psi(0)\right) = \left(\overline{x}(0), \psi(0)\right), \quad H\left(C_\beta \cap X,\ -\psi\left(\overline{T}\right)\right) = \left(\overline{x}\left(\overline{T}\right),\ -\psi\left(\overline{T}\right)\right)$$

for which the maximum condition

$$\max\left\{\left(\eta(u,t), \psi(t)\right),\ u \in U\left(\overline{x}(t),\ t\right)\right\} = \left(\eta\left(\overline{u}(t), t\right),\ \psi(t)\right),\ a.a.\ t \in \left[0, \overline{T}\right]$$

holds. Where, $U(\overline{x}(t),\ t) \subset U(t)$ is a subset, so that

$$U\left(\overline{x}(t),\ t\right) = \left\{u \in U(t) : \eta(u,t) \in \left(T\left(X, \overline{x}(t)\right) - A(t)\,\overline{x}(t)\right)\right\},\quad t \in \left[0, \overline{T}\right].$$

The proof follows easily from the proof of the theorem.

Note. For linear on x systems, an additional condition is always satisfied, since in this case

$$\frac{\partial f^*(\overline{x}(t), \overline{u}(t), t)}{\partial x} = A^*(t) = \frac{\partial f^*(x_0(t), u_0(t), t)}{\partial x}, \ a.a.\ t \in \left[0, \overline{T}\right]$$

Thus, in cases, linear with respect to the phase coordinates, we need only condition for the existence of the similar solutions.

Example. Consider the time optimal control problem with state constraint

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \quad |u| \le 1,\ x(0) = x_0,\ x(T) = 0,\ x_2 - 2x_1 \le 2, \\ T \to \min \end{cases}$$

In the example we show that the similarity condition holds, therefore there exists the absolutely continuous nontrivial adjoint function $\psi(t)$, $t \in \left[0, \overline{T}\right]$ for which the maximum condition holds.

Assume that $x_0 = \left(0, \ -\frac{\sqrt{15}}{2\sqrt{2}}\right)$.

The optimality of the control function (Fig. 1)

$$\overline{u}(t) = \begin{cases} +1, \quad t \in [0, \tau_1], \\ u(\overline{x}(t)), \quad t \in (\tau_1, \tau_2], \\ +1, \quad t \in (\tau_2, \tau_3], \\ -1, \quad t \in \left(\tau_3, \overline{T}\right] \end{cases}$$

may be shown by [4].

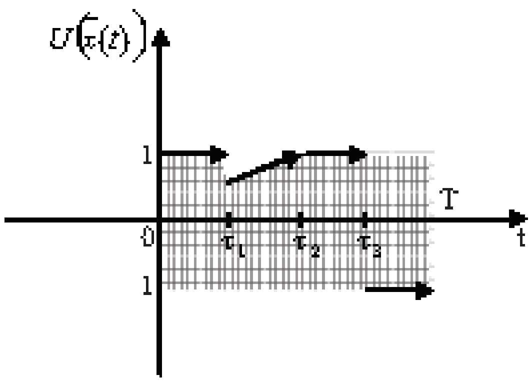

Fig. 1.

In other words, on the time interval $[0, \tau_1]$ we are moving along the curve $x_1 = \frac{1}{2}x_2^2 - \frac{15}{16}$, then on the time interval $[\tau_1, \tau_2]$ along the straight line $x_2 - 2x_1 = 2$, after that on the time interval $[\tau_2, \tau_3]$ along the curve $x_1 = \frac{1}{2}x_2^2 - \frac{7}{8}$, then on the time interval $\left[\tau_3, \overline{T}\right]$ along the parabola $x_1 = -\frac{1}{2}x_2^2$ (Fig. 2).

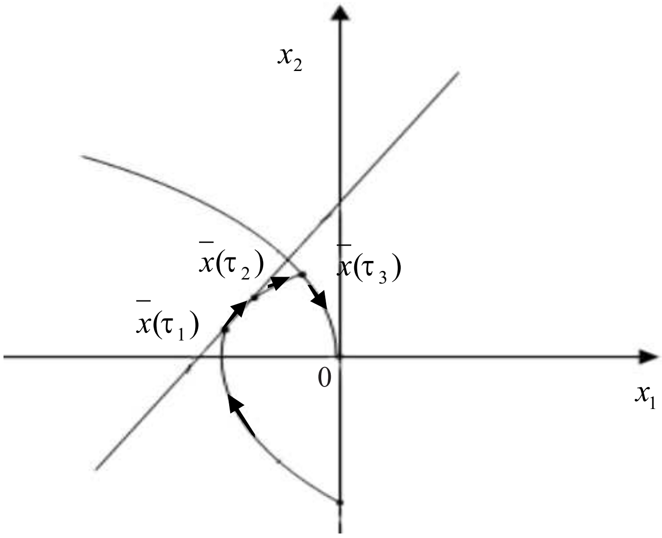

Fig. 2.

By the calculation we find that

$$\overline{x}(\tau_1)=\left(-\frac{1+3\sqrt{2}}{4\sqrt{2}},\frac{\sqrt{2}-1}{2\sqrt{2}}\right)\quad\text{and}\quad\overline{x}(\tau_2)=\left(-\frac{3}{4},\frac{1}{2}\right).$$

The corresponding tangent cone is:

$$T\left(U\left(\overline{x}\left(t\right)\right),\overline{u}\left(t\right)\right)=\begin{cases}(-\infty,0], & t\in[0,\tau_3]\\ [0,+\infty), & t\in\left[\tau_3,\overline{T}\right]\end{cases}$$

The auxiliary problem without state constraint

$$\begin{cases}\dot{x}_1=x_2,\\ \dot{x}_2=u,\quad |u|\le 1,\ x(0)=\widehat{x}_\alpha,\ x(T)=0,\\ T\to\min\end{cases}$$

has the optimal control function

$$u_0(t)=\begin{cases}1, & t\in[0,\tau_3],\\ -1, & t\in\left(\tau_3,\overline{T}\right]\end{cases}$$

which is similar to the $\overline{u}(t),\ t\in\left[0,\overline{T}\right]$, because

$$T\left(U\left(\overline{x}\left(t\right)\right),\overline{u}\left(t\right)\right)=T\left(U,u_0\left(t\right)\right),\ a.a.\ t\in\left[0,\overline{T}\right]$$

Then there exists absolutely continuous function $\psi(t)=\begin{pmatrix}1\\ \tau_3-t\end{pmatrix},\ t\in\left[0,\overline{T}\right]$ (Fig. 3), as a solution of the adjoint system of differential equations

$$\begin{cases}\dot{\psi}_1=0\\ \dot{\psi}_2=-\psi_1\end{cases}$$

for which the maximum condition

$$H\left(U\left(\overline{x}\left(t\right)\right),\psi\left(t\right)\right)=\left(\overline{u}\left(t\right),\psi\left(t\right)\right),\ a.a.\ t\in\left[0,\overline{T}\right]$$

holds.

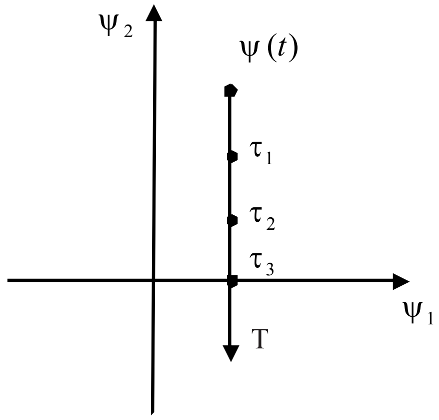

Fig. 3.

Conclusion. Note that, the results obtained in this study include the entire regular optimal trajectory, i. e. the optimal trajectory is investigated as a whole, not dividing it to the boundary or interior parts.

The advantage of this result is the fact that the adjoint equation is much simpler and has the same form as in optimal control problems without state constraints and regular trajectories in this case may be irregular for the whole set U.

A specialty of this work is also that the maximum condition is not taken on a set U, but on a subset $U\left(\overline{x}\left(t\right)\right)\subset U$, as done in [14], [15].

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