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ON NOETHER PROPERTY AND INDEX FORMULA OF A PROBLEM

Abstract

In the paper an abstract analogue of the Riemann boundary value problem generated by several operators is considered. Under definite conditions its Noether property is shown and a formula for the index is given.

Many theorems on perturbations, generally speaking, in Banach spaces are of great scientific interest from the point of view of applications in different fields of mathematics. One of these fields is the theory of bases. Apparently, such an approach for establishing bases originates in the paper of Paley, Wiener [1] on the Riesz basicity of the perturbed system of exponents $\left\{e^{i\lambda_n t}\right\}_{n\in Z}$ (Z is the set of integers, C are complex numbers) in $L_2(-\pi,\pi)$, where $\{\lambda_n\}_{n\in Z}\subset C$ is some sequence. Afterwards this direction was rapidly developed. This was elucidated in detail in the review paper [2] and monographs [3-5]. Mainly, the theorems applied here concern small in this or other sense perturbations. If, for instance, the asymptotics λ_n contains the principal part of the form $n+\alpha sign\ n$, as usual such theorems are inapplicable. In this case another investigation methods are used. One of these methods is the method of theory of boundary value problems for analytic functions. The principles of this method was laid in [6]. Further, the authors of the papers [7-12] successfully used this method. In the suggested paper, an abstract generalization of the method of boundary value problems is given.

1. Necessary notion and facts. Briefly we'll call the Banach space a B-space. By $L(X;Y)$ we denote B-space of bounded operators acting from X to Y; $L(X)\equiv L(X;X)$. $KerT$ is a kernel of the operator $T: KerT\equiv\{x: Tx=0\}$. $\dim(\cdot)$ is dimensionality (linear) $(\cdot)$, $\overline{M}$ is a closure of the set M in the corresponding space, X^* is a space conjugated to X, D_T is a domain of definition of the operator T, $R(T)$ is a space of values of the operator T.

Let X,Y be some B-spaces, $T: X\to Y$ a linear operator.

The factor-space $Y\Big/\overline{R(T)}$ is called a co-kernal of the operator T and is denoted as $coKerT$. Assume $\alpha(T)=\dim KerT$, $\beta(T)=\dim coKerT$. The ordered pair $(\alpha(T),\beta(T))$ is said to be d - characteristic of the operator T. If one of the components of this pair is finite, the difference $\alpha(T)-\beta(T)$ is called the index of the operator T and is denoted as $indT$.

Definition 1. *The operator $T: X\to Y$ is called Hausdorff normally solvable if the equation $Tx=y$ for the given $y\in Y$ has at least one solution $x\in X$ iff $f(y)=0$, $\forall f\in(R(T))^{\perp}$, where*

$$(R(T))^{\perp}=\{f\in Y^*: f(Tx)=0,\quad \forall x\in D_T\}.$$

Definition 2. *The closed, normally solvable operator T is said to be Noether or Φ - operator if its d-characteristic is finite.*

2. Abstract analogues of boundary value problems. Let X be some B - space; $X^{\pm}$ its subspace, and it hold the direct expansion

$$X = X^{+} \dot{+} X^{-}.$$

Assume that $T^{\pm} \in L\left(X^{\pm}; X\right)$ are some operators. Take $\forall y \in X$ and consider the equation

$$T^{+}x^{+} + T^{-}x^{-} = y, \tag{1}$$

with respect to $(x^{+}; x^{-})$, i.e. we look for a pair $(x^{+}; x^{-}) \in X^{+} \times X^{-}$ that satisfies equality (1).

Define $T \in L(X) : Tx = T^{+}x^{+} + T^{-}x^{-}$, $\forall x = x^{+} + x^{-}$, $x^{\pm} \in X^{\pm}$.

We call this problem pertaining to Noether if the followings are fulfilled:

a) $R(T^{+})$ and $R(T^{-})$ are closed;

b) $\dim co\ker T < +\infty$;

c) $\dim\left[R(T^{+}) \cap R(T^{-})\right] < +\infty$;

d) $\dim\left[KerT^{\pm}\right] < +\infty$.

From a) it follows that $R(T^{+}) + R(T^{-})$ is closed in X. Therefore, equation (1) is solvable iff $y \in \left[R(T^{+}) + R(T^{-})\right]$. Equation (1) has a unique solution iff $R(T^{+}) \cap R(T^{-}) = \{0\}$, $KerT^{+} = KerT^{-} = \{0\}$. Indeed, let $T^{+}x_k^{+} + T^{-}x_k^{-} = y, k = 1, 2$. Then $y^{+} = T^{+}\left(x_2^{+} - x_1^{+}\right) = -T^{-}\left(x_2^{-} - x_1^{-}\right) = y^{-} \in R(T^{+}) \cap R(T^{-})$. If $KerT^{+}\,(KerT^{-}) \neq \{0\}$, it is clear that the uniqueness of the solution doesn't hold. If $R(T^{+}) \cap R(T^{-}) \neq \{0\}$, then $\exists y_0 \neq 0$, $y_0 \in R(T^{+}) \cap R(T^{-})$. Let $T^{\pm}x_0^{\pm} = y_0$. It is clear that $x_0^{\pm} \neq 0$. Then it is obvious that alongside with $(x^{+}; x^{-})$ the pair $\left(x^{+} + x^{+}; x^{-} - x_0^{-}\right)$ also is a solution.

Denote $\alpha^{+} = \dim coKer\left[R(T^{+}) + R(T^{-})\right]$, $\alpha^{-} = \dim\left[R(T^{+}) \cap R(T^{-})\right] + $ $+ \dim KerT^{+} + \dim KerT^{-}$. α^{+} is called defect, α^{-} excess of equation (1). The number $\alpha = \alpha^{+} - \alpha^{-}$ is said to be an index of equation (1). Equation (1) is called Fredholm if $\alpha = 0$, i.e. $\alpha^{+} = \alpha^{-}$.

Consider the operator $T : X^{+} \times X^{-} \to X$ determined by the expression $T\left[(x^{+}; x^{-})\right] = T^{+}x^{+} + T^{-}x^{-}$, $\forall (x^{+}; x^{-}) \in X^{+} \times X^{-}$. We give the norm in $X^{+} \times X^{-}$ in such a way that

$$\left\|\left(x^{+}; x^{-}\right)\right\|_p = \left(\left\|x^{+}\right\|^p + \left\|x^{-}\right\|^p\right)^{1/p}, \; p \geq 1,$$

and denote the corresponding B - space by $X^{+} \otimes_p X^{-}$. It is obvious that $T \in L\left(X^{+} \otimes_p X^{-}; X\right)$. We can write equation (1) in the form:

$$Tx = y, \; x \in X^{+} \otimes_p X^{-}. \tag{2}$$

From the conditions a) -d) it follows that T is a Noether operator. Therefore, all the theory for Noether equation (2) may be taken to the case (1).

Case (1) is the abstract analogue of the known boundary value problems of theory of analytic functions in the Hardy $H_p^{\pm}$ and Smirnov $E_p^{\pm}$ classes. In this case let us take the subspaces H_p^+ and H_p^- (E_p^+ and E_p^-), of Lebesgue spaces L_p instead of the subspaces X^+ and X^-, respectively ($p > 1$).

3. General case. We'll consider the general case. So, let B-space $(X; \|\cdot\|)$ have the direct expansion

$$X = X_1 \dot{+} X_2 \dot{+} ... \dot{+} X_r = \sum_{k=1}^{r} (\dot{+} X_k),$$

in subspaces X_k, $k = \overline{1,r}$. Let $T_k \in L(X_k; X)$, $k = \overline{1,r}$ be some operators. Take $y \in X$ and consider the following equation

$$\sum_{k=1}^{r} T_k x_k = y, x_k \in X_k, k = \overline{1,r}. \tag{3}$$

Determine the operator $T : X \to X$:

$$Tx = \sum_{k=1}^{r} T_k x_k, \forall x = \sum_{k=1}^{r} x_k, x_k \in X_k.$$

To simplify the formula for $\dim KerT$ we accept some notation.

Assume

$$X^{(m)} = X_1 \dot{+} X_2 \dot{+} ... \dot{+} X_m, \quad m = \overline{1,r}.$$

Determine the operator

$$T^{(m)} x^{(m)} = T_1 x_1 + ... + T_m x_m,$$

where

$$x^{(m)} = x_1 + ... + x_m, \quad x_k \in X_k, \quad k = \overline{1,m}.$$

This problem is said to be Noether if the following conditions are fulfilled:

a) $R(T_k)$, $k = \overline{1,r}$ are closed;

b) $\dim co\ker T < +\infty$;

c) $\dim [R(T_i) \cap R(T_j)] < +\infty$, $\forall i \neq j$;

d) $\dim KerT_k < +\infty$, $k = \overline{1,r}$.

Notice that under $\sum_{k=1}^{r} R(T_k)$ we understand an ordinary sum of subspaces $R(T_k)$, $k = \overline{1,r}$. Equation (3) is solvable iff $y \in \sum_{k=1}^{r} R(T_k)$. Similar to case (1) we show that it has a unique solution iff $KerT_k = \{0\}$, $\forall k = \overline{1,r}$ and $R(T_i) \cap R(T_j) = \{0\}$, $\forall i \neq j$.

Assume $a^+ = \dim coKer \sum_{k=1}^{r} R(T_k)$ and $\alpha^- = \dim KerT = \dim KerT^{(r-1)} + \dim KerT_r + \dim \left[R\left(T^{(r-1)}\right) \cap R(T_r)\right]$. The number α^+ (α^-) is said to be defect (excess) of equation (3). The number $\alpha = \alpha^{+} - \alpha^{-}$ is called an index of equation (3). Equation (3) for $\alpha = 0$ is called Fredholm.

It is clear that $Ker\,(T_i) \cap Ker\,(T_j) = \{0\}$, $\forall i \neq j$.

Thus, we can represent equation (3) in the form

$$Tx = y \tag{4}$$

From conditions a)-d) it follows that the operator T is Noether. Show that the following formula is valid

$$\dim KerT = \dim KerT^{(r-1)} + \dim KerT_r + \dim \left[R\left(T^{(r-1)}\right) \cap R\,(T_r)\right],$$

It is enough to give the proof for the case $r = 2$. It is easily taken to the general case by the mathematical induction method.

So, let $X = X_1 \dot{+} X_2$ and $Tx = T_1x_1 + T_2x_2$, $x = x_1 + x_2$, $x_k \in X_k$, $k = 1, 2$.

For simplicity of the statement, carry out the proof step by step.

1) Let $\exists T_2^{-1}$ and T_1 maps X_1 onto $X^{\cap} \equiv R\,(T_1) \cap R\,(T_2)$ one-to-one, i.e. $\dim X_1 = \dim X^{\cap}$, $T_1 : X_1 \leftrightarrow X^{\cap}$. Consider the operator $J : X_1 \to X$:

$J = I - T_2^{-1}T_1$, where $I \in L\,(X)$ is a unit operator. Take $\forall x_1 \in X_1$:

$Jx_1 = x_1 - T_2^{-1}T_1x_1 = x_1 + x_2 = x$, where $x_2 = T_2^{-1}T_1x_1$.

We have

$$Tx = T_1x_1 + T_2x_2 = T_1x_1 - T_1x_1 = 0 \Rightarrow x \in KerT.$$

Consequently, $J : X_1 \to KerT$.

Take $\forall x \in KerT$, $x = x_1 + x_2$. We have

$$Tx = 0 \Rightarrow T_1x_1 = -T_2x_2 \Rightarrow x_2 = -T_2^{-1}T_1x_1 \Rightarrow x = x_1 + x_2 = \left(I - T_2^{-1}T_1\right)x_1 = Jx.$$

Thus, J maps X_1 onto $KerT$.

Let $Jx_1 = 0$, $x_1 \in X_1$. Then $x_1 = T_2^{-1}T_1x_1 \in X_2$. Since $X_1 \cap X_2 = \{0\}$, if follows that $x_1 = 0$. Consequently, J one-to-one maps X_1 onto $KerT$ and it holds

$$\dim KerT = \dim X_1 = \dim X^{\cap}.$$

2) $\exists T_2^{-1}$, $k = 1, 2$. Assume $X_1^{\cap} = T_1^{-1}\,(X_1^{\cap})$. It is clear that $X_1^{\cap}$ is a finite-dimensional subspace of X_1. Consider the operator $J_1 : X_1^{\cap} \to X : J_1 = I - T_2^{-1}T_1$. It is easy to see that $R\,(J_1) \subset KerT$. Let $x \in Ker\not{T}$, $x = x_1 + x_2 \Rightarrow T_1x_1 = -T_2x_2 \in X^{\cap} \Rightarrow x = J_1x_1 \Rightarrow R\,(J_1) \equiv KerT$. Take $\forall x_1 \in KerJ_1 \Rightarrow x_1 = T_2^{-1}T_1x_1 \in X_2 \Rightarrow x_1 = 0$, since $x_1 \in X_1^{\cap} \subset X_1$. As a result, $X_1^{\cap}$ is isomorphic to $KerT$ and so $\dim KerT = \dim X_1^{\cap} = \dim X^{\cap}$ is valid.

3) $\exists T_2^{-1}$; $\dim KerT_1 < +\infty$. Consider the direct expansion X_1:

$$X_1 = KerT_1 \dot{+} \widetilde{X}_1.$$

Thus,

$$X = KerT_1 \dot{+} \widetilde{X}_1 \dot{+} X_2 = KerT_1 \dot{+} \widetilde{X}.$$

It is obvious that $KerT_1 \subset KerT$. By $\widetilde{T}$ denote the contraction of the operator T on $\widetilde{X}$, i. e. $\widetilde{T} = T/_{\widetilde{X}}$. It is clear that it holds

$$KerT = KerT_1 \dot{+} Ker\widetilde{T} \Rightarrow \dim KerT = \dim KerT_1 + \dim Ker\widetilde{T}.$$

It is easy to see that the contraction $\widetilde{T}_1$ of the operator T_1 on $\widetilde{X}_1$ has the inverse $\widetilde{T}_1^{-1}$ as the operator $\widetilde{T}_1 : \widetilde{X}_1 \to X$. Furthermore, it holds $R\left(\widetilde{T}_1\right) = R(T_1) \Rightarrow R\left(\widetilde{T}_1\right) \cap R(T_2) = X^{\cap}$. Then, according to case 2) we have

$$\dim Ker\widetilde{T} = \dim X^{\cap},$$

and it is valid

$$\dim KerT = \dim KerT_1 + \dim X^{\cap}.$$

4) $\dim KerT_k < +\infty$, $k = 1, 2$. We successively carry out the reasoning of case 3) and get the formula

$$\dim KerT = \dim KerT_1 + \dim KerT_2 + \dim X^{\cap},$$

i.e.

$$\dim KerT = \dim KerT_1 + \dim KerT_2 + \dim \left[R(T_1) \cap R(T_2)\right].$$

Thus, the following main theorem is proved.

Theorem 1. *Let B - space X have the direct expansion*

$$X = \sum_{k=1}^{r} \left(\dot{+} X_k\right).$$

It the operators $T_k \in L(X_k; X), k = \overline{1, r}$ satisfy the conditions a) – d) , then for the operator $T : X \to X$:

$$Tx = \sum_{k=1}^{r} T_k x_k, x = \sum_{k=1}^{r} x_k, x_k \in X_k, k = \overline{1, r},$$

the following formula is true:

$$\dim KerT = \dim KerT^{(r-1)} + \dim KerT_r + \dim \left[R\left(T^{(r-1)}\right) \cap R(T_r)\right].$$

Show a method for constructing a basis in $KerT$ when $r = 2$. Let $\left\{e_k^i\right\}_{k=1}^{m_i}$ form a basis in $KerT_i$, $i = 1, 2$, where $m_i = \dim KerT_i$. Take an arbitrary basis $\{y_k\}_1^m$ in $R(T_1) \cap R(T_2)$, $m = \dim \left[R(T_1) \cap R(T_2)\right]$. Let $T_i x_k^i = y_k$, $k = \overline{1, m}$; $i = 1, 2$. Consider the system $\{x_k\}_1^m$, where $x_k = x_k^1 - x_k^2$, $k = \overline{1, m}$. It is easy to see that the elements of the systems $\left\{x_k^i\right\}_1^m$, $i = 1, 2$; are linearly independent. From $x_k^i \in X_i$, $i = 1, 2$; it directly follows that x_k, $k = \overline{1, m}$ are also linearly independent. Show that the set

$$E = \left\{e_k^1\right\}_1^{m_1} \cup \left\{e_k^2\right\}_1^{m_2} \cup \{x_k\}_1^m, \tag{5}$$

is linearly independent. Let

$$\sum_{k=1}^{m_1}\alpha_k e_k^1+\sum_{k=1}^{m_2}\beta_k e_k^2+\sum_{k=1}^{m}\gamma_k x_k=0.$$

We have

$$\sum_{k=1}^{m_1}\alpha_k e_k^1+\sum_{k=1}^{m_2}\gamma_k x_k^1=\sum_{k=1}^{m}\gamma_k x_k^2-\sum_{k=1}^{m_2}\beta_k e_k^2.$$

Consequently,

$$\sum_{k=1}^{m_1}\alpha_k e_k^1+\sum_{k=1}^{m}\gamma_k x_k^1=0,\qquad \sum_{k=1}^{m}\gamma_k x_k^2-\sum_{k=1}^{m_2}\beta_k e_k^2=0.$$

Hence we get

$$T\left(\sum_{k=1}^{m_1}\alpha_k e_k^1+\sum_{k=1}^{m}\gamma_k x_k^1\right)=\sum_{k=1}^{m}\gamma_k y_k=0\Rightarrow\gamma_k=0,\ \forall k=\overline{1,m}.$$

From the previous equalities it follows that $\alpha_k=0$, $\forall k=\overline{1,m}$ and $\beta_k=0$, $m=\overline{1,m_2}$. Thus, (5) is a linearly independent system. It is clear that the linear span $L[E]$ belongs to $KerT : L[E]\subset KerT$. Then, the following statement directly follows from theorem 1

Statement 1. Let a B - space X have the direct expansion $X=X^+\dot{+}X^-$ the operators $T^\pm\in L(X^\pm;X)$ satisfy the conditions a) – d). Determine $T\in L(X)$: $Tx=T^+x^++T^-x^-$, $\forall x=x^++x^-$, $x^\pm\in X^\pm$. Then system (5) constructed above forms a basis in $KerT$.

We can apply the obtained results to concrete cases. For example, let $N_k^\pm$, $k=\overline{1,r^\pm}$ be subsets of natural numbers N ordered in the usual way. Assume that it holds the expansion

$$N=\bigcup_{k=1}^{r^\pm}N_k^\pm,N_i^\pm\bigcap N_j^\pm=\varnothing\text{ for }i\neq j.$$

By H_p^+, $1\le p<+\infty$ denote the Hardy classic class of functions analytic interior to a unit circle ω. Non-tangential boundary values on a unit circle $\partial\omega$ of the function $f\in H_p^+$ from within a unit circle denote by f^+. The Fourier coefficients of the function $f^+\left(e^{it}\right)$ in the system $\left\{e^{int}\right\}_{n\in Z}$ denote by $\{f_n^+\}_{n\in Z}$. Assume

$$H_{p,k}^+\equiv\left\{f\in H_p^+ : f_n^+=0,\forall n\notin N_k^+\right\},\quad k=\overline{1,r^+}.$$

It is clear that $H_{p,k}^+$, $k=\overline{1,r^+}$ is a subspace in H_p^+ and $H_{p,i}^+\cap H_{p,j}^+=\{0\}$ for $i\neq j$. $\{z^n\}_{n\in N_k^+}$, $k=\overline{1,r^+}$ is a basis of the subspace H_p^+. It is easy to see that it holds the direct expansion

$$H_p^+=C\dot{+}\sum_{k=1}^{r^+}\left(\dot{+}H_{p,k}^+\right).$$

Let H_p^- be a class of Hardy functions that are analytic exterior to unit circle and disappear at infinity. By f^- denote non-tangential boundary values of the function $f \in H_p^-$ on $\partial\omega$. Assume

$$H_{p,k}^- \equiv \left\{ f \in H_p^- : f_n^- = 0, \forall n \in N_k^- \right\}, \quad k = \overline{1, r^-},$$

where $\{f_n^-\}_{n \geq 1}$ are the Fourier coefficients of the function $f^-\left(e^{it}\right)$ in the system $\left\{e^{int}\right\}_{n \in Z}$. It holds

$$H_p^- = \sum_{k=1}^{r^-} \left(\dot{+} H_{p,k}^+ \right).$$

The direct sum $C \dot{+} H_{p,k_0}^+$ for an arbitrary fixed $k_0 \in \{1, ..., r^+\}$ again denote by H_{p,k_0}^+. Let $L_p \equiv L_p(-\pi, \pi)$ be an ordinary Lebesgue space of p-th degree summable functions on $(-\pi, \pi)$. It is clear that the following direct expansion is valid:

$$L_p = \sum_{k=1}^{r^+} \left(\dot{+} H_{p,k}^+ \right) \dot{+} \sum_{k=1}^{r^-} \left(\dot{+} H_{p,k}^- \right).$$

Enumerate the union of $\left\{ H_{p,k}^+ \right\}_1^{r^+} \cup \left\{ H_{p,k}^- \right\}_1^{r^-}$ and denote it by $\{H_{p,k}\}_1^r$. Thus

$$L_p = \sum_{k=1}^{r} \dot{+} H_{p,k}.$$

Let $T_k \in L(H_{p,k}; L_p) : T_k F = A_k F$, $k = \overline{1, r}$, where

1) $A_k^{\pm 1} \in L_\infty$, $\forall k = \overline{1, r}$ are some functions.

Let $f \in L_p$ be some function. State the problem

$$\left. \begin{array}{c} \sum\limits_{k=1}^{r} A_k F_k = f, \\ F_k \in H_{p,k}, \quad k = \overline{1, r}, \end{array} \right\} \tag{6}$$

i.e. we look for the analytic functions $F_k \in H_{p,k}$ whose boundary values on $\partial\omega$ a.e. satisfy relation (6). For $r = 2$ it is a classic conjugation problem in the classes $H_p^\pm$ and the theory of this problem has been studied well (see [13]).

It is clear that if the following condition is fulfilled:

2) $\inf \underset{(-\pi,\pi)}{\nu rai} |A_k| > 0$, $\forall k = \overline{1, r}$, then $KerT_k = \{0\}$, $k = \overline{1, r}$. From conditions 1), 2) it directly follows that the operators T_k are closed in L_p. Thus, if conditions a) - d) are fulfilled for the operators T_k, problem (6) is Noether and its index æ equals

$$æ = \dim coKerT - \dim \left[R\left(T^{(r-1)}\right) \cap R(T_r) \right],$$

where

$$T^{(r-1)} = T_1 + ... + T_{r-1}.$$

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